When exactly is Scott sober?

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This is where I come from ...

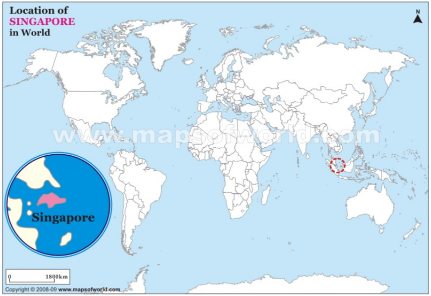

Figure: Location of Singapore in the world map

This is where I come from ...

Figure: National Institute of Education, Singapore

Outline

1. Some jargon

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2. The problem

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Dcpo

Definition (Directed subsets)

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Dcpo

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Trivial example of a directed set: A chain.

Dcpo

Definition (Dcpo)

A poset is a *dcpo* if every directed subset has a supremum.

Dcpo

Here are some dcpo's:

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Example

1. Finite posets

Dcpo

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2. Complete lattices

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4. $\mathrm{Irr}(L)$ where L is a complete lattice

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3. $(\mathrm{Id}(P), \subseteq)$ and $(\mathrm{Filt}(P), \subseteq)$
4. $\mathrm{Irr}(L)$ where L is a complete lattice

We shall add one more important example later.

T_0-space

Definition (T_0-space)

A topological space X is T_0 if for every pair of distinct points $x \neq y$, there exists an open set U that contains exactly one of them.

Scott topology

Definition (Scott open)

Let P be a poset and $U \subseteq P$.

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Scott topology

Let P be a poset.

Definition (Scott topology)

The collection of all Scott open subsets of P is called the *Scott topology* of P, and is denoted by $\sigma(P)$.

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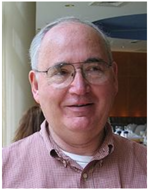

Figure: Dana Scott

Scott topology

Proposition

The Scott topology on any dcpo is T_0.

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Proposition

With respect to the Scott topology, the closure of singleton x *is precisely the principal ideal generated by it, i.e.,*

$$\mathrm{cl}\left(\{x\}\right) = \downarrow x.$$

Sober space

Definition (Irreducible subsets)

A nonempty subset A of a topological space X is *irreducible* if $A \subseteq B \cup C$ for closed subsets B and C implies $A \subseteq B$ or $A \subseteq C$.

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Point closures $\mathrm{cl}(\{p\})$ are always irreducible closed subsets.

Sober space

Definition (Sober space)

A topological space X is *sober* if for every irreducible closed set C, there exists a unique $x \in X$ such that $\mathrm{cl}(\{x\}) = C$.

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A topological space X is *sober* if for every irreducible closed set C, there exists a unique $x \in X$ such that $\mathrm{cl}(\{x\}) = C$.

Remark.

A sober space is automatically T_0 because $\mathrm{cl}(\{x\}) = \mathrm{cl}(\{y\})$ always implies $x = y$.

Specialization order

Proposition (Specialization order)

Let (X, τ) *be a* T_0*-space.*

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Define the relation $\leq_\tau$ on X as follows:

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Then $\leq_\tau$ is a partial order on X, known as the specialization order.

Scott is not always sober

We now add one more example of a dcpo.

Proposition

Let X *be a sober topological space. Then*

Scott is not always sober

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Let X be a sober topological space. Then

1. *the specialization order on X is a dcpo, and*

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Proposition

Let X be a sober topological space. Then

1. *the specialization order on X is a dcpo, and*
2. *the topology on X is contained in the Scott topology for the specialization order on X.*

Scott is not always sober

This naturally leads one to ask in the opposite direction:

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Question

Is the Scott topology on a dcpo always sober?

Scott is not always sober

In December 1978, P.T. Johnstone discovered a counterexample that answers this question in the negative.

Scott is not always sober

Example (Johnstone's construction)

Let $P = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$.

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Let $P = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$. We order P as follows:

$$(m, n) \leq (m', n') \iff \text{either } m = m' \ \& \ n \leq n' \text{ or } n' = \infty \ \& \ n \leq m'.$$

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1. P is a poset whose maximal elements are (m, ∞).
2. A directed subset of L either has a greatest element or is contained in $\{m\} \times (\mathbb{N} \cup \{\infty\})$ for some m.
3. P is a dcpo.
4. Any two non-empty Scott open subsets of P have a nonempty intersection.

