Does that ring a bell?

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Outline

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1 Introduction

2 Bell curve

3 Double integral

Outline

10 Mind Blowing Mathematical Equations

Among the top 10 "mind blowing mathematical equations", the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

ranks No. 3.

Today's journey begins with the idea of

Taking Chances

Taking chances

Suppose you wake up one morning and realized you need to sit for a Multiple Choice Questions (MCQ) test for a nasty subject you have forgotten to prepare.

Taking chances

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Question

What will you do?

Taking chances

Solution

You decide to take chances and see if

Taking chances

Solution

You decide to take chances and see if

smiles on you.

Taking chances

You begin to do your sums ...

Taking chances

Each MCQ offers 5 options, of which only one is the right answer.

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Analysis

- Each guess has 2 outcomes: right or wrong.

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What are your odds?

Taking chances

Suppose there are a total of 10 questions, and I am lucky enough to get 7 right.

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Analysis

A typical score-sheet looks like this:

Question	1	2	3	4	5	6	7	8	9	10
Outcome	R	W	R	R	R	R	W	W	R	R

Taking chances

Analysis

For each typical score-sheet such as:

Question	1	2	3	4	5	6	7	8	9	10
Outcome	R	W	R	R	R	R	W	W	R	R

Analysis

For each typical score-sheet such as:

Question	1	2	3	4	5	6	7	8	9	10
Outcome	R	W	R	R	R	R	W	W	R	R

the probability of getting this is

$$\left(\frac{1}{5}\right)^7 \cdot \left(\frac{4}{5}\right)^3 .$$

Taking chances

But there are many ways by which one can get 7 right and 3 wrong.

Taking chances

But there are many ways by which one can get 7 right and 3 wrong.

Analysis

Marking out all the R's and W's as distinct, we can arrange

$$R_1, R_2, R_3, \ldots, R_7, W_1, W_2, W_3$$

in a linear fashion via a total of

$$10 \times 9 \times 8 \times \cdots \times 1 = 10! \text{ ways.}$$

Taking chances

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Taking chances

But there are many ways by which one can get 7 right and 3 wrong.

Analysis

Blurring out the distinctions first for the R's, i.e.,

$$R, R, R, \ldots, R, W_1, W_2, W_3$$

collapses the numbers to

$$\frac{10!}{7!}.$$

Taking chances

Analysis

Blurring out the distinctions next for W's, i.e.,

$$R, R, R, \ldots, R, W, W, W$$

further collapses the total number of arrangements to

$$\frac{10!}{7!3!} := \binom{10}{7}.$$

Taking chances

Analysis

Since each score sheet is equally likely to happen, we have a chance of

$$\binom{10}{7}\left(\frac{1}{5}\right)^7\left(\frac{4}{5}\right)^3$$

getting 7 questions correct out of 10 by mere random guess.

Taking chances

Since

$$\binom{10}{7}\left(\frac{1}{5}\right)^7\left(\frac{4}{5}\right)^3 \approx 0.000786432 > 0,$$

Taking chances

Since

$$\binom{10}{7}\left(\frac{1}{5}\right)^7\left(\frac{4}{5}\right)^3 \approx 0.000786432 > 0,$$

it is better than not going and getting ...

for sure.

Binomial event

Suppose an event satisfies the following conditions:

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We are interested in the number of successful trials out of n.

Binomial event

Theorem

Let X be the number of successful trials out of n. Then, the probability that $X = r$ is given by

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r},$$

where $r = 0, 1, \ldots, n$.

Binomial event

Definition (Binomial distribution)

Let X be a random variable whose probability distribution function is given by

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}, \qquad r = 0, \ldots, n,$$

where n is a positive integer and p a positive constant.

Binomial event

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where n is a positive integer and p a positive constant. We say that X has a *binomial distribution*.

