Are you getting the right signal?

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National Institute of Education, NTU

27 July 2012

Outline

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Music to your ears

I used to wonder how an entire orchestra can fit into my head-set . . .

Music to your ears

Let us try to ...

Explain how this works and get to the mathematics of it.

Music to your ears

What is perhaps surprising is that it has to do with ...

Music to your ears

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VECTORS

Part I

Vectorial background

Vectors

Definition (Näive)

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- magnitude, and

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A vector is a mathematical entity that has both

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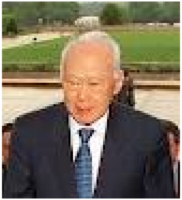
Vectors

Your first encounter with vectors would probably acquaint you with some “you-must-know” friends, i.e.,



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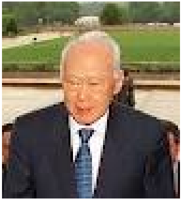

$\mathbf{0}, \ -\mathbf{a}, \ \hat{\mathbf{a}}$

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The zero vector *behaves like* 0 in $\mathbb{R}$.

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Negative vectors *behave like* negative numbers in $\mathbb{R}$.

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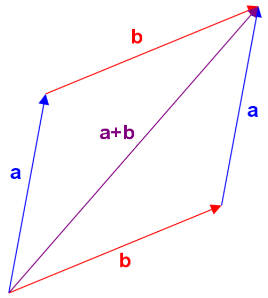
Vector addition, $+$



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Example

Let $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$.

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Let $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

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Then, for any vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, we have:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

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So, the set

$$\{\mathbf{i},\ \mathbf{j},\ \mathbf{k}\}$$

forms the basic building blocks – *basis*.

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The unit vector in the same direction as $\mathbf{a}$ is defined to be

$$\hat{\mathbf{a}} := \frac{1}{|\mathbf{a}|}\mathbf{a}$$

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$$\hat{\mathbf{a}} := \frac{1}{|\mathbf{a}|}\mathbf{a} \text{ or } \frac{\mathbf{a}}{|\mathbf{a}|}.$$

Here, $|\mathbf{a}|$ is the magnitude or length of $\mathbf{a}$.

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- magnitude or length of a vector

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Scalar multiplication of a vector $\mathbf{a}$ to a real scalar α dilates it:

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Scalar multiplication of a vector **a** to a real scalar α dilates it:

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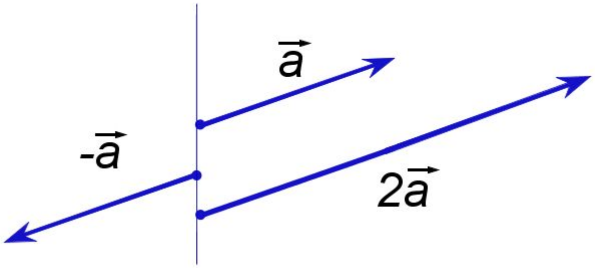

Figure: $\alpha = -1,\ 2$

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Example

As λ varies, the point P with position vector

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$$

traces out the straight line through $A(1, 2, -1)$ parallel to $\begin{pmatrix} -1 \\ -2 \\ 3 \end{pmatrix}$.

Enlarging the scope

Figure: Who are these?

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Paradigm shift

Any set of mathematical entities, endowed with $+$ and $\cdot$, which satisfies all these conditions is called a *vector space*, and the elements of which *vectors*.

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Any set of mathematical entities, endowed with $+$ and $\cdot$, which satisfies all these conditions is called a *vector space*, and the elements of which *vectors*.

We make a new definition out of properties observed in a special case!

Enlarging the scope

Example

Let $\mathcal{C}$ be the set of continuous real-valued functions

$$f : \mathbb{R} \longrightarrow \mathbb{R}.$$

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Define $+$ on $\mathcal{C}$ as follows:

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Graphically, addition can be interpreted below:

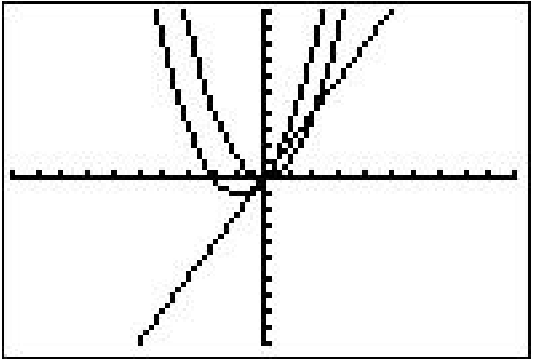

Figure: Pointwise addition

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Graphically, scalar multiplication can be interpreted below:

