Poset reconstruction problem

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Outline

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1. Object: poset
2. Morphism: monotone function
3. Poset reconstruction problem

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2. Morphism: monotone function
3. Poset reconstruction problem
4. Conclusion

Introduction

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Introduction

I work

1. as an Assistant Professor in NIE,
2. as an academia in the mathematics academic group, and
3. in areas like general topology, order theory, domain theory, algebra, programming semantics and so on.

Highlights

Today's talk is about an open problem in order theory, which I am quite interested in.

Highlights

Many people have heard about

Chaos Theory.

Highlights

Many people have heard about

Chaos Theory.

But few have heard about

Order Theory.

Highlights

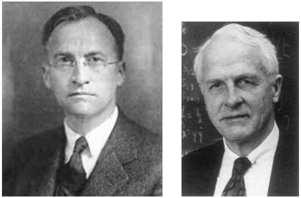

Figure: The Birkhoffs

Chaos Theory in 1927 – George David Birkhoff's "3-Body Problem".

Highlights


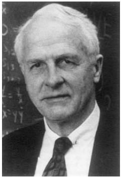

Figure: The Birkhoffs

Chaos Theory in 1927 – George David Birkhoff's "3-Body Problem".

Order theory in 1948 – Garrett Birkhoff's "Lattice Theory".

Highlights

Before describing the problem, we need to build up some vocabulary of order theory.

Highlights

We shall touch on

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1. Partial orders

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2. Sups and infs

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3. Lattices, complete lattices and dcpo's

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before we explain the reconstruction problem.

Order everywhere

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- ranking based on results

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- lexicographical order in dictionary
- system of reporting officers in a department

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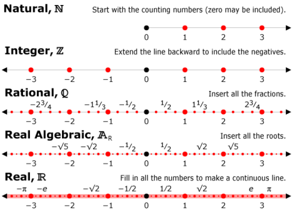
We see more later ...

Natural ordering on the real line



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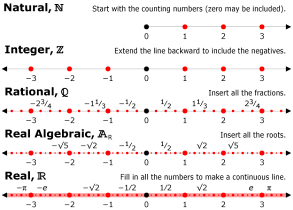

The most basic comparison of numbers can be done on $\mathbb{R}$ with the

$\leq$: less than or equal.

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Pop-quiz 1 (1 min)

Without the use of calculators, fill in the box with the correct inequality sign ($\leq$ or $\geq$):

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$$e^{\pi} \ \square \ \pi^{e}.$$

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- If $a \leq b$ and $b \leq c$, then

$$a \leq c.$$

Subset inclusion

Another relation which displays similar properties is the subset inclusion relation on sets:

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Exercise

Which are the three corresponding properties enjoyed by $\subseteq$?

Partial order

We can abstract ourselves from the above motivating examples by giving the definition of a *partially ordered set* (a.k.a. *poset*, for short).

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Why?

This abstraction allows us to crystallize the essential bits of the relevant information without losing too many details.

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Definition

A *partial order* is a relation $\leq$ on a set P which is

1. reflexive,
2. antisymmetric and
3. transitive.

We say that $(P, \leq)$ is a *partially ordered set* or a *poset*, for short.

Reflexive

For any element $x \in P$, we have

$$x \leq x.$$

Antisymmetric

If $x \leq y$ and $y \leq x$ in P, then

$$x = y.$$

Transitive

If $x \leq y$ and $y \leq z$ in P, then

$$x \leq z.$$

Examples

Set theory

Let X be a set. Then the powerset $\mathcal{P}(X)$, the set of all subsets of X, is partially ordered by the subset inclusion $\subseteq$.

Examples

Number theory

Let $\mathbb{Z}$ be the set of integers. We define the relation $|$ on $\mathbb{Z}$ as follows:

$$a \mid b \iff b \text{ is an integer multiple of } a.$$

Then $(\mathbb{Z}, |)$ is a poset.

