An Introduction to Topology and its Applications: A New Approach

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Weng Kin Ho
wengkin.ho@nie.edu.sg

National Institute of Education, Nanyang Technological University

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Outline

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The word "topology"

Etymology of the word

$$\lambda o \gamma o \zeta \quad \tau o \tau \tau o \zeta$$

The word “topology”

Etymology of the word

$$\underbrace{\lambda o \gamma o \zeta}_{\text{study}} \quad \underbrace{\tau o \tau \tau o \zeta}_{\text{place}}$$

The word "topology"

Slogan

Topology = The *study* of *locality*

Back to Königsberg

In 1736, Leonhard Euler solved the problem of

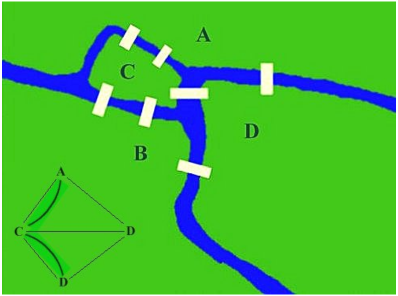

Figure: The Seven Bridges of Königsberg

Back to Königsberg

The main idea

Quantitative geometry does not count, what counts is the

Qualitative Geometry.

Topology – the modern term

The present vocabulary "topology" which replaces "Qualitative Geometry" was invented by

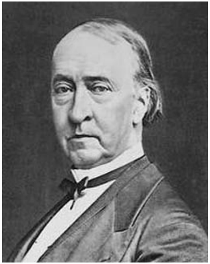

Figure: Johann Benedict Listing (25 July 1808 24 December 1882)

Point-set topology

Deals with ...

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- foundational issues of topology, and

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- foundational issues of topology, and
- studies topological properties inherent to spaces which are invariant under homeomorphisms

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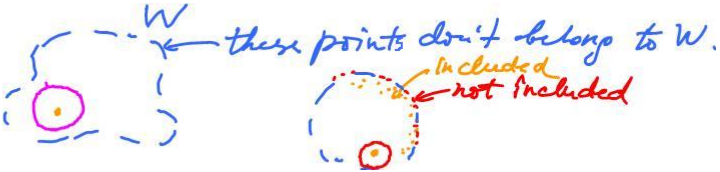

Algebraic topology

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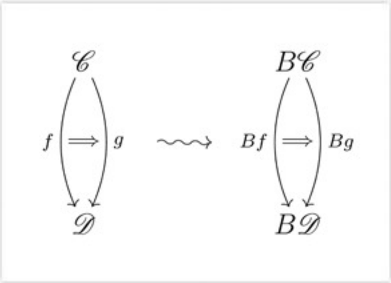
- the use of tools of algebra to study topological spaces, and

Algebraic topology

Deals with ...

- the use of tools of algebra to study topological spaces, and
- homotopy and homology.



Geometric topology

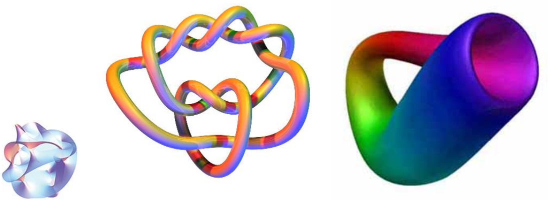


Figure: Topological manifolds

Topology

Topology is often popularized as *Rubber-sheet Geometry*:

Figure: Coffee-mug = Donut

A new approach to topology

Today, my talk introduces topology and its applications from a completely fresh perspective.

Computational viewpoint

Let us begin by computing the value of the real number

$$\alpha = \sqrt{2}.$$

Computational viewpoint

Rely on technology offered by a scientific programming language.

Computational viewpoint

Rely on technology offered by a scientific programming language.

Life demonstration

Let us do a real-time demonstration of this calculation using

MATLAB.

Computational viewpoint

Shocking fact

Computing a real number is not about computing a point!

Computational viewpoint

Big idea

Computing a real number is about *affirming its locality*.

Affirmable observation

We expect that ...

An affirmable observation is one ...

- that says 'Yes' if a certain statement/property holds,

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We expect that ...

An affirmable observation is one ...

- that says 'Yes' if a certain statement/property holds,
- but not obliged to say 'No' otherwise.

Affirmable observation

We expect that ...

An affirmable observation is one ...

of finite nature, i.e., one that takes finite time and finite steps.

Affirmable observation

Example

An affirmable observation, involved in the calculation of $\sqrt{2}$, is just one which affirms the locality of $\sqrt{2}$.

