Rank Commutators

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Outline

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1. $\text{rank}(A) = \text{rank}(A^T)$.
2. *For any invertible matrix B of size n,* $\text{rank}(AB) = \text{rank}(BA)$.

Common misconception

Many beginners believe that ...

For any square matrices A and B of the same size, it holds that

$$\text{rank}(AB) = \text{rank}(BA).$$

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Counterexample

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Counterexample

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then,

$$\text{rank}(AB) = 0 \text{ while } \text{rank}(BA) = 1.$$

A natural question to ask

What exactly are those square matrices A such that

$$\text{rank}(AB) = \text{rank}(BA)$$

for *all* real $n \times n$ matrices B?

Question generalised

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$$\text{rank}(AB) = \text{rank}(BA).$$

The set of $\mathcal{L}$-rank commutators is denoted by $\mathcal{L}^*$.

Basic results

Lemma

For any $n \times n$ matrices A and B,

$$\text{rank}(A) + \text{rank}(B) - n \leq \text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}.$$

Basic results

Lemma

For any $n \times n$ *matrices* A *and* B,

$$0 \leq |\operatorname{rank}(AB) - \operatorname{rank}(BA)| \leq \min\{r, s, n-r, n-s\},$$

where $\operatorname{rank}(A) = r$ *and* $\operatorname{rank}(B) = s$.

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Lemma

For any $n \times n$ *matrix* B *with rank* r *and any integer* i *with* $|i| \leq \min\{r, n-r\}$, *there is a matrix* A *such that*

$$\text{rank}(AB) - \text{rank}(BA) = i.$$

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Theorem

For any $n \times n$ matrix B with rank r and any integer i with $1 \leq i \leq \min\{r, n-r\}$, the following conditions hold:

1. *There is a matrix A such that $\text{rank}(AB) = \text{rank}(A) = i$ and BA is a zero-matrix.*
2. *There is a matrix A' such that $\text{rank}(BA') = \text{rank}(A') = i$ and $A'B$ is a zero-matrix.*

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Moreover, the bound $\min\{r, n-r\}$ is sharp.

Basic results

Corollary

For any integers $n > 0$ *and* i *with* $0 \leq i \leq \lfloor n/2 \rfloor$, *there are* $n \times n$ *matrices* A *and* B *such that*

$$|\operatorname{rank}(AB) - \operatorname{rank}(BA)| = i.$$

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Furthermore, the bound $\lfloor n/2 \rfloor$ is sharp.

Some notations

Throughout this talk, we adopt the following notations:

$\mathcal{M}_{m\times n}$	the set of $m \times n$ matrices
$\mathcal{M}_n$	the set of non-zero square matrices of size n
$\mathcal{GL}_n$	the set of invertible square matrices of size n
$\mathcal{SL}_n$	the set of square matrices of size n whose determinant is 1
$\mathcal{S}_n$	the set of non-zero symmetric matrices of size n
$\mathcal{D}_n$	the set of non-zero diagonal matrices of size n
$\mathcal{DL}_n$	the set of non-zero diagonalizable matrices of size n
$\mathcal{R}_{n,i}$	the set of non-zero square matrices of size n and of rank i
I_n	the identity matrix of size n
$0_{m,n}$	the $m \times n$ zero matrix

General framework

For a given binary relation R on a set X, define a pair of endofunctions on $\mathcal{P}(X)$ as follows:

$$(-)^* : \mathcal{P}(X) \longrightarrow \mathcal{P}(X),\ M \mapsto M^*,$$

where $M^* := \{x \in X \mid \forall m \in M.\ m\ R\ x\}$, and

$$(-)_* : \mathcal{P}(X) \longrightarrow \mathcal{P}(X),\ N \mapsto N_*,$$

where $N_* := \{x \in X \mid \forall n \in N.\ x\ R\ n\}$.

It can be shown that ...

Theorem

For any $M, N \in \mathcal{P}(X)$,

$$M \subseteq N_* \iff M^* \supseteq N.$$

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1. $(-)^*$ preserves arbitrary suprema in $(\mathcal{P}(X), \subseteq)$, i.e., for any family $(M_i)_{i \in I}$ of subsets of X, it holds that

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2. $(-)_*$ preserves arbitrary infima in $(\mathcal{P}(X), \supseteq)$, i.e., for any family $(N_i)_{i\in I}$ of subsets of X, it holds that

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1. $M \subseteq (M^*)_*$ *and* $M \subseteq (M_*)^*$.
2. $M^* = ((M^*)_*)^*$ *and* $M_* = ((M_*)^*)_*$.