Examples

Biology

Let S be a species of animals. Define the relation $\prec$ on S as follows:

$$x \prec y \iff \text{either } x = y \text{ or } x \text{ is an offspring of } y.$$

Examples

Recall that a partition of an interval $[a, b]$ is a finite subset $\{a_i\}_{i=0}^{n}$ of it with

$$a = a_0 < a_1 < a_2 < \cdots < a_n = b.$$

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Integration theory

Let $\mathcal{P}[a, b]$ be the collection of all partitions on $[a, b]$. Define $\leq$ on this set as follows:

$$P \leq Q$$

if P contains all the points of Q and possibly some other points.

Pictorial representation of posets

We can use a Hasse diagram to represent a poset.

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Question

What is a Hasse diagram?

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For a partially ordered set $(S, \leq)$, one

- represents each element of S as a vertex on the page, and
- draws a line segment or curve that goes upward from x to y if $x < y$, and
- there is no z such that $x < z < y$ (here, $<$ is obtained from $\leq$ by removing elements (x, x) for all x).

Hasse diagram

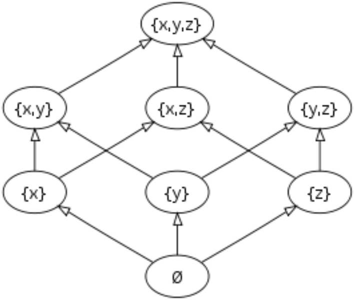


Figure: $(\mathcal{P}(\{x, y, z\}), \subseteq)$

Hasse diagram

Pop-quiz 2 (3 min)

Let $A = \{n \in \mathbb{Z}^+ : n \text{ is a factor of } 60\}$.
Draw the Hasse diagram that represents the poset $(A, |)$.

Hasse diagram

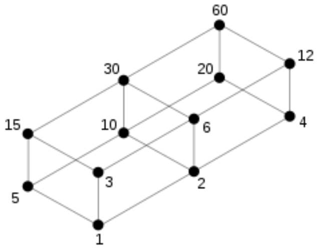


Figure: $(A, |)$

Hasse diagram

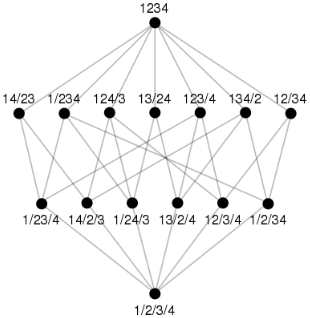


Figure: Partition of 4

Dual

Definition (Dual of a poset)

The *dual* of a poset P, denoted by P^{op}, has the same underlying set but has all its order relations reversed, i.e., its relation $\leq^{\mathrm{op}}$ is defined by

$$x \leq^{\mathrm{op}} y \iff y \leq x.$$

Upper and lower bound

Pop-quiz 3 (2 min)

A sequence u_n of real numbers is defined as follows:

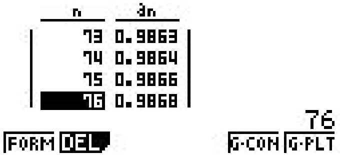
$$u_n = 1 - \frac{1}{n}, \ n = 1, 2, \ldots$$

Can you find two numbers $m, \ M$ such that

$$m \leq u_n \leq M$$

for all $n = 1, 2, \ldots$?

Upper and lower bound



Upper and lower bound

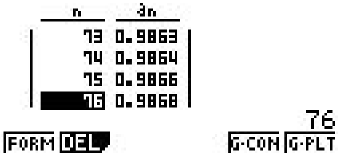
n	an
73	0.9863
74	0.9864
75	0.9866
76	0.9868

76

FORM DEL G·CON G·PLT

In the above example, let $U = \{u_n : n = 1, 2, \ldots\}$. We call

Upper and lower bound



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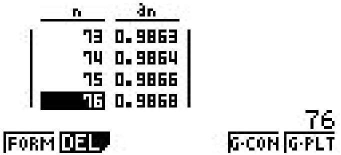

In the above example, let $U = \{u_n : n = 1, 2, \ldots\}$. We call

- m: a lower bound of U, and
- M: an upper bound of U.

Upper and lower bound

Definition

Let S be a poset and $T \subseteq S$.

1. u is called an *upper bound* of T iff

$$u \in \mathrm{ub}(T) \iff \text{for all } t \in T,\ t \leq u.$$

Upper and lower bound

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2. l is called an *lower bound* of T iff

$$l \in \mathrm{lb}(T) \iff \text{for all } t \in T,\ l \leq t.$$

Least upper bound

In the preceding example,

$$M = 1$$

is the least upper bound of U.