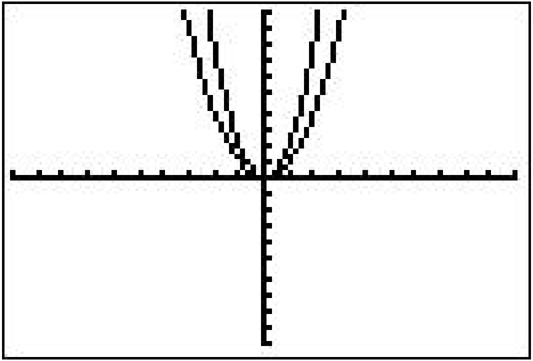

Figure: Pointwise scaling

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8. $\lambda \cdot (f + g) = \lambda \cdot f + \lambda \cdot g$

Enlarging the scope

Theorem

The set of continuous functions $\mathcal{C}$*, together with addition scalar multiplication of functions, forms a vector space* $\langle \mathcal{C}, +, \cdot \rangle$.

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Proof.

The zero vector is the zero function

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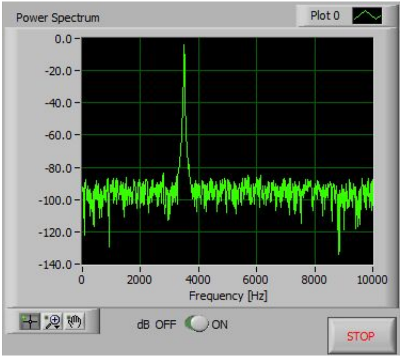
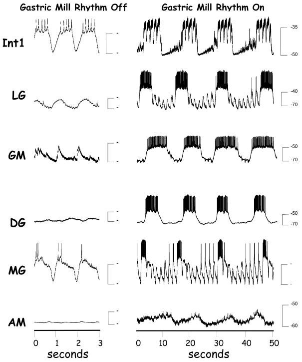
$$-f : \mathbb{R} \longrightarrow \mathbb{R}, \ x \mapsto -f(x), \ x \in \mathbb{R}.$$

□

Paradigm shift

Physical variations such as signals and seismic waves





are actually (continuous) functions of time and space.

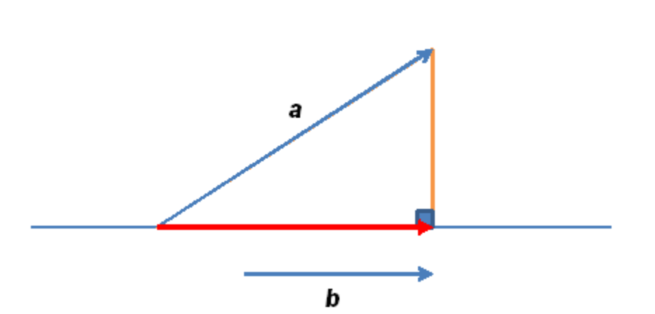
Motivation

Because an arbitrary continuous signal is hard to code mathematically, it may perhaps be good to break it down into (several) basic components.

Projection problem

Question

Find the projection of the vector $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ onto the vector $\mathbf{b} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$.



Projection problem

To tackle the projection problem, we need a piece of vector machinery called

Dot Product.

Dot product

Definition (Dot product)

The dot product (a.k.a. scalar product or inner product) of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is defined to be the real scalar

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It appears that the above definition of $\mathbf{a} \cdot \mathbf{b}$ is dependent on the coordinate basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

Dot product

Suppose the position vectors $\mathbf{a}$ and $\mathbf{b}$ of the points A and B with respect to the origin O.

Dot product

Suppose the position vectors $\mathbf{a}$ and $\mathbf{b}$ of the points A and B with respect to the origin O.
Writing $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$,

$$\vec{AB} = \mathbf{b} - \mathbf{a} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}.$$

Dot product

Suppose the position vectors $\mathbf{a}$ and $\mathbf{b}$ of the points A and B with respect to the origin O.
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$$\vec{AB} = \mathbf{b} - \mathbf{a} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}.$$

Using the Pythagoras Theorem,

$$AB^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2.$$

Dot product

On the other hand, we can make use of the angle $\angle AOB = \theta$ and the cosine rule

$$AB^2 = OA^2 + OB^2 - 2(OA)(OB)\cos\theta.$$

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This yields

$$\begin{aligned} AB^2 = & (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) \\ & - 2\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}\cos\theta \end{aligned}$$

Dot product

Equating the two expressions for AB^2, we have

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2 = (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2) \\ - 2\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}\cos\theta.$$

Dot product

Expanding the left hand side and cancelling the squares, we have

$$-2(a_1b_1 + a_2b_2 + a_3b_3) = -2\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}\cos\theta,$$

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Thus,

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta.$$

Dot product

Given any points $O,\ A$ and B, all the three quantities

- OA,
- OB, and
- $\angle AOB$

are fixed and thus independent of any coordinated basis.