Scott is not always sober

P fails to be sober because ...

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The quest for a characterization

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Problem

Find an order-theoretic characterization for those dcpo's for which their Scott topology is sober.

– Continuous lattices and domains, p.155

Where I started

Previous work

Characterize those complete lattices of the form:

$$\Gamma(P) := \Sigma^{\mathrm{op}}(P),$$

for a complete semilattice P.

The idea I

Characterize those dcpo's of the form

$$H(P) := \mathrm{Irr}(\Gamma(P))$$

for some kind of dcpo P.

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$\mathrm{Irr}(L)$ = the dcpo of join-irreducible elements of a (distributive) lattice L.

Notes

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1. In 2006, X. Mao and L. Xu have considered $H(P)$ (they used the notation $c(P)$) to study FS-posets as a generalization of FS-domains.
2. They called $H(P)$ the *directed completion* of P.
3. Its order-theoretic properties were not studied for non-continuous P.

The idea II

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$$H(P)$$

for *some (kind of)* dcpo P, then one should be able to

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If one can characterize those dcpo's Q of the form

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for *some (kind of)* dcpo P, then one should be able to

- recover P from Q by
- identifying the 'compact' elements of Q.

Denote the set of 'compact' elements of a dcpo Q by $K(Q)$.

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Since the 'compact' elements of $H(P)$ are *supposedly* the principal ideals, we can deduce the following:

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Every element of P is ‘compact’.

A crucial comparison

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I left out *H-algebraic* on purpose.

H-below

Definition

Let P be a poset. Define $x \blacktriangleleft y$ as

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For any $a \in P$, we denote $\{x \in P \mid x \blacktriangleleft a\}$ by

$$H(a).$$

H-below

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Let P be a poset. We defined $x \blacktriangleleft y$ as

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What is $\blacktriangleleft$ when P is sober w.r.t. its Scott topology?

H-continuity

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Note

$H(a)$ is always Scott-closed but may not be irreducible.

Union completeness

Theorem

For any poset P and any $\mathcal{C} \in H(H(P))$, the following holds:

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Theorem

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This results in:

Corollary

For any poset P, the dcpo $H(P)$ has suprema of all irreducible Scott closed subsets.

We call this kind of completeness *ISC completeness*.

H-compact elements

Definition (H-compactness)

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We use $K(P)$ to denote the set of all H-compact elements of P.

$K(M)$ as a sub-dcpo of M

Proposition

If M is a dcpo and $K(M) \neq \varnothing$, then $K(M)$ is a sub-dcpo with respect to the order inherited from M.

Principal ideals are H-compact

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Warning

Not every H-compact element of $H(P)$ is a principal ideal.

H-algebraic poset

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1. $a = \bigsqcup\{x \in K(P) \mid x \le a\}$,

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3. $\{x \in K(P) \mid x \leq a\} \in H(K(P))$.

Passage between M and $K(M)$

To facilitate the transport of lower sets between M and $K(M)$, we create a two-way passage:

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$$\mathcal{L}(M) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} \mathcal{L}(K(M))$$

$\flat$ and $\sharp$

Definition ($\flat$ & $\sharp$)

Let M be a poset and S be any non-empty subset of M.

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and for each $B \subseteq S$, a corresponding subset of M given by

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and for each $B \subseteq S$, a corresponding subset of M given by

$$B^\flat := \{m \in M \mid \exists b \in B.m \leq B\}.$$

$\flat$ & $\sharp$ as e-p pair

Proposition

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Proposition

1. *$\langle \flat, \sharp \rangle$ forms an e-p pair pair between the lattices of lower subsets of M and that of S.*

$\flat$ & $\sharp$ as e-p pair

Proposition

1. $\langle \flat, \sharp \rangle$ *forms an e-p pair pair between the lattices of lower subsets of* M *and that of* S.
2. *For any* $x \in S$, *it holds that*

$$(\downarrow_S x)^{\flat} = \downarrow_M x \ \& \ (\downarrow_M x)^{\sharp} = \downarrow_S x.$$

$\flat$ & $\sharp$

From now on, when we take $\flat$ and $\sharp$, we always restrict to $S = K(M)$.