Affirmable observation

The closed intervals $[a_n, b_n]$'s

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- can give rise to affirmable observations.

What is an affirmable observation?

Definition (Affirmable observation)

- An *affirmable observation* is one which can be used to affirm the locality (or some *property*) of a number (data point), i.e., obliged to say 'yes' if the observation offers certainty about the presence of the point, but not obliged to say 'no' if it is beyond its ability to do so.
- Properties which can be affirmed by an affirmable observation are termed as *affirmable properties*.

Affirmable observation

Open intervals

An open interval with rational endpoints

$$(r, s) := \{x \in \mathbb{R} \mid r < x < s\}$$

is an affirmable observation of real numbers.

Affirmable observations

One helpful way to think of an affirmable observations is to think of a test or experiment carried out to test some property.

Extreme examples of affirmable observations

Let us begin with a space of data points

$$X$$

in which we wish to make observations.

Extreme examples of affirmable observations

Given a point $x \in X$, there is always one trivial observation of it, i.e., do nothing at all to observe it!

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The point x is out there somewhere ... where?

It is in X.

Extreme examples of affirmable observations

Theorem

The entire space X is an affirmable observation.

Extreme examples of affirmable observations

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The entire space X is an affirmable observation.

Plain lazy!

Extreme examples of affirmable observations

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Extreme examples of affirmable observations

Another extreme example of an affirmable observation is the famous

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because it is vacuously useless to affirm anything!

Extreme examples of affirmable observations

Theorem

The following are the two extreme possible affirmable observations of X*:*

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- *the entire space X, and*

Extreme examples of affirmable observations

Theorem

The following are the two extreme possible affirmable observations of X:

- *the entire space X, and*
- *the empty set $\emptyset$.*

Affirmable observations

For any given space of data, there may be different affirmable observations available.

Affirmable observations

Given that we have some affirmable observations available to us, there are somethings that can be said of these affirmable observations *collectively*.

Affirmable observations

Given a collection of affirmable observations U_i's, is the set-union

$$\bigcup_{i \in I} U_i$$

another affirmable one?

Affirmable observations

The set union affirms the locality of a point $x \in X$ if and only if there is some affirmable observation that affirms that.

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Run all observations in parallel to locate x. We can affirm its presence as long as one of the observation affirms it.

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So, $\bigcup_{i \in I} U_i$ is an affirmable observation.

Affirmable observations

Suppose that we have two affirmable observations U and V.

Question

How do we ensure that a data point is affirmed by both the observations?

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How do we ensure that a data point is affirmed by both the observations?

Just use make the observations one after another.

Affirmable observations

Notice that arbitrary intersection of affirmable observations may not be an affirmable observation.

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$$U_1 \cap U_2 \cap U_3 \cap \cdots$$

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It might take forever!

Affirmable observations

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$$\bigcup_{i \in I} U_i \in \tau.$$

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3. *For any U and $V \in \tau$, $U \cap V \in \tau$.*

Topology motivated

Definition (Topology)

Let τ be a collection of subsets of a set X. Suppose we have:

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We sometimes use the symbol $\mathcal{O}X$ to denote the set of opens of X.

Opens as functions

Think of an affirmable observation as a computer program P trying to affirm a certain property U for a given data point $x \in X$.

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- P must be of a finite nature.
- P needs to affirm that x satisfies U if and only if x satisfies it.
- P is not obliged to return a negative answer in the event that it cannot affirm that x satisfies U.

Opens as functions

Such a test-program

$$P : X \longrightarrow \Sigma$$

meets the following specification:

$$P(x) = \begin{cases} 1 & \text{if } x \in U; \\ 0 & \text{otherwise.} \end{cases}$$

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Here, $\Sigma := \{0, 1\}$.

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The only affirmable observations present in the space Σ are

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Definition (Observation data type)

The only affirmable observations present in the space Σ are

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This special topological space is termed as the *Sierpinski's space*.

Opens as functions

Theorem

Let P be a program used to affirm a property U. Then

$$U = P^{-1}(1).$$

Opens as functions

Theorem

Let P be a program used to affirm a property U. Then

$$U = P^{-1}(1).$$

Certainly, P is just a semi-decision.

Opens as functions

Idea

A property or observation is affirmable if and only if it can be realized by a semi-decision.

Opens as functions

Since the only non-trivial open in Σ is $\{1\}$, it follows that

$$U \in \mathcal{O}X \iff \chi_U \in [X \longrightarrow \Sigma],$$

where $[X \longrightarrow \Sigma]$ is the collection of functions f for which

$$f^{-1}(U)$$

for all open sets in Σ.