Dot product

This then gives rise to a coordinates-independent definition of the dot product.

Dot product

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Definition (Dot product)

The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined to be the real scalar

$$\mathbf{a} \cdot \mathbf{b} := |\mathbf{a}||\mathbf{b}| \cos \theta,$$

where $\theta = \angle AOB$.

Dot product

Theorem

Let $\mathbf{a}$ *and* $\mathbf{b}$ *be given vectors, and the angle between them* θ*. Then we have the following:*

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Let $\mathbf{a}$ *and* $\mathbf{b}$ *be given vectors, and the angle between them* θ*. Then we have the following:*

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2. $\mathbf{a} \cdot \mathbf{b} = 0 \iff \theta = \frac{\pi}{2}$.

Dot product

Theorem

Let $\mathbf{a}$ *and* $\mathbf{b}$ *be given vectors, and the angle between them* θ*. Then we have the following:*

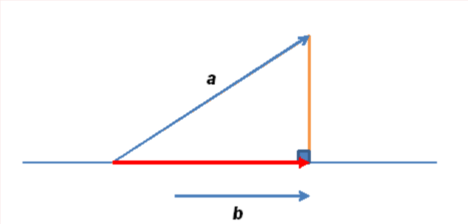
1. $\mathbf{a} \cdot \mathbf{b} > 0 \iff 0 \leq \theta < \frac{\pi}{2}$.
2. $\mathbf{a} \cdot \mathbf{b} = 0 \iff \theta = \frac{\pi}{2}$.
3. $\mathbf{a} \cdot \mathbf{b} < 0 \iff \frac{\pi}{2} < \theta \leq \pi$.

Projection problem

We are now ready to solve the original projection problem:

Question

Find the projection of the vector $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ onto the vector $\mathbf{b} = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{k})$.



Projection problem

Solution.

By the definition of dot product, we have:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta,$$

where $\theta = \angle AOB$.

Projection problem

Solution.

By the definition of dot product, we have:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta,$$

where $\theta = \angle AOB$.

But we can work out $\mathbf{a} \cdot \mathbf{b}$ using their coordinates, i.e.,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}} [(2)(1) + (-3)(0) + (1)(-1)] = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}. \end{aligned}$$

Projection problem

From the fact that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta,$$

the positive value of $\mathbf{a} \cdot \mathbf{b}$ implies that

$$\cos\theta > 0$$

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So the diagram we drew is correct.

Projection problem

Solution.

The length of the projection of $\mathbf{a}$ unto $\mathbf{b}$ is

$$||\mathbf{a}|\cos\theta| = \left|\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2},$$

since $\mathbf{b}$ is a unit vector.

Projection problem

Solution.

The length of the projection of **a** unto **b** is

$$||\mathbf{a}|\cos\theta| = \left|\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{|\mathbf{b}|} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2},$$

since **b** is a unit vector. So, the required projection (vector) is

$$\frac{\sqrt{2}}{2}\hat{\mathbf{b}} = \frac{\sqrt{2}}{2}\mathbf{b} = \frac{1}{2}(\mathbf{i}-\mathbf{k}).$$

□

Projection problem

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *be three mutually perpendicular unit vectors. Then any 3D vector* $\mathbf{v}$ *can be decomposed into*

$$\mathbf{v} = \alpha_1 \mathbf{v_1} + \alpha_2 \mathbf{v_2} + \alpha_3 \mathbf{v_3},$$

where

$$\alpha_k = \mathbf{v} \cdot \mathbf{v}_k, \quad k = 1, \ 2, \ 3.$$

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where

$$\alpha_k = \mathbf{v} \cdot \mathbf{v}_k, \ \ k = 1, \ 2, \ 3.$$

Proof.

Easy: Just dot the vector equation by $\mathbf{v}_k$ $(k = 1, 2, 3)$. □

Part II

Applications of vectors

Signal processing

To tell you the truth, we shall be using:

Signal processing

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- Functions

Signal processing

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- Graphing techniques

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Signal processing

To tell you the truth, we shall be using:

- Functions
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- Integration

which you have learnt or will be learning in H2 Mathematics.

