Scott Co-topology and Sober Scott Spaces

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This is where I come from ...

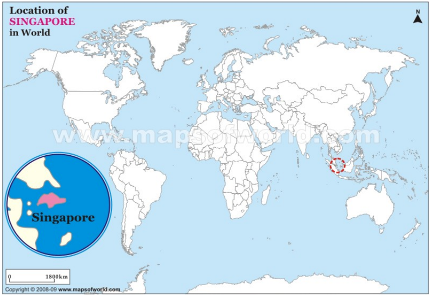

Figure: Location of Singapore in the world map

This is where I come from ...

Figure: National Institute of Education, Singapore

This is where I come from ...

Mathematics and Mathematics Education,
National Institute of Education,
Nanyang Technological University

Research interests

I do many cool things, such as:

Research interests

I do many cool things, such as:

- Partial orders and lattices

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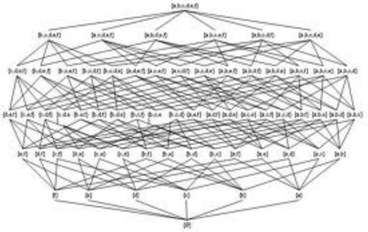
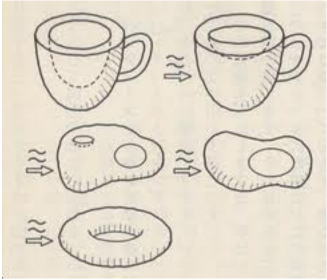
- Domain theory and topology





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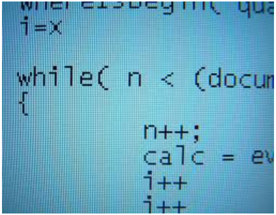
- Real exact arithmetic



Research interests

Along the way, I also play with:

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- Number theory: Fourier series applied to the ζ function

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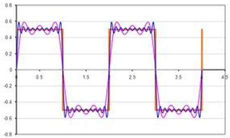
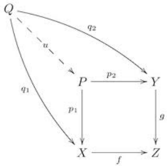
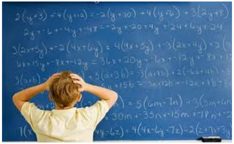
- Algebra: Universal algebra and Category theory
- Mathematics education







My Ph.D. Genealogy

Here are my ascending chains of supervisors:

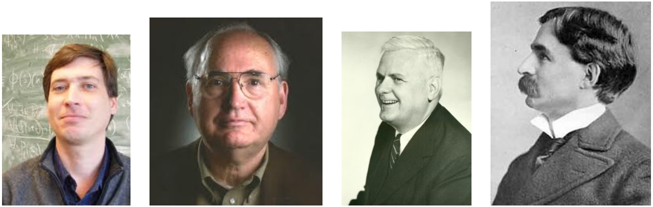


Figure: M.H. Escardó, D.S. Scott, A. Church, E. Moore

Structure of talk

In this talk, we touch on two topics:

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- Scott co-topology and its representation theory

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- Scott co-topology and its representation theory
- Sober Scott spaces

Part I

Title

Scott co-topology and its representation theory

Structures of discussion

We talk about a special topology on a special kind structure:

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Partially Ordered Sets

Partial order

Definition

A *partial order* is a relation $\leq$ on a set P which is

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1. reflexive,

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Partial order

Definition

A *partial order* is a relation $\leq$ on a set P which is

1. reflexive,
2. antisymmetric and
3. transitive.

We say that $(P, \leq)$ is a *partially ordered set* or a *poset*, for short.

Reflexive

For any $x \in P$, we have

Reflexive

For any $x \in P$, we have

$$x \leq x.$$

Antisymmetric

If $x \leq y$ and $y \leq x$ in P, then

Antisymmetric

If $x \leq y$ and $y \leq x$ in P, then

$$x = y.$$

Transitive

If $x \leq y$ and $y \leq z$ in P, then

Transitive

If $x \leq y$ and $y \leq z$ in P, then

$$x \leq z.$$

Examples of posets

Set theory

Let X be a set.
Then the powerset

$$(\mathcal{P}(X), \subseteq),$$

the set of all subsets of X, is a poset.

Hasse diagram

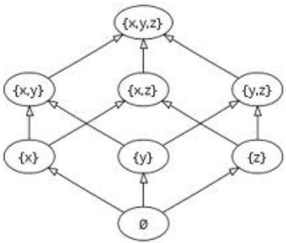

Figure: $(\mathcal{P}(\{x, y, z\}), \subseteq)$

Hasse diagram

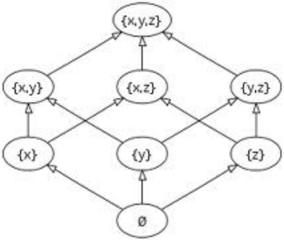

Figure: $(\mathcal{P}(\{x,y,z\}), \subseteq)$

Here $X = \{x, y, z\}$ is finite.