Specializing the results

For the special case where $X = \mathcal{M}_n$, define the binary relation $\sim$ on X as follows:

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Since $\sim$ is symmetric, $(-)^* = (-)_*$ and use only one notation $(-)^*$

One trivial result

Lemma

For any positive integer n,

$$\mathcal{GL}_n^* = \mathcal{SL}_n^* = \mathcal{M}_n.$$

Another trivial result

Lemma

For any positive integer n and any $\mathcal{L} \subseteq \mathcal{M}_n$, it holds that

$$\mathcal{GL}_n \subseteq \mathcal{L}^*.$$

$\mathcal{M}_n$-rank commutators

Theorem

For any positive integer n,

$$\mathcal{M}_n^* = \mathcal{GL}_n.$$

$\mathcal{R}_{n,j}$-rank commutators

Theorem

For any positive integers n *and* $1 < i < n$,

$$\mathcal{R}^*_{n,i} = \mathcal{GL}_n.$$

Fixed points of $(-)^*$

Notation

The ith row of a given matrix $A \in \mathcal{M}_n$ is denoted by $\vec{r_i}$, i.e.,

$$\vec{r_i} := \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix},$$

while its jth column is denoted by $\vec{c_j}$, i.e.,

$$\vec{c_j} := \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}.$$

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We say that $A \in \mathcal{M}_n$ is *rank-symmetric* if for any non-empty subset $K \subseteq \{1, \cdots, n\}$, the following are equivalent:

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(ii) The set $\{\vec{c_k} \mid k \in K\}$ is linearly independent.

The set of all rank-symmetric matrices of size n is denoted by $\mathcal{T}_n$.

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(i) A *is rank-symmetric.*

(ii) *For every non-empty subset* $K \subseteq \{1, \cdots, n\}$,

$$\dim\langle \vec{r_k} \mid k \in K\rangle = \dim\langle \vec{c_k} \mid k \in K\rangle.$$

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(iii) A *is either invertible or of the form* $C \begin{pmatrix} X & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{pmatrix} C^T$ *for some invertible matrices* C *and* X*, where* $\text{rank}(X) = r < n$.

Fixed point of $(-)^*$

Theorem

For any positive integer n, we have

$$\mathcal{D}_n^* = \mathcal{S}_n^* = \mathcal{T}_n^* = \mathcal{T}_n.$$

Transformation theorems

Lemma

For any $\mathcal{L} \subseteq \mathcal{M}_n$, *we have* $(\mathcal{L}^T)^* = (\mathcal{L}^*)^T$.

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Lemma

For any $\mathcal{L} \subseteq \mathcal{M}_n$ *and any invertible matrix* C, *it holds that*

$$(\mathcal{L}C)^* = C^{-1}\mathcal{L}^* \text{ and } (C\mathcal{L})^* = \mathcal{L}C^{-1}.$$

Transformation theorems

Theorem

For any invertible matrix C, we have

$$(\mathcal{T}_n C)^* = \mathcal{T}_n C^T \text{ and } (C\mathcal{T}_n)^* = C^T \mathcal{T}_n.$$

Other fixed points of $(-)^*$

Corollary

For any symmetric and invertible matrix C, it holds that

$$(\mathcal{T}_n C)^* = \mathcal{T}_n C \text{ and } (C\mathcal{T}_n)^* = C\mathcal{T}_n.$$

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Corollary

Let $C_1, C_2, \cdots, C_k$ be square matrices, each of which obtained from I_n by exchanging two columns. If $C = (C_1 C_2 \cdots C_k)^T$, then

$$(\mathcal{T}_n C)^* = \mathcal{T}_n C^{-1}.$$

$\mathcal{DL}_n$-rank commutators

Theorem

For any positive integer n,

$$\mathcal{DL}_n^* = \mathcal{GL}_n.$$

Some open problems

Let $\mathcal{U}_n$, $\mathcal{AS}_n$, $\mathcal{H}_n$ denote the sets of upper triangular, anti-symmetric, and Hadamard matrices of sizes n respectively.

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Question 1

Determine

$$\mathcal{U}_n^*,\ \mathcal{AS}_n^*,\ \mathcal{H}_n^*.$$

Some open problems

Question 2

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Note that the above is true for $\mathcal{L} = \mathcal{T}_n,\ \mathcal{D}_n,\ \mathcal{S}_n$.

Some open problems

Question 3

Are there any other interesting real-valued matrix functions other than rank for which the same game can be played?

References

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The End

Thank you!