Signal processing

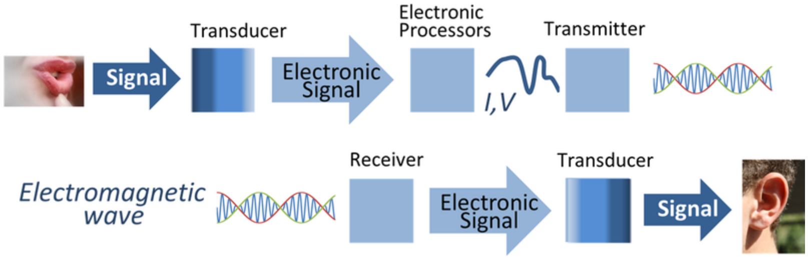

Figure: A diagrammatic summary of signal processing

Signal processing

In this lecture, we only have time to take a glimpse at

$$\text{Signal} \longrightarrow \text{Simpler sine/cosine signals}$$

via Fourier series.

Signal processing

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$$\text{Signal} \longrightarrow \text{Simpler sine/cosine signals}$$

via Fourier series.
But the motivation comes from the reverse direction.

Combining sinusoidal waves

We start with a simple Graphing Calculator activity.

Combining sinusoidal waves

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GC activity

Sketch the following graphs in succession:

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- Y2: $y = -2\sin(t) + \sin(2t)$

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- Y4: $y = -2\sin(t) + \sin(2t) - \frac{2}{3}\sin(3t) + \frac{1}{2}\sin(4t)$
- Y5: $y = -2\sin(t) + \sin(2t) - \frac{2}{3}\sin(3t) + \frac{1}{2}\sin(4t) - \frac{2}{5}\sin(5t)$

GC Activity

Exercise

Write down, in summation notation, the Nth function.

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Write down, in summation notation, the Nth function.

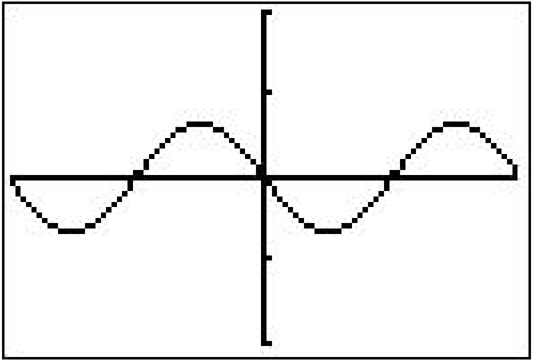
Solution.

$$f_N(t) = \sum_{n=1}^{N} \frac{2(-1)^n}{n} \sin(nt).$$

□

GC Activity

This produces a sequence of graphs:



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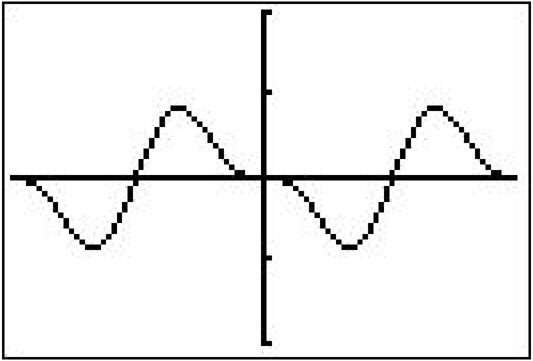

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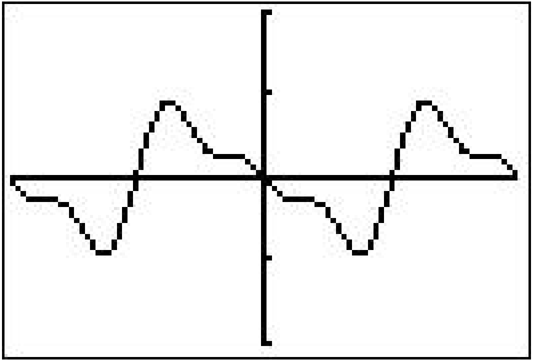

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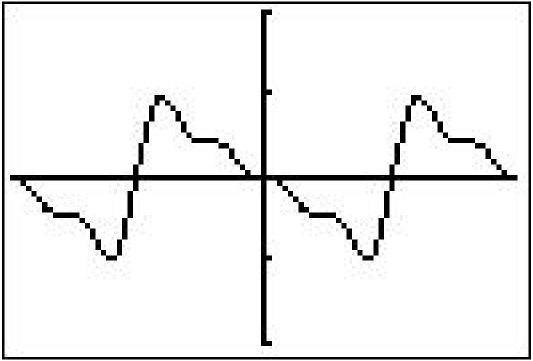

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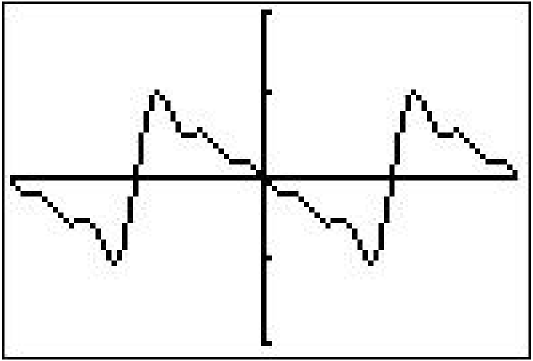