Hasse diagram

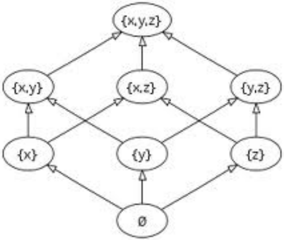

Figure: $(\mathcal{P}(\{x, y, z\}), \subseteq)$

Here $X = \{x, y, z\}$ is finite.
Of course, X may be infinite.

Food for thought

500,000 Rupiah-Question

Given the poset $Q = (\mathcal{P}(X), \subseteq)$ for a set X, how can I tell whether a particular element $A \in \mathcal{P}(X)$ is finite or not using only information about the structure of Q?

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You cannot count the number of elements in A since you have no access to its elements.

Examples of posets

Number theory

Let $\mathbb{N} := \{0, 1, 2, \dots\}$ be the set of natural numbers.
We define the relation $\prec$ on $\mathbb{N}$ as follows:

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Then $(\mathbb{N}, <)$ is a poset.

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We define the relation $\prec$ on $\mathbb{N}$ as follows:

$$a \prec, b \iff b \text{ is an integer multiple of } a.$$

Then $(\mathbb{N}, \prec)$ is a poset.

Number theorists use the well-known symbol $a \mid b$ for $a \prec b$.

Examples of posets

Recall that a partition of an interval $[a, b]$ is a finite subset $\{a_i\}_{i=0}^{n}$ of it with

$$a = a_0 < a_1 < a_2 < \cdots < a_n = b.$$

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Integration theory

Let $\mathcal{P}[a, b]$ be the collection of all partitions on $[a, b]$. Define $\leq$ on this set as follows:

$$P \leq Q$$

if P contains all the points of Q and possibly some other points.

Dcpo

Definition (Directed subsets)

A subset D of a poset P is *directed* if for any d_1, $d_2 \in D$,

Dcpo

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Directed subsets which are *lower* are called *ideals*.

Dcpo

Directed subsets

1. Every subset of a chain is directed.

Dcpo

Directed subsets

1. Every subset of a chain is directed.
2. Every subset of the form

$$\downarrow x := \{p \in P \mid p \leq x\}, \quad x \in P$$

in a poset P is an ideal.

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Definition (Dcpo)

A poset is a *dcpo* if every directed subset has a supremum.

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Recall from our experience in real analysis that

$$\text{supremum} = \text{least upper bound}$$

Dcpo

Here are some dcpo's:

Dcpo

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Example

1. Finite posets

Dcpo

Here are some dcpo's:

Example

1. Finite posets
2. Complete lattices

Dcpo

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Example

- $(\mathrm{Id}(P), \subseteq)$, the set of ideals of a poset P

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Example

- $(\mathrm{Id}(P), \subseteq)$, the set of ideals of a poset P
- $(\mathrm{Filt}(P), \subseteq)$, the set of filters of a poset P
- $\mathrm{Irr}(L)$, the poset of join-irreducible elements of a complete lattice L

We shall add one more important example later.

Order isomorphism

We regard two posets to be the 'same' if they have the same structure.

Order isomorphism

Definition

Posets P and Q are isomorphic, denoted by

$$P \cong Q$$

if there is an order-preserving bijection between them.

Order isomorphism

Example

Two isomorphic posets

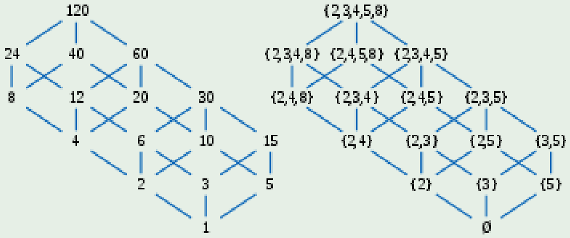

Figure: $P \cong Q$

T_0-space

Definition (T_0-space)

A topological space X is T_0 if for every pair of distinct points $x \neq y$, there exists an open set U that contains exactly one of them.

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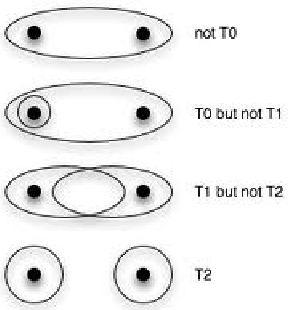

Scott topology

Definition (Scott open)

Let P be a poset and $U \subseteq P$.
U is *Scott open* if

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Let P be a poset and $U \subseteq P$.
U is *Scott open* if

- U is upper, and
- U is inaccessible by directed suprema.

Scott topology

In other words,

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Let P be a poset and $U \subseteq P$.
U is *Scott open* if

- $\uparrow U = U$, and
- $\forall$ directed $D \subseteq P.\ \bigsqcup D \in U \implies D \cap U \neq \emptyset$.