Empirical vs Theoretical

For $n = 10$ and $p = 0.2$, we can chart out the theoretical relative frequency, $P(X = r)$, as follows:

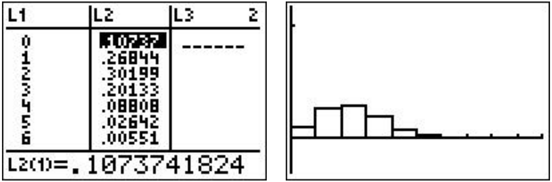

Figure: Probability distribution

Empirical vs Theoretical

We can use the following applet:

`http://members.shaw.ca/ron.blond/TLE/Bin.APPLET/EX/index.html`

to get a feeling about the differences between

- empirical, and
- theoretical frequencies.

Emergence of the bell

What we gather ...

When n gets very large, the frequency chart takes on a bell shape.

Sample total

- Suppose X is a random variable with mean μ and variance σ^2.

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Sample total

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- We now take a sample of size n of occurrences of X's, i.e., the *independent and identically distributed* random variables
$$X_1, X_2, \ldots, X_n.$$
Then we compute the sample total
$$X_1 + X_2 + \cdots + X_n.$$

Sample total

Problem

What is the probability distribution function of the random variable

$$S := X_1 + \cdots + X_n?$$

Sample total

Again relying on simulations, we make a preliminary investigation:

http://www.stat.sc.edu/~west/javahtml/CLT.html

Sample mean

For large sample size n, the distribution can again be estimated by a bell curve.

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Consequently,

Theorem (Central Limit Theorem)

When n is large, the random variable

$$\overline{X} := \frac{X_1 + \cdots + X_n}{n}$$

known as the sample mean also has a distribution that is roughly a bell curve, irrespective of the original distribution that X's have.

Examination marks and grades

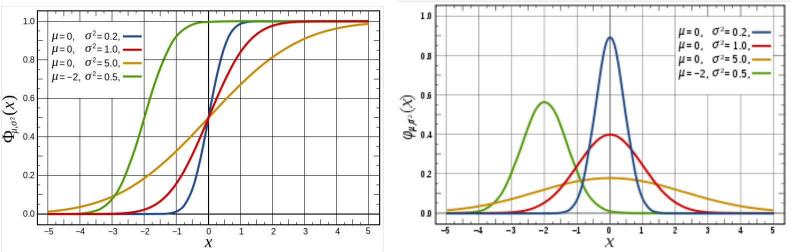

Figure: Grading process

Bell curve

Attention

There is something significant about the bell curve in probability and statistics!

Many candidates but one choice

There are many different choices of functions that can take the shape of a bell curve.

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2. $y = \dfrac{1}{e^{x^2}}$

Many candidates but one choice

There are many different choices of functions that can take the shape of a bell curve.

1. $y = \dfrac{1}{1+x^2}$
2. $y = \dfrac{1}{e^{x^2}}$
3. $y = \dfrac{1}{g(x)}$, where $g(x)$ is a positively-valued even function that has a U-shape.

Many candidates but one choice

It turns out that the most suitable candidate is one of the form

$$y = e^{-x^2},$$

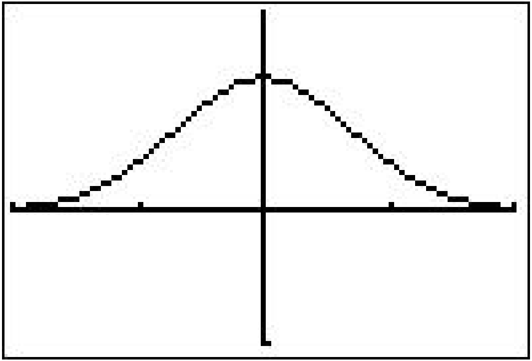
which is the Gaussian bell curve.