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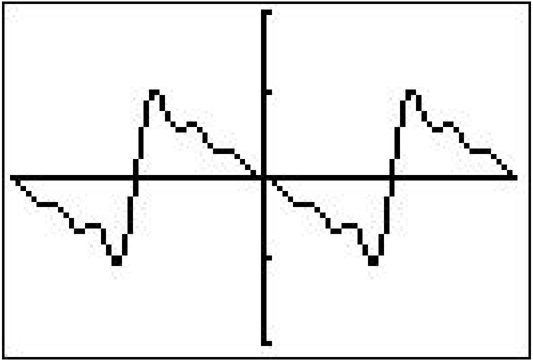

GC Activity

Question

Where is this sequence of functions limiting to?

GC Activity

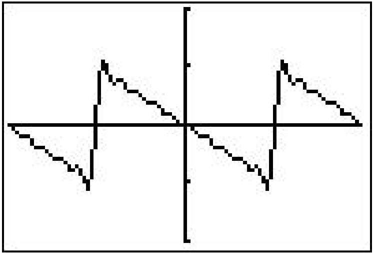

Figure: Graph of $f_{10}(t) = \sum_{n=1}^{10} \frac{2(-1)^n}{n} \sin(nt)$

GC Activity

GC suggests that on the interval $-\pi \leq t \leq \pi$ the sequence $f_n(t)$ tends to

$$\sum_{n=1}^{\infty} \frac{2(-1)^n}{n} \sin(nt) = -t := f(t),$$

and

$$f(t + 2\pi) = f(t).$$

Problem solving

In order to prove that on $[-\pi, \pi]$,

$$\sum_{n=1}^{\infty} \frac{2(-1)^n}{n} \sin(nt) = -t,$$

one may proceed in this way.

Problem solving

We generalize to the following problem:

Problem

Given a function f defined on $[-\pi, \pi]$, find coefficients a_n's and b_n's such that

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt).$$

Making connections

If we treat continuous functions on $[-\pi, \pi]$ as vectors, then looking for a_n's and b_n's amounts to a projection problem!

Making connections

If we treat continuous functions on $[-\pi, \pi]$ as vectors, then looking for a_n's and b_n's amounts to a projection problem!

$$f(t) = a_0 1 + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt),$$

where the infinite basis is given by $\{1, \cos(nt), \sin(nt)\}_{n=1}^{\infty}$.

Making connections

But recall the solution to the projection problem:

Theorem

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three mutually perpendicular unit vectors. Then any 3D vector $\mathbf{v}$ can be decomposed into

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3,$$

where

$$\alpha_k = \mathbf{v} \cdot \mathbf{v}_k, \;\; k = 1,\; 2,\; 3.$$

Making connections

What do the following terms mean in our present context of the vector space of functions?

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2. mutually perpendicular vectors

Making connections

What do the following terms mean in our present context of the vector space of functions?

1. 3D
2. mutually perpendicular vectors
3. unit vectors

Making connections

The above terms make sense only if we have a sensible way of defining the dot product for the vector space

$$\mathcal{C}[-\pi,\pi] := \{f \mid f : [-\pi,\pi] \longrightarrow \mathbb{R} \text{ is continuous}\}.$$

Making connections

To do so, one must define the dot product abstractly from its properties.

Making connections

Theorem

Let $\cdot$ be the dot product of 3D vectors. Then the following properties are satisfied for all vectors $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$, *and for all real number* α:

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4. $\mathbf{a} \cdot \mathbf{a} \geq 0$ *with equality if and only if* $\mathbf{a} = \mathbf{0}$.

Making connections

We generalize this to invent a new definition:

Definition (Dot product space)

A vector space $\langle V, +, \cdot \rangle$ is a *dot product space* (better known as *inner product space* if the following conditions are satisfied for all vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, and for all real number α:

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4. $\mathbf{a} \cdot \mathbf{a} \geq 0$ with equality holding if and only if $\mathbf{a} = \mathbf{0}$.

Making connections

For 3D vectors, the usual dot product has been defined by

$$\begin{pmatrix} \mathbf{u}(1) \\ \mathbf{u}(2) \\ \mathbf{u}(3) \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v}(1) \\ \mathbf{v}(2) \\ \mathbf{v}(3) \end{pmatrix} := \mathbf{u}(1)\mathbf{v}(1) + \mathbf{u}(2)\mathbf{v}(2) + \mathbf{u}(3)\mathbf{v}(3)$$

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Rewriting the above as

$$\mathbf{u} \cdot \mathbf{v} := \sum_{i \in \{1,2,3\}} \mathbf{u}(i)\mathbf{v}(i).$$

Making connections

For functions, the dot product should therefore involve some kind of infinite summation over real numbers, i.e., it should look like:

Making connections

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$$f \cdot g := \sum_{x \in [-\pi,\pi]} f(x)g(x).$$

Making connections

Big idea is ...