Scott topology

Let P be a poset.

Definition (Scott topology)

The collection of all Scott open subsets of P is called the *Scott topology* of P, and is denoted by $\sigma(P)$.

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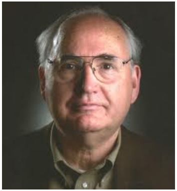


Figure: Dana Scott

Scott topology

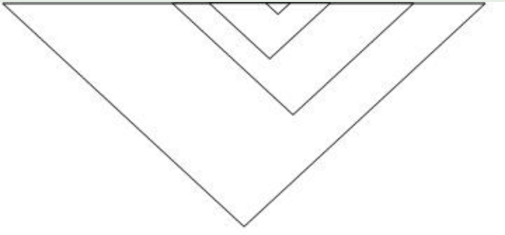
The Scott topology can capture the notion of

observability by finite means.

Example of Scott topology

Example (Interval domain)

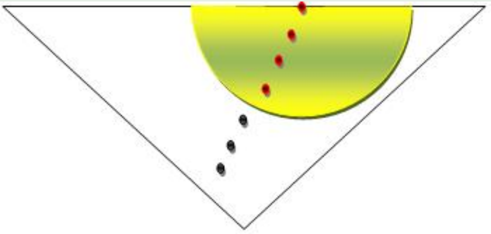
The interval domain $\mathbb{IR}$ consists of elements the closed intervals $[a, b] \subseteq \mathbb{R}$, ordered by reverse inclusion.



Example of Scott topology

Example (Interval domain)

A Scott-open set looks like this:



Example of a Scott open set

Example (Decimal representations not equal to π)

Think of real numbers in decimal representations as streams of natural numbers.
Consider the observation (by a machine or computer program) that can decide that an input stream (real number) is not equal to π.

Scott topology

Proposition

The Scott topology on any dcpo is T_0.

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Proposition

With respect to the Scott topology, the closure of singleton x is precisely the principal ideal generated by it, i.e.,

$$\mathrm{cl}\left(\{x\}\right) = \downarrow x.$$

Scott co-topology

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A subset C of a poset P is *Scott-closed* if

1. C is lower, and
2. C is closed under the formation of directed suprema.

Lattice of Scott-closed sets

Definition ($\Gamma(P)$)

We denote the lattice of Scott-closed sets, ordered by inclusion, of P by

$$(\Gamma(P), \subseteq)$$

Sober space

Definition (Irreducible subsets)

A nonempty subset A of a topological space X is *irreducible* if $A \subseteq B \cup C$ for closed subsets B and C implies $A \subseteq B$ or $A \subseteq C$.

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Example

Point closures $\mathrm{cl}(\{p\})$ are always irreducible closed subsets.

Sober space

Definition (Sober space)

A topological space X is *sober* if for every irreducible closed set C, there exists a unique $x \in X$ such that $\mathrm{cl}(\{x\}) = C$.

Sober space

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A topological space X is *sober* if for every irreducible closed set C, there exists a unique $x \in X$ such that $\mathrm{cl}(\{x\}) = C$.

Remark.

A sober space is automatically T_0 because $\mathrm{cl}(\{x\}) = \mathrm{cl}(\{y\})$ always implies $x = y$.

Specialization order

Proposition (Specialization order)

Let (X, τ) be a T_0-space.

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$$x \leq_\tau y \iff \forall U \in \tau.(x \in U \implies y \in U).$$

Then $\leq_\tau$ is a partial order on X, known as the specialization order.

Specialization order

Example (Scott topology)

The specialization order of the Scott topology coincides with the underlying order.

Scott is not always sober

We now add one more example of a dcpo.

Proposition

Let X be a sober topological space. Then

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Proposition

Let X be a sober topological space. Then

1. *the specialization order on X is a dcpo, and*

Scott is not always sober

We now add one more example of a dcpo.

Proposition

Let X be a sober topological space. Then

1. *the specialization order on X is a dcpo, and*
2. *the topology on X is contained in the Scott topology for the specialization order on X.*

Scott is not always sober

This naturally leads one to ask in the opposite direction:

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Question

Is the Scott topology on a dcpo always sober?

Scott is not always sober

In December 1978, P.T. Johnstone discovered a counterexample that answers this question in the negative.

Scott is not always sober

Example (Johnstone's construction)

Let $P = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$.

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Scott is not always sober

Example (Johnstone's construction)

Let $P = \mathbb{N} \times (\mathbb{N} \cup \{\infty\})$. We order P as follows:

$$(m, n) \leq (m', n') \iff \text{either } m = m' \;\&\; n \leq n' \text{ or } n' = \infty \;\&\; n \leq m'.$$

Key features of Johnstone's dcpo

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1. P is a poset whose maximal elements are (m, ∞).