Bell curve

Consider the equation

$$f(x) = k \cdot e^{-x^2}$$

for $x \in \mathbb{R}$, whose curve has the following shape:



Bell curve

Probability distribution (a.k.a. relative frequency) bears upon us that

$$P(-\infty < X < \infty) = 1$$

i.e., the total area under the graph of the distribution function f must be unity.

Bell curve

So, to determine the value of k for which

$$\int_{-\infty}^{\infty} k \cdot e^{-x^2}\, dx = 1$$

Bell curve

So, to determine the value of k for which

$$\int_{-\infty}^{\infty} k \cdot e^{-x^2}\ dx = 1$$

one must be able to calculate the exact value of the Gaussian integral:

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx.$$

Bell curve

We now set our mind to solve the following problem:

Problem

Find the exact value of the improper integral

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx.$$

A trick

Instead of obtaining an anti-derivative for the function

$$g(t) = e^{-t^2},$$

we approach this area-problem by considering a seemingly unrelated volume-problem.

Volume of revolution

Consider the volume generated by rotating the region R, bounded by the bell curve $y = e^{-x^2}$ and the coordinate-axes, through 4 right angles about the vertical axis.

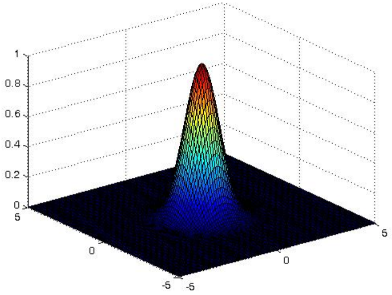


Figure: 3D-sketch of the bell

Volume of revolution

We can find this volume in two different ways.

Elemental method

Method 1

We consider making vertical slices, cut parallel to the $y - z$ plane $x = 0$.

Elemental method

Method 2

We consider making horizontal slices, cut parallel to the $x - y$ plane $z = 0$.

Elemental method and integration

One central ideal behind the integration theory in the sense of Newton and Leibniz is that of

accumulation of the infinitesimal elements .

Volume of revolution

Because of rotational symmetry, the equation of the surface of the bell is given by

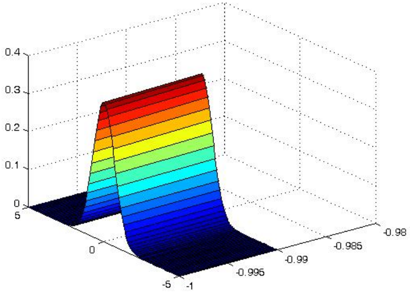
$$z = e^{-(x^2+y^2)},$$

as a result of the Pythagoras theorem.

Volume of revolution

According to Method 1, when we make a vertical cut at x and another one at $x + \delta x$, we produce an elemental slice (of bread) whose volume is approximately

$$\left(\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dy\right) \cdot \delta x.$$



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To find the desired volume, we sum up the volume of all the slices via integration, i.e.,

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$$\begin{aligned} V &= \lim_{\delta x \to 0} \sum_{x \in \mathbb{R}} \left(\int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dy \right) \delta x \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dy \right) dx \end{aligned}.$$

Volume of revolution

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$$V = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\ dy \right)\ dx$$
$$= \int_{-\infty}^{\infty} e^{-x^2} \left(\int_{-\infty}^{\infty} e^{-y^2}\ dy \right)\ dx$$

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\begin{aligned}
V &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-(x^2+y^2)} \, dy \right) \, dx \\
&= \int_{-\infty}^{\infty} e^{-x^2} \left(\int_{-\infty}^{\infty} e^{-y^2} \, dy \right) \, dx \\
&= \int_{-\infty}^{\infty} e^{-x^2} \, dx \cdot \int_{-\infty}^{\infty} e^{-y^2} \, dy
\end{aligned}
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Volume of revolution

On the other hand, Method 2 cuts out horizontal circular discs of thickness δz and of cross-sectional area $e^{-(x^2+y^2)}$.