Opens as functions

Definition (Continuous functions)

A function $f : X \longrightarrow Y$ between topological spaces X and Y is said to be *continuous* if

$$f^{-1}(U) \in \mathcal{O}X$$

for all $U \in \mathcal{O}Y$.

Computer programs as tests

Let us now consider some computer programs which are used for the purpose of decision and semi-decision.

Computer programs as tests

We use the functional language HASKELL to demonstrate this portion of the talk.
This language can be downloaded at

`http://www.haskell.org`

Computer programs as tests

Exercise

Test whether a positive integer n is an even number or not.

Computer programs as tests

Exercise

Test whether a positive integer n is an even number or not.

The property that n is an even number is both affirmable and reputable, i.e., it is *decidable*.

Computer programs as tests

Exercise

Test whether a binary stream has some 1's in it or not.

Computer programs as tests

Exercise

Test whether a binary stream has some 1's in it or not.

The property that a binary stream has some 1's in it is affirmable but not refutable, i.e., it is *semi-decidable*.

Computer programs as tests

From our functional perspective, we can ...

declare

the semi-decidable sets to be *open*.

Computer programs as continuous functions

Suppose σ and τ are two data types (i.e., collection of data sharing the same attributes), and

$$f : \sigma \longrightarrow \tau$$

be a program.

Computer programs as continuous functions

Suppose σ and τ are two data types (i.e., collection of data sharing the same attributes), and

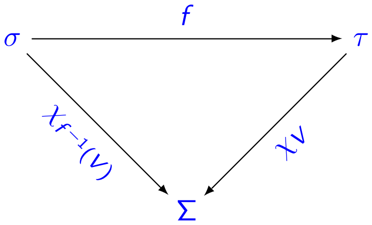
$$f : \sigma \longrightarrow \tau$$

be a program.
Given that $V \subseteq \tau$ is an open set (i.e., semi-decible set), what can we say of the set

$$f^{-1}(V) \subseteq \sigma?$$

Computer programs as continuous functions

Let us study the following commutative diagram:



Computer programs as continuous functions

Proof.

For any $x \in \sigma$,

$$\begin{aligned} x \in f^{-1}(V) &\iff f(x) \in V \\ &\iff \chi_V(f(x)) = 1 \\ &\iff (\chi_V \circ f)(x) = 1 \end{aligned}$$

so that

$$\chi_{f^{-1}(V)} = \chi_V \circ f.$$

□

Topology via computation

Using the current approach based on affirmable observations/semi-decidable properties/computer programs, there are a number of pedagogical advantages/implications.

Topology via computation

Our approach offers a more direct and concrete mode of representation for topology.

Topology via computation

1. Opens are affirmable properties/observations, while closed sets are refutable properties.

Topology via computation

1. Opens are affirmable properties/observations, while closed sets are refutable properties.
2. One can avoid conflicts with natural language about the seemingly bizarre usage of ‘open’ and ‘closed’, e.g., not open is not the same as closed.

Topology via computation

1. Continuous functions are thought of as computer programs.

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2. Avoid connecting with the notorious $\epsilon - \delta$ definition of continuous functions.

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2. Avoid connecting with the notorious $\epsilon - \delta$ definition of continuous functions.
3. Certain properties of continuous functions are immediate with this paradigm.

Topology via computation

Theorem

The composition of two continuous functions yields a continuous function.

Topology via computation

Theorem

The composition of two continuous functions yields a continuous function.

Proof.

The sequential composition of two computer programs is a computer program. □

Topology via computation

One need not be restricted to the diagrammatic ways of illustrating topological concepts.

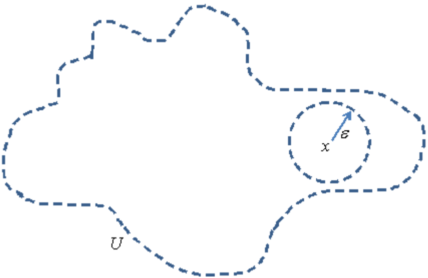

Figure: Diagrammatic illustration of an open set

Topology via computation

Question

Try to illustrate the concept of a limit point by diagrams.

Topology via computation

Warning

This pictorial approach can be dangerous since not all spaces are Hausdorff.

Topology via computation

Warning

This pictorial approach can be dangerous since not all spaces are Hausdorff.

The computational approach does not make use of diagrams of this kind.

New areas in topology

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In truth, it is a *synthetic* or *operational* topology.

New areas in topology

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In truth, it is a *synthetic* or *operational* topology.
The core of topology can developed along this line of consideration.

References

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The End

Thank you!