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Key features of Johnstone's dcpo

1. P is a poset whose maximal elements are (m, ∞).
2. A directed subset of L either has a greatest element or is contained in $\{m\} \times (\mathbb{N} \cup \{\infty\})$ for some m.
3. P is a dcpo.
4. Any two non-empty Scott open subsets of P have a nonempty intersection.

Scott is not always sober

P fails to be sober because ...

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- P itself is irreducible.

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The quest for a characterization

The next natural question is:

The quest for a characterization

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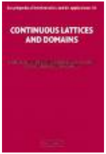
Open Problem

Find an order-theoretic characterization for those dcpo's for which their Scott topology is sober.

– Continuous lattices and domains, p.155



Where I started

Previous work

Characterize those complete lattices of the form:

$$\Gamma(P) := \sigma^{\mathrm{op}}(P),$$

for a complete semi-lattice P.

Where I started

In other words,

Representation theory of Scott co-topology

Which exactly are those complete lattices M that are isomorphic to

$$\Gamma(P)$$

for some complete semi-lattice P?

Parallel theories

This runs parallel to the classical representation theory for

$$\mathsf{Id}(P)$$

for some poset P.

Representation theory

Suppose a complete lattice $M = \Gamma(P)$ for some complete semi-lattice P.

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Representation theory

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Task of recovery

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Representation theory

To understand how this is done, we recall the

500, 000 Rupiah-Question

Given the poset $Q = (\mathcal{P}(X), \subseteq)$ for a set X, how can I tell whether a particular element $A \in \mathcal{P}(X)$ is finite or not using only information about the structure of Q?

Representation theory

Theorem (Kuratowsky's characterization of finiteness)

The following statements are equivalent:

1. $A \in \mathcal{P}(X)$ *is finite.*

Representation theory

Theorem (Kuratowsky's characterization of finiteness)

The following statements are equivalent:

1. $A \in \mathcal{P}(X)$ *is finite.*
2. *For every directed family* $\mathcal{D}$ *in* $\mathcal{P}(X)$*, whenever* $\bigcup \mathcal{D} \supseteq A$*, then there is already* $D \in \mathcal{D}$ *with* $D \supseteq A$*.*

We use the symbol $A \ll A$ to denote (2).

Representation theory

Theorem

The following statements are equivalent:

1. $A \in \mathrm{Id}(P)$ *is a principal ideal.*
2. *For every directed family* $\mathcal{D}$ *in* $\mathrm{Id}(P)$, *whenever* $\bigcup \mathcal{D} \supseteq A$, *then there is already* $D \in \mathcal{D}$ *with* $D \supseteq A$.

We use the symbol $A \ll A$ to denote (2).

Representation theory

Theorem

The following statements are equivalent:

1. $A \in \Gamma(P)$ *is a principal ideal.*
2. *For every Scott closed family* $\mathcal{D}$ *in* $\mathsf{Id}(P)$, *whenever* $\bigcup \mathcal{D} \supseteq A$, *then there is already* $D \in \mathcal{D}$ *with* $D \supseteq A$.

We use the symbol $A \prec A$ to denote (2).

Representation theory

Big idea

Given $M = \Gamma(P)$, we recover P by setting

$$P = K(M),$$

where

$$K(M) := \{m \in M \mid m \prec m\}.$$

Break

Let us have a 10 minutes break.

Part II

Title

Sober Scott spaces

The quest for a characterization

Recall that the problem we are about to attack is:

The quest for a characterization

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Open Problem

Find an order-theoretic characterization for those dcpo's for which their Scott topology is sober.

– Continuous lattices and domains, p.155

The idea I

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for some kind of dcpo P.

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Characterize those dcpo's of the form

$$H(P) := \mathsf{Irr}(\Gamma(P))$$

for some kind of dcpo P.

$\mathsf{Irr}(L)$ = the dcpo of join-irreducible elements of a (distributive) lattice L.

Notes

Notes

1. In 2006, X. Mao and L. Xu have considered $H(P)$ (they used the notation $c(P)$) to study FS-posets as a generalization of FS-domains.

Notes

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1. In 2006, X. Mao and L. Xu have considered $H(P)$ (they used the notation $c(P)$) to study FS-posets as a generalization of FS-domains.
2. They called $H(P)$ the *directed completion* of P.
3. Its order-theoretic properties were not studied for non-continuous P.

The idea II

If one can characterize those dcpo's Q of the form

$$H(P)$$

for *some (kind of)* dcpo P, then one should be able to

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for *some (kind of)* dcpo P, then one should be able to

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Denote the set of 'compact' elements of a dcpo Q by $K(Q)$.

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Since the 'compact' elements of $H(P)$ are *supposedly* the principal ideals, we can deduce the following:

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Every element of P is 'compact'.

A crucial comparison

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I left out *H-algebraic* on purpose.