Least upper bound

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We call l.u.b. the *supremum*.

Greatest lower bound

In that example,

$$m = 0$$

is the greatest upper bound of U.

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We call the g.l.b. the *infimum*.

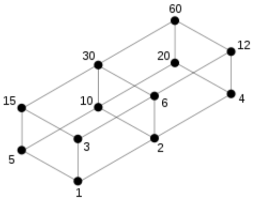
Sup and inf

In general, if $T \subseteq S$, we use the following notations:

$$\sup T := \bigvee T, \ \inf T := \bigwedge T.$$

Finding sups and infs

In the following poset,



Finding sups and infs

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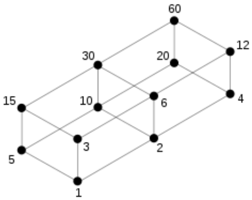

find

(i) $\bigvee\{6, 15\}$.

Finding sups and infs

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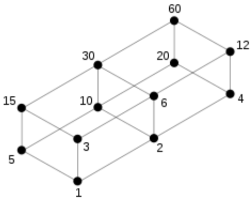


find

(i) $\bigvee\{6, 15\}$.

(ii) $\bigwedge\{4, 10\}$.

Lattice

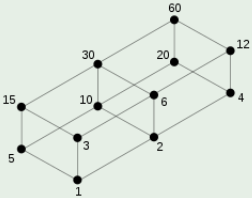
Definition

A poset L where every finite set has a sup and an inf is called a *lattice*.

Lattice

Example

The divisibility relation defines a lattice.



The inf is gcd (or hcf) and sup is lcm.

Complete lattice

Definition

A lattice L is a *complete lattice* if for every subset S both

$$\bigvee S \text{ and } \bigwedge S$$

exists.

Complete lattice

Example

The unit interval $[0,1]$ is a complete lattice w.r.t. the usual order $\leq$.

Complete lattice

Example

The powerset lattice $(\mathcal{P}(X), \subseteq)$ of X is complete since for any subset $\mathcal{A} \subseteq \mathcal{P}(X)$, we have

$$\bigvee \mathcal{A} = \bigcup \mathcal{A} \text{ and } \bigwedge \mathcal{A} = \bigcap \mathcal{A}.$$

Monotone functions

Like any other algebraic structures, one is interested in certain kinds of mappings between posets which preserve the structure. Here one naturally looks at a basic requirement:

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Like any other algebraic structures, one is interested in certain kinds of mappings between posets which preserve the structure. Here one naturally looks at a basic requirement:

Definition

A mapping $f : P \longrightarrow Q$ between posets is *monotone* if

$$a \leq b \implies f(a) \leq f(b).$$

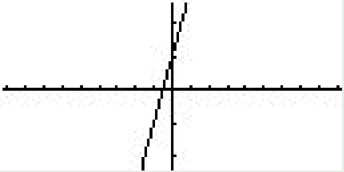
Monotone function

Example (Linear map)

The function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \ x \mapsto 2x + 1$$

is monotone.



Monotone function

Example (Probability)

The probability measure

$$P : \mathcal{F} \longrightarrow [0,1],\ E \mapsto P(E)$$

is a monotone mapping from a σ-algebra $\mathcal{F}$ on the sample space Ω to the closed unit interval.

Monotone function

Example (Integration)

The integral operator

$$\int : \mathcal{C}[0,1] \to \mathbb{R}, \ f \mapsto \int_0^1 f(x) \, \mathrm{d}x$$

is monotone.

Monotone function

Example (Integration)

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is monotone.

Notice that if $\{f_n\}_{n=1}^{\infty}$ is a monotone increasing sequence of functions, then we can say more:

$$\sup_n \int_0^1 f_n(x) \ \mathrm{d}x = \int_0^1 \sup f_n(x) \ \mathrm{d}x.$$

Non-example

An anti-tone function

Consider the function

$$h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \ x \mapsto \frac{1}{x}.$$

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Consider the function

$$h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+, \; x \mapsto \frac{1}{x}.$$

The function h is clearly order-reversing since

$$x_1 < x_2$$
$$\iff \frac{1}{x_1} > \frac{1}{x_2}$$

for any $x_1, \; x_2 \in \mathbb{R}^+$.