Infinite summation over real numbers is

Making connections

Big idea is ...

Infinite summation over real numbers is **INTEGRATION**.

Making connections

Definition (Dot product of functions)

Define the dot product of two functions f and $g \in \mathcal{C}[-\pi, \pi]$ as follows:

$$f \cdot g := \int_{-\pi}^{\pi} f(x)g(x)\ dx.$$

Making connections

With this definition, one can check that all the conditions of a (generalized) dot product are satisfied, i.e.,

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Theorem

For all functions f, g and h in $\mathcal{C}[-\pi, \pi]$, and for all real number α, the following conditions hold:

Making connections

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Theorem

For all functions f, g *and* h *in* $\mathcal{C}[-\pi, \pi]$, *and for all real number* α, *the following conditions hold:*

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2. $(\alpha f) \cdot g = \alpha(f \cdot g)$.

Making connections

With this definition, one can check that all the conditions of a (generalized) dot product are satisfied, i.e.,

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For all functions f, g *and* h *in* $\mathcal{C}[-\pi,\pi]$, *and for all real number* α, *the following conditions hold:*

1. $f \cdot g = g \cdot f$.
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3. $f \cdot (g + h) = f \cdot g + f \cdot h$.

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2. $(\alpha f) \cdot g = \alpha(f \cdot g)$.
3. $f \cdot (g + h) = f \cdot g + f \cdot h$.
4. $f \cdot f \geq 0$ *with equality holding if and only if* $f = 0$.

Sinusoidal waves

There is something natural about the choice of the functions

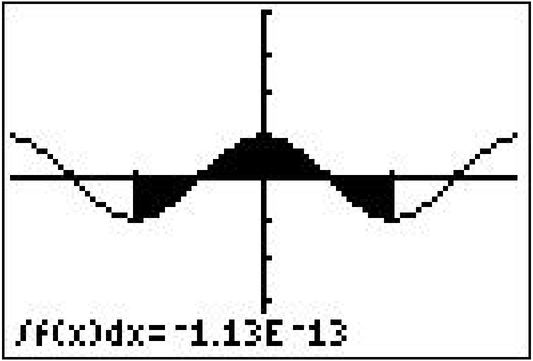
$$1, \cos(nx), \sin(nx), \qquad n = 1, 2, 3, \cdots$$

with regards to this dot product of functions.

Sinusoidal waves

The dot product of any pair of distinct waves must be zero.

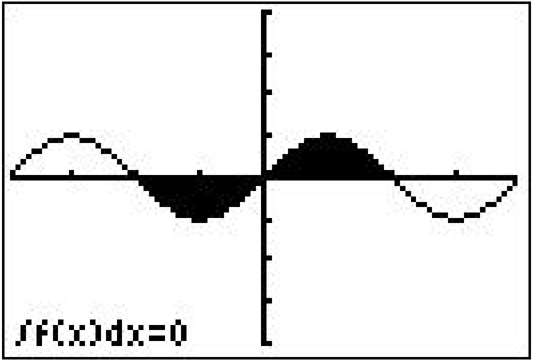
$$1 \cdot \cos(nx) = \int_{-\pi}^{\pi} \cos(nx)\,dx = 0.$$



Sinusoidal waves

The dot product of any pair of distinct waves must be zero.

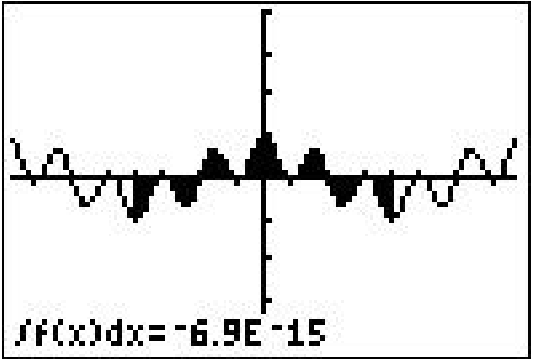
$$1 \cdot \sin(nx) = \int_{-\pi}^{\pi} \sin(nx)\,dx = 0.$$



Sinusoidal waves

The dot product of any pair of distinct waves must be zero.

$$\cos(mx) \cdot \cos(nx) = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = 0.$$