H-below

Definition

Let P be a poset. Define $x \blacktriangleleft y$ as

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For any $a \in P$, we denote $\{x \in P \mid x \blacktriangleleft a\}$ by

$$H(a).$$

H-below

Pause and think

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Pause and think

Let P be a poset. We defined $x \blacktriangleleft y$ as

$$\forall C \in H(P).(\bigsqcup C \geq y \implies x \in C).$$

What is $\blacktriangleleft$ when P is sober w.r.t. its Scott topology?

H-continuity

Definition

A poset P is *H-continuous* if for all $a \in P$,

H-continuity

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1. $H(a) \in H(P)$, and

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A poset P is *H-continuous* if for all $a \in P$,

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2. $\bigsqcup H(a) = a$.

Note

$H(a)$ is always Scott-closed but may not be irreducible.

Union completeness

Theorem

For any poset P and any $\mathcal{C} \in H(H(P))$, the following holds:

$$\bigsqcup_{H(P)} \mathcal{C} = \bigcup \mathcal{C}.$$

Union completeness

Theorem

For any poset P and any $\mathcal{C} \in H(H(P))$, the following holds:

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This results in:

Corollary

For any poset P, the dcpo $H(P)$ has suprema of all irreducible Scott closed subsets.

We call this kind of completeness *ISC completeness*.

H-compact elements

Definition (H-compactness)

We define the *H-compact elements* of a poset P to be those $x \in P$ for which

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We define the *H-compact elements* of a poset P to be those $x \in P$ for which

$$x \blacktriangleleft x.$$

We use $K(P)$ to denote the set of all H-compact elements of P.

$K(M)$ as a sub-dcpo of M

Proposition

If M is a dcpo and $K(M) \neq \emptyset$, then $K(M)$ is a sub-dcpo with respect to the order inherited from M.

Principal ideals are H-compact

Thanks to union-completeness of H, we have:

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Let P be a poset. Then for each $x \in P$,

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Warning

Not every H-compact element of $H(P)$ is a principal ideal.

H-algebraic poset

Definition (H-algebraic dcpo)

A poset P is said to be *H-algebraic* if for each $a \in P$,

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1. $a = \bigsqcup \{x \in K(P) \mid x \leq a\}$,

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3. $\{x \in K(P) \mid x \leq a\} \in H(K(P))$.

Passage between M and $K(M)$

To facilitate the transport of lower sets between M and $K(M)$, we create a two-way passage:

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$$\mathcal{L}(M) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} \mathcal{L}(K(M))$$

$\flat$ and $\sharp$

Definition ($\flat$ & $\sharp$)

Let M be a poset and S be any non-empty subset of M.

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Let M be a poset and S be any non-empty subset of M. Define for each $A \subseteq M$, a corresponding subset $A^\sharp \subseteq S$ given by

$$A^\sharp := A \cap S$$

and for each $B \subseteq S$, a corresponding subset of M given by

$$B^\flat := \{m \in M \mid \exists b \in B. m \leq B\}.$$

$\flat$ and $\sharp$ as e-p pair

Proposition

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Proposition

1. $\langle \flat, \sharp \rangle$ *forms an e-p pair pair between the lattices of lower subsets of* M *and that of* S.

$\flat$ and $\sharp$ as e-p pair

Proposition

1. $\langle \flat, \sharp \rangle$ *forms an e-p pair pair between the lattices of lower subsets of* M *and that of* S.
2. *For any* $x \in S$, *it holds that*

$$(\downarrow_S x)^{\flat} = \downarrow_M x \ \& \ (\downarrow_M x)^{\sharp} = \downarrow_S x.$$

$\flat$ and $\sharp$

From now on, when we take $\flat$ and $\sharp$, we always restrict to $S = K(M)$.

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$$\mathcal{L}(M) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} \mathcal{L}(K(M))$$

$\flat$, $\sharp$ & H-algebraicity

Lemma

Let P be an H-algebraic poset.

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Lemma

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$\flat$, $\sharp$ & H-algebraicity

Lemma

Let P be an H-algebraic poset. Then for any $a \in P$, it holds that

$$(H(a))^{\sharp\flat} = H(a).$$

H-algebraicity implies H-continuity

The preceding lemma is used to obtain the following result:

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Proposition

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H-algebraicity implies H-continuity

The preceding lemma is used to obtain the following result:

Proposition

Let P be an H-algebraic poset. Then P is H-continuous.

Dominated dcpo

Let us motivate a crucial concept: dominated dcpo's.

Dominated dcpo

Let us motivate a crucial concept: dominated dcpo's.
We must reverse-engineer!

Recovery of P from $H(P)$

One wishes to have a theorem that looks like this:

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The converse is always true for any poset, as mentioned earlier.

Recovery of P from $H(P)$

Given that $X \in K(H(P))$, we hope to have some $x \in X$ such that $X \subseteq \downarrow x$.

Recovery of P from $H(P)$

To make use of the definition of $X \in K(H(P))$, we must produce some $\mathcal{C}_X \in H(H(P))$ such that $\bigsqcup_{H(P)} \mathcal{C}_X \supseteq X$.