Poset isomorphism

We wish to say something about posets which one regards the same, i.e., they have the same order structure.

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Definition

$f : P \longrightarrow Q$ is an *isomorphism* if f is a bijection between P and Q such that

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$$p_1 \leq p_2 \iff f(p_1) \leq f(p_2).$$

In this case, we say that P is *isomorphic* to Q, denoted by

$$P \cong Q.$$

Poset isomorphism

Pop-quiz 4 (1 min)

Explain why the following two posets are isomorphic.

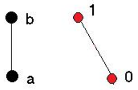

Figure: Two isomorphic posets: Sierpinski space Σ

Poset isomorphism

Consider the number $x = 120$ and the set $X = \{2, 3, 4, 5, 8\}$.

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Consider the number $x = 120$ and the set $X = \{2, 3, 4, 5, 8\}$.

1. $P = \{n \in \mathbb{Z}^+ : n \,|\, x\}$
2. $Q = \mathcal{P}(X)$

Poset isomorphism

It turns out that $P \cong Q$.

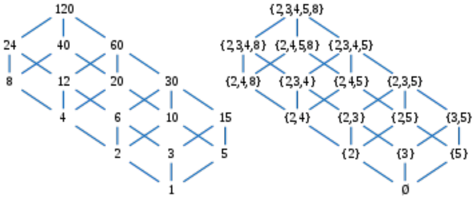

Figure: Two isomorphic lattices

Subposet

Definition (Subposet)

Let $(P, \leq)$ be a poset and a non-empty subset $S \subseteq P$. Define $\leq_S$ as follows:

$$s_1 \leq_S s_2 \iff s_1 \leq s_2 \text{ in } P.$$

Then $(S, \leq_S)$ is a poset called a *subposet* of P.

One-point-deleted subposet

Definition (One-point-deleted subposet)

Let $x \in P$ be given. Then the one-point-deleted subposet induced by x is defined to be

$$D_x := (P - \{x\}, \leq_{P-\{x\}}).$$

One-point-deleted subposet

Example

Consider the chain

$$1 < 2 < 3 < \cdots < 10.$$

One-point-deleted subposet

Example

Consider the chain

$$1 < 2 < 3 < \cdots < 10.$$

Then

$$D_3 = \{1 < 2 < 4 < \cdots < 10\}.$$

One-point-deleted subposet

Pop-quiz 5 (3 min)

Construct all the one-point-deleted subposets of the poset N_5:

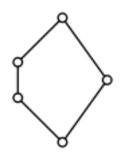

Figure: The famous non-distributive lattice N_5

Note that we do keep record of repetitions.

Deck

Definition (Deck)

The *deck* of P is the collection (i.e., multi-set)

$$\{P - \{x\} : x \in P\}$$

of its (unlabelled) one-element-deleted subposets (called *cards* of P).

Group activity

Activity (5 min)

For the given cards of a mystery poset P, make a reconstruction of the original poset P.

Be aware the orientation of the card w.r.t. to the card number printed at the bottom-right-corner.

Poset reconstruction

Definition (Reconstruction)

A poset Q is a reconstruction of P if there is a bijection $\sigma : P \to Q$ such that for every $x \in P$, it holds that

$$P - \{x\} \cong Q - \{\sigma(x)\}.$$

We say that P and Q have the same deck.

Reconstructible

Definition (Reconstructible)

A poset P is said to be *reconstructible* if every reconstruction of P is isomorphic to P.

Poset reconstruction problem

Figure: Stanislaw Ulam (1909-1984)

Conjecture [P.J. Kelly and S. Ulam, 1960]

Every finite poset P of more than three elements is reconstructible.

Poset reconstruction problem

Activity (2 min)

Is the condition of having more than three elements necessary?

Conclusion

In this talk, I have spoke about

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1. Partial orders and Lattices, and the

Conclusion

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1. Partial orders and Lattices, and the
2. Poset reconstruction problem

References

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Acknowledgment

Thank you!