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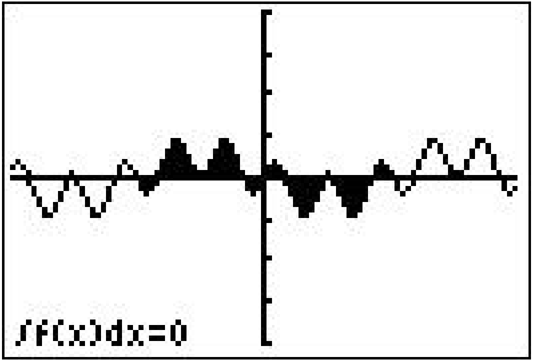
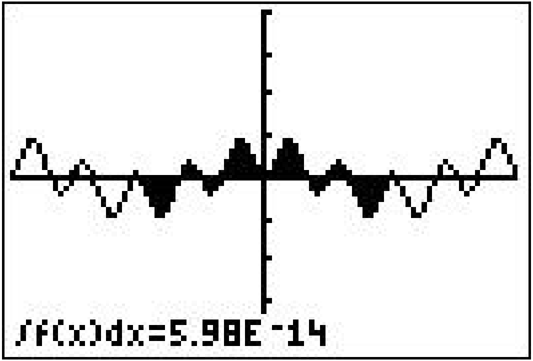


Sinusoidal waves

The dot product of any pair of distinct waves must be zero.

$$\sin(mx) \cdot \sin(nx) = \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx = 0.$$



Sinusoidal waves

Now we have to check that each of the waves are of unit magnitude.

Sinusoidal waves

For the first flat wave, we have:

Sinusoidal waves

For the first flat wave, we have:

$$\begin{aligned} 1 \cdot 1 &= \int_{-\pi}^{\pi} (1)(1) dx \\ &= \int_{-\pi}^{\pi} 1 dx \\ &= 2\pi. \end{aligned}$$

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Thus,

$$|1| = \sqrt{1 \cdot 1} = \sqrt{2\pi}.$$

Sinusoidal waves

So we obtain the first unit wave:

$$\frac{1}{\sqrt{2\pi}}.$$

Sinusoidal waves

For the cosine waves, we have:

Sinusoidal waves

For the cosine waves, we have:

$$
\begin{aligned}
\cos(nx) \cdot \cos(nx) &= \int_{-\pi}^{\pi} \cos^2(nx)\,dx \\
&= \int_{-\pi}^{\pi} \frac{1+\cos(2nx)}{2}\,dx \\
&= \frac{1}{2}\left[x + \frac{\sin(2nx)}{2n}\right]_{-\pi}^{\pi} \\
&= \frac{1}{2}(2\pi) \\
&= \pi .
\end{aligned}
$$

Sinusoidal waves

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&= \frac{1}{2}(2\pi) \\
&= \pi.
\end{aligned}$$

Thus,

$$|\cos(nx)| = \sqrt{\pi}.$$

Sinusoidal waves

So we obtain the second class of unit (cosine) waves:

$$\frac{\cos(nx)}{\sqrt{\pi}}, \qquad n = 1, 2, 3, \cdots$$

Sinusoidal waves

Similarly, we obtain the third class of unit (sine) waves:

$$\frac{\sin(nx)}{\sqrt{\pi}}, \qquad n = 1, 2, 3, \cdots$$

Sinusoidal waves

We thus obtain the following basis for $\mathcal{C}[-\pi, \pi]$:

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}} \right\}_{n=1}^{\infty},$$

which are mutually perpendicular unit vectors.

Projection problem

The projection problem of obtain a_n's and b_n's is easy to solve now.

Projection problem

This is equivalent to solving for A_n's and B_n's such that

$$f(x) = A_0 \left(\frac{1}{\sqrt{2\pi}} \right) + \sum_{n=1}^{\infty} A_n \left(\frac{\cos(nx)}{\sqrt{\pi}} \right) + B_n \left(\frac{\sin(nx)}{\sqrt{\pi}} \right).$$

Projection problem

To find A_0, we project f along the flat wave $\frac{1}{\sqrt{2\pi}}$:

$$A_0 = f \cdot \frac{1}{\sqrt{2\pi}} = \int_{-\pi}^{\pi} f(x) \left(\frac{1}{\sqrt{2\pi}} \right) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx.$$

Projection problem

To find A_n, we project f along the class of cosine waves $\frac{\cos(nx)}{\sqrt{\pi}}$:

$$A_n = f \cdot \frac{\cos(nx)}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos(nx) dx.$$

Projection problem

To find B_n, we project f along the class of cosine waves $\frac{\sin(nx)}{\sqrt{\pi}}$:

$$B_n = f \cdot \frac{\sin(nx)}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \sin(nx) dx.$$

Projection problem

Definition (Fourier series)

Let f be a continuous function on $[-\pi, \pi]$.