Recovery of P from $H(P)$

To make use of the definition of $X \in K(H(P))$, we must produce some $\mathcal{C}_X \in H(H(P))$ such that $\bigsqcup_{H(P)} \mathcal{C}_X \supseteq X$.

But this choice of $\mathcal{C}_X$ must result in the existence of $x \in X$ such that $X \subseteq \downarrow x$.

Recovery of P from $H(P)$

To make use of the definition of $X \in K(H(P))$, we must produce some $\mathcal{C}_X \in H(H(P))$ such that $\bigsqcup_{H(P)} \mathcal{C}_X \supseteq X$.

But this choice of $\mathcal{C}_X$ must result in the existence of $x \in X$ such that $X \subseteq \downarrow x$.

Crucially, $X \in \mathcal{C}_X$.

Recovery of P from $H(P)$

This forces upon us the choice:

$$\mathcal{C}_X := \{F \in H(P) \mid \exists x \in X . F \subseteq \downarrow x\}.$$

Recovery of P from $H(P)$

This forces upon us the choice:

$$\mathcal{C}_X := \{F \in H(P) \mid \exists x \in X . F \subseteq \downarrow x\}.$$

Note that

$$\bigsqcup_{H(P)} \mathcal{C}_X = X.$$

Recovery of P from $H(P)$

Question

Do we always have:

$$\mathcal{C}_X \in H(H(P))$$

for any irreducible Scott closed set $X \subseteq P$?

Recovery of P from $H(P)$

Question

Do we always have:

$$\mathcal{C}_X \in H(H(P))$$

for any irreducible Scott closed set $X \subseteq P$?

We cannot even be sure that $\mathcal{C}_X \in \Gamma(H(P))$!

Dominated dcpo

Definition (X-dominated)

Let P be a poset and $X \subseteq P$ be a nonempty.

Dominated dcpo

Definition (X-dominated)

Let P be a poset and $X \subseteq P$ be a nonempty.

- A subset F of P is *X-dominated* if there exists $x \in X$ such that $F \subseteq \downarrow x$.

Dominated dcpo

Definition (X-dominated)

Let P be a poset and $X \subseteq P$ be a nonempty.

- A subset F of P is *X-dominated* if there exists $x \in X$ such that $F \subseteq \downarrow x$.
- A family $\mathcal{F}$ of subsets of P is *X-dominated* if every $F \in \mathcal{F}$ is X-dominated.

Dominated dcpo

Definition (Dominated poset)

P is *dominated* if for every irreducible Scott closed set X of P and for every directed X-dominated family $\mathcal{D}$ of irreducible Scott closed subsets of P,

$$\bigcup \mathcal{D} \text{ is } X\text{-dominated.}$$

Dominated dcpo

Proposition

Every dominated poset is a dcpo.

Henceforth, we only use the terminology *dominated dcpo*.

Dominated dcpo

Example

1. Complete lattices and complete semilattices.

Dominated dcpo

Example

1. Complete lattices and complete semilattices.
2. ISC complete dcpo's.

Dominated dcpo

Example

1. Complete lattices and complete semilattices.
2. ISC complete dcpo's.
3. Dcpo's whose Scott topology is sober.

Equivalent formulations for a dominated dcpo

Let P be a poset. Then t.f.a.e.:

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1. P is dominated.

Equivalent formulations for a dominated dcpo

Let P be a poset. Then t.f.a.e.:

1. P is dominated.
2. For every irreducible Scott closed subset X of P, $\bigsqcup_{H(P)} \mathcal{D}$ is X-dominated for all directed X-dominated family $\mathcal{D}$ of irreducible Scott closed subsets of P.

Equivalent formulations for a dominated dcpo

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1. P is dominated.
2. For every irreducible Scott closed subset X of P, $\bigsqcup_{H(P)} \mathcal{D}$ is X-dominated for all directed X-dominated family $\mathcal{D}$ of irreducible Scott closed subsets of P.
3. For every irreducible Scott closed subset X of P, it holds that

$$\mathcal{C}_X \in H(H(P)).$$

Equivalent formulations for a dominated dcpo

Let P be a poset. Then t.f.a.e.:

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3. For every irreducible Scott closed subset X of P, it holds that

$$\mathcal{C}_X \in H(H(P)).$$

4. For every irreducible Scott closed subset X of P, it holds that

$$\mathcal{C}_X \in \Gamma(H(P)).$$

Open problem

Unsolved problem

Give an example of a dcpo, if any, which is not dominated.

Properties of $H(P)$ for dominated P

Theorem

Let P be a dominated dcpo.

Properties of $H(P)$ for dominated P

Theorem

Let P be a dominated dcpo.
Then $X \in K(H(P))$ if and only if X is principal.

Properties of $H(P)$ for dominated P

Corollary

Let P be a dominated dcpo.