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$$\mathcal{L}(M) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} \mathcal{L}(K(M))$$

$\flat$, $\sharp$ & H-algebraicity

Lemma

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$$(H(a))^{\sharp\,\flat} = H(a).$$

H-algebraicity implies H-continuity

The preceding lemma is used to obtain the following result:

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H-algebraicity implies H-continuity

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Proposition

Let P be an H-algebraic poset. Then P is H-continuous.

Dominated dcpo

Let us motivate a crucial concept: dominated dcpo's.

Dominated dcpo

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We work backwards: Time for some board work.

Dominated dcpo

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Let P be a poset and $X \subseteq P$ be a nonempty.

- A subset F of P is *X-dominated* if there exists $x \in X$ such that $F \subseteq \downarrow x$.

Dominated dcpo

Definition (X-dominated)

Let P be a poset and $X \subseteq P$ be a nonempty.

- A subset F of P is *X-dominated* if there exists $x \in X$ such that $F \subseteq \downarrow x$.
- A family $\mathcal{F}$ of subsets of P is *X-dominated* if every $F \in \mathcal{F}$ is X-dominated.

Dominated dcpo

Definition (Dominated poset/dcpo)

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- P is *dominated* if for every irreducible Scott closed set X of P and for every directed X-dominated family $\mathcal{D}$ of irreducible Scott closed subsets of P,

$$\bigcup \mathcal{D} \text{ is } X\text{-dominated.}$$

Dominated dcpo

Definition (Dominated poset/dcpo)

- P is *dominated* if for every irreducible Scott closed set X of P and for every directed X-dominated family $\mathcal{D}$ of irreducible Scott closed subsets of P,

$$\bigcup \mathcal{D} \text{ is } X\text{-dominated.}$$

- A dcpo which is dominated is called a *dominated dcpo*.

Dominated dcpo

Example

1. Complete lattices and complete semilattices.

Dominated dcpo

Example

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2. ISC complete dcpo's.

Dominated dcpo

Example

1. Complete lattices and complete semilattices.
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3. Dcpo's whose Scott topology is sober.

Equivalent formulations for a dominated dcpo

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Let P be a poset. Then t.f.a.e.:

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2. For every irreducible Scott closed subset X of P, $\bigsqcup_{H(P)} \mathcal{D}$ is X-dominated for all directed X-dominated family $\mathcal{D}$ of irreducible Scott closed subsets of P.

Equivalent formulations for a dominated dcpo

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4. For every irreducible Scott closed subset X of P, it holds that

$$\mathcal{C}_X \in \Gamma(H(P)).$$

Properties of $H(P)$ for dominated P

Theorem

Let P be a dominated dcpo.

Properties of $H(P)$ for dominated P

Theorem

Let P be a dominated dcpo.
Then $X \in K(H(P))$ if and only if X is principal.

Properties of $H(P)$ for dominated P

Corollary

Let P be a dominated dcpo.

Properties of $H(P)$ for dominated P

Corollary

Let P be a dominated dcpo. Then

$$P \cong K(H(P)).$$

Properties of $H(P)$ for dominated P

Corollary

$H(P)$ is H-algebraic for any dominated dcpo P.

An equivalent condition for dominated P

Proposition

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An equivalent condition for dominated P

Proposition

Let P be a dcpo. Then t.f.a.e.:

1. *P is dominated.*
2. *For every irreducible Scott-closed subset $\mathcal{C}$ of $H(P)$, it holds that*

$$(\mathcal{C})^{\sharp\flat} \in H(H(P)).$$

Dcpo's with sober Scott topology as fixed point

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Dcpo's with sober Scott topology as fixed point

Proposition

Let P be a dcpo. Then t.f.a.e.:

(i) $(P, \sigma(P))$ *is sober.*

Dcpo's with sober Scott topology as fixed point

Proposition

Let P *be a dcpo. Then t.f.a.e.:*

(i) $(P, \sigma(P))$ *is sober.*

(ii) *For every irreducible Scott-closed subset* $\mathcal{C}$ *of* $H(P)$*, it holds that*

$$\mathcal{C} = (\mathcal{C}^{\sharp})^{\flat}.$$

$\flat$-stable dcpo

Definition ($\flat$-stability)

Let M be a dcpo. We say that M is *$\flat$-stable* if for every $B \in H(K(M))$, it holds that

$$B^\flat \in H(M).$$

Fundamental lemma

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Lemma

For an H-algebraic dominated dcpo M, t.f.a.e.:

(i) M *is* $\flat$*-stable.*

(ii) $K(M)$ *is dominated.*

An immediate consequence of this is:

Proposition

$H(P)$ *is* $\flat$*-stable for any dominated dcpo* P.