Projection problem

Definition (Fourier series)

Let f be a continuous function on $[-\pi, \pi]$.
The *Fourier series* of f is given by

$$f(x) = A_0\left(\frac{1}{\sqrt{2\pi}}\right) + \sum_{n=1}^{\infty}(A_n\frac{\cos(nx)}{\sqrt{\pi}} + B_n\frac{\sin(nx)}{\sqrt{\pi}}),$$

where

$$A_0 = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(x)dx, \ A_n = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,$$

and

$$B_n = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\sin(nx)dx.$$

Projection problem

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Let f be a continuous function on $[-\pi, \pi]$.
The *Fourier series* of f is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)dx,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)dx.$$

Projection problem

Example

Consider the function f defined by

$$f(t) = -t, \qquad -\pi \leq t \leq \pi.$$

Projection problem

We calculate a_n's and b_n's as follows.

Projection problem

$$
\begin{aligned}
a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} -t dt \\
&= \frac{1}{2\pi}\left[-\frac{t^2}{2}\right]_{-\pi}^{\pi} \\
&= 0.
\end{aligned}
$$

Projection problem

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} -t \cos(nt) dt$$

Projection problem

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} -t\cos(nt)\,dt$$
$$= 0$$

since the function $-t\cos(nt)$ is odd, i.e., has rotational symmetry about O (of order 2).

Projection problem

Making use of the result that

$$\frac{d}{dt}\left(t\cos\left(nt\right)\right) = -nt\sin\left(nt\right) + \sin\left(nt\right),$$

we evaluate

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} -t\sin\left(nt\right)dt \\ &= \frac{1}{\pi}\cdot\frac{1}{n}\left(\left[t\cos\left(nt\right)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\cos\left(nt\right)\right) \\ &= \frac{1}{\pi}\cdot\frac{1}{n}\left(2\pi(-1)^n - \frac{1}{n}\left[\sin\left(nt\right)\right]_{-\pi}^{\pi}\right) \\ &= \frac{2(-1)^n}{n}. \end{aligned}$$

Projection problem

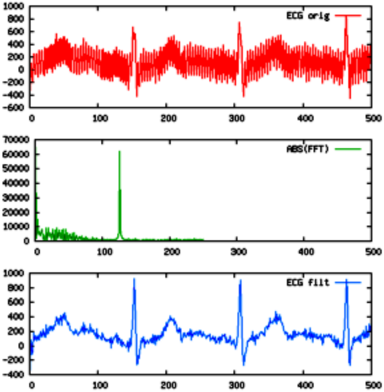
Consequently, we have shown that on $[-\pi, \pi]$ it holds that

$$-t = \sum_{n=1}^{\infty} = \frac{2(-1)^n}{n} \sin(nt),$$

as desired.

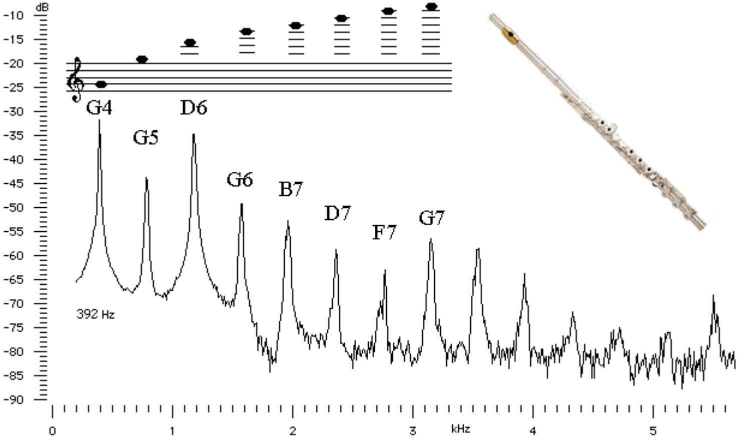
Back to signals

Physical signals are in general more irregular such as:



Back to signals

Music also comes in this form:



Back to signals

The theory developed so far guarantees that no matter how irregular the continuous (time dependent) signal is, it can be decomposed into its cosine and sine components of varying amplitudes a_n and b_n, depending on the frequency n.

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Fourier transform

Time domain $\longrightarrow$ Frequency domain

Back to signals

There is a long way from the orchestra music to your head-set.

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Sound waves $\longrightarrow$ Discrete sampling of signals (time domain) $\longrightarrow$ Discrete data in frequency domain $\longrightarrow$ Transmission/Storage $\longrightarrow$ Decoded sound waves

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In practice, the second arrow involved DFT (Discrete Fourier Transform).

Back to signals

Let us see how experience how vectors make music:

http://falstad.com/fourier/

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