Properties of $H(P)$ for dominated P

Corollary

Let P be a dominated dcpo. Then

$$P \cong K(H(P)).$$

Properties of $H(P)$ for dominated P

Corollary

$H(P)$ is H-algebraic for any dominated dcpo P.

An equivalent condition for dominated P

Proposition

Let P be a dcpo. Then t.f.a.e.:

1. *P is dominated.*

An equivalent condition for dominated P

Proposition

Let P be a dcpo. Then t.f.a.e.:

1. *P is dominated.*
2. *For every irreducible Scott-closed subset $\mathcal{C}$ of $H(P)$, it holds that*

$$(\mathcal{C})^{\sharp\flat} \in H(H(P)).$$

Dcpo's with sober Scott topology as fixed point

Proposition

Let P *be a dcpo. Then t.f.a.e.:*

Dcpo's with sober Scott topology as fixed point

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Let P *be a dcpo. Then t.f.a.e.:*

(i) $(P, \sigma(P))$ *is sober.*

Dcpo's with sober Scott topology as fixed point

Proposition

Let P be a dcpo. Then t.f.a.e.:

(i) $(P, \sigma(P))$ *is sober.*

(ii) *For every irreducible Scott-closed subset $\mathcal{C}$ of $H(P)$, it holds that*

$$\mathcal{C} = (\mathcal{C}^{\sharp})^{\flat}.$$

$\flat$-stable dcpo

Definition ($\flat$-stability)

Let M be a dcpo. We say that M is *$\flat$-stable* if for every $B \in H(K(M))$, it holds that

$$B^\flat \in H(M).$$

Fundamental lemma

Lemma

For an H-algebraic dominated dcpo M, t.f.a.e.:

Fundamental lemma

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(i) M *is* $\flat$*-stable.*

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(ii) $K(M)$ *is dominated.*

Fundamental lemma

Lemma

For an H-algebraic dominated dcpo M, t.f.a.e.:

(i) *M is $\flat$-stable.*

(ii) *$K(M)$ is dominated.*

An immediate consequence of this is:

Proposition

$H(P)$ is $\flat$-stable for any dominated dcpo P.

Coherent dcpo

Definition

A dcpo M which has all suprema of irreducible Scott-closed subsets of $K(M)$ is called *coherent*.

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Remark.

$H(P)$ is coherent for any dominated dcpo P.

Stably H-algebraic dcpo

Definition (Stably H-algebraic dcpo)

A $\flat$-stable, coherent, ISC-complete H-algebraic dcpo will be called a *stably H-algebraic* dcpo, for short.

Characterization of $H(P)$

Theorem

A dcpo M is isomorphic to $H(P)$ for a dominated dcpo P if and only if M is stably H-algebraic.

Strongly H-algebraic dcpo

Definition (Strongly H-algebraic)

A stably H-algebraic dcpo P with $K(P) = P$ will be called *strongly H-algebraic*.

Main theorem

Theorem

Let P be a dcpo. The following statements are equivalent:

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(i) $(P, \sigma(P))$ *is sober.*

Main theorem

Theorem

Let P be a dcpo. The following statements are equivalent:

(i) $(P, \sigma(P))$ *is sober.*

(ii) P *is strongly* H*-algebraic.*

Proof

($\Longrightarrow$):

- Since P is sober, $K(H(P)) = H(P)$.

Proof

$(\implies)$:

- Since P is sober, $K(H(P)) = H(P)$.
- Since P is sober, it is dominated.

Proof

($\Longrightarrow$):

- Since P is sober, $K(H(P)) = H(P)$.
- Since P is sober, it is dominated.
- Since P is dominated, $K(H(P)) \cong P$.

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- Since P is sober, $K(H(P)) = H(P)$.
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- So $P \cong H(P)$.

Proof

$(\implies)$:

- Since P is sober, $K(H(P)) = H(P)$.
- Since P is sober, it is dominated.
- Since P is dominated, $K(H(P)) \cong P$.
- So $P \cong H(P)$.

So P takes on all the properties of $H(P)$, for a dominated P.

Proof

($\Longleftarrow$):

- Since P is stably H-algebraic, $P \cong H(K(P))$.

Proof

($\Longleftarrow$):

- Since P is stably H-algebraic, $P \cong H(K(P))$.
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Proof

($\Longleftarrow$):

- Since P is stably H-algebraic, $P \cong H(K(P))$.
- Since $P = K(P)$, $P \cong H(P)$.
- Since P is ISC-complete, it is dominated.

Proof

($\Longleftarrow$):

- Since P is stably H-algebraic, $P \cong H(K(P))$.
- Since $P = K(P)$, $P \cong H(P)$.
- Since P is ISC-complete, it is dominated.
- Since P is dominated, $P \cong K(H(P))$.