Coherent dcpo

Definition

A dcpo M which has all suprema of irreducible Scott-closed subsets of $K(M)$ is called *coherent*.

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Remark.

$H(P)$ is coherent for any dominated dcpo P.

Stably H-algebraic dcpo

Definition (Stably H-algebraic dcpo)

A $\flat$-stable, coherent, ISC-complete H-algebraic dcpo will be called a *stably H-algebraic* dcpo, for short.

Characterization of $H(P)$

Theorem

A dcpo M is isomorphic to $H(P)$ for a dominated dcpo P if and only if M is stably H-algebraic.

Strongly H-algebraic dcpo

Definition (Strongly H-algebraic)

A stably H-algebraic dcpo P with $K(P) = P$ will be called *strongly H-algebraic*.

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(i) $(P, \sigma(P))$ *is sober.*

(ii) P *is strongly* H*-algebraic.*

Proof

($\Longrightarrow$):

- Since P is sober, $K(H(P)) = H(P)$.

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- Since P is sober, $K(H(P)) = H(P)$.
- Since P is sober, it is dominated.

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- Since P is dominated, $K(H(P)) \cong P$.

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- Since P is sober, it is dominated.
- Since P is dominated, $K(H(P)) \cong P$.
- So $P \cong H(P)$.

So P takes on all the properties of $H(P)$, for a dominated P.

Proof

($\Longleftarrow$):

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- Since P is ISC-complete, it is dominated.

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- Since $P = K(P)$, $P \cong H(P)$.
- Since P is ISC-complete, it is dominated.
- Since P is dominated, $P \cong K(H(P))$.
- Thus $H(P) = K(H(P))$.

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A dcpo P is sober if and only if P

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Usefulness of the Main Theorem

The main theorem asserts that:
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1. is ISC-complete,
2. H-algebraic,
3. $\flat$-stable,
4. coherent, and
5. contains nothing but H-compact elements.

Johnstone's construction

Johnstone's dcpo fails to be sober because it is not ISC-complete.

Isbell's construction

Isbell constructed a complete lattice L for which the Scott topology is not sober.

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Isbell's construction

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In that example, there exists an irreducible Scott closed set X for which $X \neq \downarrow \bigsqcup X$. Thus,

$$\bigsqcup X \notin X.$$

So L contains at least one element which is not H-compact, namely, $\bigsqcup X$.

An extension of Isbell's construction

Let L be Isbell's complete lattice.

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We claim that $H(L)$ is also not sober.

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Again since L is complete and thus dominated, we have

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This implies that $L \cong H(L)$, a contradiction since L is not sober w.r.t. its Scott topology.

An extension of Isbell's construction

An extension of Isbell's construction

Consider the following facts:

Fact	Reason
$H(L)$ is ISC-complete	
$H(L)$ is H-algebraic	L is dominated
$H(L)$ is $\flat$-stable	$K(H(L)) \cong L$ since L is dominated
$H(L)$ is coherent	L is dominated

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$H(L)$ is $\flat$-stable	$K(H(L)) \cong L$ since L is dominated
$H(L)$ is coherent	L is dominated

That $H(L)$ fails to be sober indicates that $H(L)$ contains some element which is not H-compact.

New construction of non-sober dcpo's

Example

Let $P_0 = L$, where L is Isbell's construction.
Define a sequence of dcpo's P_n as follows:

$$P_0 = L \text{ and } P_{n+1} = H(P_n), \ \ n \in \mathbb{N}.$$

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Let $P_0 = L$, where L is Isbell's construction.
Define a sequence of dcpo's P_n as follows:

$$P_0 = L \text{ and } P_{n+1} = H(P_n), \ \ n \in \mathbb{N}.$$

Then we have a countable infinitude of pairwise non-isomorphic dominated dcpo's which are not sober w.r.t. their Scott topology.

Conclusion

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1. $\blacktriangleleft$-relation,
2. H-continuous and H-algebraic dcpo's,
3. dominated dcpo's
4. $\flat$-stability and coherence, and
5. necessary and sufficient conditions for a dcpo to be sober w.r.t. its Scott topology.

Conclusion

Thank you!

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