Proof

($\Longleftarrow$):

- Since P is stably H-algebraic, $P \cong H(K(P))$.
- Since $P = K(P)$, $P \cong H(P)$.
- Since P is ISC-complete, it is dominated.
- Since P is dominated, $P \cong K(H(P))$.
- Thus $H(P) = K(H(P))$.

Usefulness of the Main Theorem

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Usefulness of the Main Theorem

The main theorem asserts that:
A dcpo P is sober if and only if P is

1. ISC-complete,
2. H-algebraic,
3. $\flat$-stable,
4. coherent, and
5. contains nothing but H-compact elements.

Johnstone's construction

Johnstone's dcpo fails to be sober because it is not ISC-complete.

Isbell's construction

Isbell constructed a complete lattice L for which the Scott topology is not sober.

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In that example, there exists an irreducible Scott closed set X for which $X \neq \downarrow \bigsqcup X$.

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Isbell's construction

Isbell constructed a complete lattice L for which the Scott topology is not sober.
In that example, there exists an irreducible Scott closed set X for which $X \neq \downarrow \bigsqcup X$. Thus,

$$\bigsqcup X \notin X.$$

So L contains at least one element which is not H-compact, namely, $\bigsqcup X$.

An extension of Isbell's construction

Let L be Isbell's complete lattice.

An extension of Isbell's construction

Let L be Isbell's complete lattice.
We claim that $H(L)$ is also not sober.

An extension of Isbell's construction

Suppose not, i.e., $H(L)$ is sober w.r.t. its Scott topology.

An extension of Isbell's construction

Suppose not, i.e., $H(L)$ is sober w.r.t. its Scott topology. Because L is complete and thus dominated, the H-compact elements of $H(L)$ are precisely the principal ideals, i.e.,

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$$K(H(L)) = H(L).$$

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Suppose not, i.e., $H(L)$ is sober w.r.t. its Scott topology. Because L is complete and thus dominated, the H-compact elements of $H(L)$ are precisely the principal ideals, i.e.,

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Again since L is complete and thus dominated, we have

An extension of Isbell's construction

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$$K(H(L)) = H(L).$$

Again since L is complete and thus dominated, we have

$$K(H(L)) \cong L.$$

An extension of Isbell's construction

Suppose not, i.e., $H(L)$ is sober w.r.t. its Scott topology. Because L is complete and thus dominated, the H-compact elements of $H(L)$ are precisely the principal ideals, i.e.,

$$K(H(L)) = H(L).$$

Again since L is complete and thus dominated, we have

$$K(H(L)) \cong L.$$

This implies that $L \cong H(L)$, a contradiction since L is not sober w.r.t. its Scott topology.

An extension of Isbell's construction

An extension of Isbell's construction

Consider the following facts:

Fact	Reason
$H(L)$ is ISC-complete	Union-completeness of H
$H(L)$ is H-algebraic	L is dominated
$H(L)$ is $\flat$-stable	$K(H(L)) \cong L$ since L is dominated
$H(L)$ is coherent	L is dominated

An extension of Isbell's construction

Consider the following facts:

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$H(L)$ is H-algebraic	L is dominated
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$H(L)$ is coherent	L is dominated

That $H(L)$ fails to be sober indicates that $H(L)$ contains some element which is not H-compact.

New construction of non-sober dcpo's

Example

Let $P_0 = J$, where J is Johnstone's construction.
Define a sequence of dcpo's P_n as follows:

$$P_0 = J \text{ and } P_{n+1} = H(P_n), \ \ n \in \mathbb{N}.$$

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Then we have a countable infinitude of pairwise non-isomorphic dominated dcpo's which are not sober w.r.t. their Scott topology.

Conclusion

In this talk, I have spoke about

1. the Scott topology and co-topology,

Conclusion

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2. the problem of Scott sobriety,

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1. the Scott topology and co-topology,
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Conclusion

In this talk, I have spoke about

1. the Scott topology and co-topology,
2. the problem of Scott sobriety,
3. $\blacktriangleleft$-relation,
4. H-continuous and H-algebraic dcpo's,
5. dominated dcpo's
6. $\flat$-stability and coherence, and
7. necessary and sufficient conditions for a dcpo to be sober w.r.t. its Scott topology.

Conclusion

Thank you for your patience!

References

1. M. Erné. Z-continuous posets and their topological manifestation. *Applied Categorical Structures*, 7:31-70, 1999.
2. G. Gierz, *et al*. Continuous Lattices and Domains. *Encyclopedia of Mathematics and its Applications*, 93, Cambridge University Press, 2003.
3. W.K. Ho and D. Zhao. Lattices of Scott-closed sets. *Commentationes Mathematicae Universitatis Carolinae*, 50(2):297-314, 2009.
4. J. Isbell. Completion of a construction of Johnstone. *Proceedings of American Mathematical Society*, 85(3):333-334, July 1982.
5. P.T. Johnstone. Scott is not always sober. *Lecture Notes in Mathematics*, 871, 1981.