Volume of revolution

Let $r^2 = x^2 + y^2$, and so

$$z = e^{-(x^2+y^2)} = e^{-r^2}.$$

Volume of revolution

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$$z = e^{-(x^2+y^2)} = e^{-r^2}.$$

Thus,

$$\frac{dz}{dr} = -2re^{-r^2}.$$

Volume of revolution

To find the desired volume, sum up the volumes of all the discs, i.e.,

$$V = \lim_{\delta z \to 0} \sum_{z \in (0,1]} \pi r^2 \delta z$$

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At this juncture, we must resort to the *method of substitution*.

Volume of revolution

When $z \to 0$, since $r > 0$, it follows that

$$z = e^{-r^2} \to 0 \iff r \to +\infty.$$

Volume of revolution

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When $z = 1$,

$$z = e^{-r^2} = 1 \iff r = 0.$$

Volume of revolution

Gathering all the information, we make the substitution below:

$$V = \pi \int_0^1 r^2 \, dz$$

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\end{aligned}
$$

Volume of revolution

Gathering all the information, we make the substitution below:

$$\begin{aligned} V &= \pi \int_0^1 r^2 \, dz \\ &= \pi \int_\infty^0 r^2 \cdot \left(-2re^{-r^2}\right) \, dr \\ &= -\pi \int_0^\infty r^2 \cdot \left(-2re^{-r^2}\right) \, dr \end{aligned}$$

Volume of revolution

Now we employ integration by parts. Let

$$u = r^2 \text{ and } \frac{dv}{dr} = -2re^{-r^2}.$$

Volume of revolution

Now we employ integration by parts. Let

$$u = r^2 \text{ and } \frac{dv}{dr} = -2re^{-r^2}.$$

Thus,

$$\frac{du}{dr} = 2r \text{ and } v = e^{-2r}.$$

Volume of revolution

It follows that

$$V = -\pi \int_0^\infty r^2 \cdot (-2re^{-r^2})\ dr$$

Volume of revolution

It follows that

$$\begin{aligned} V &= -\pi \int_0^\infty r^2 \cdot \left(-2re^{-r^2}\right)\, dr \\ &= -\pi \left\{ \left[r^2 e^{-r^2} \right]_0^\infty - \int_0^\infty 2re^{-r^2} \right\} \end{aligned}$$

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Volume of revolution

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&= 2\pi \int_0^\infty re^{-r^2}\, dr \\
&= \pi \left[-e^{-r^2} \right]_0^\infty \\
&= \pi \cdot 1 \\
&= \pi
\end{aligned}$$

Volume of revolution

Exercise

Find the volume of revolution, V, using the formula

$$\int_0^1 \pi x^2 \; dy,$$

where $y = e^{-x^2}$.

Volume of revolution

Since the two methods both yield the volume of revolution, it follows that

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\ dx\right)^2 = \pi.$$

Volume of revolution

Since the two methods both yield the volume of revolution, it follows that

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\ dx\right)^2 = \pi.$$

So, we finally have

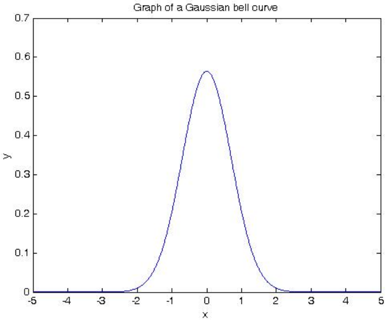
Theorem (Gaussian integral)

$$\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}.$$

Gaussian bell curve

All the integration techniques result in the following explicit equation:

$$\phi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}.$$



Normal distribution

Of course, taking care to make sure that the variance of the random variable is unity would result in a scaling of $z = \sqrt{2}x$:

Definition (Standard normal distribution)

A continuous random variable Z whose probability distribution function is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

is said to be *normally distributed* with mean 0 and variance 1^2, i.e., denoted by

$$Z \sim N(0, 1^2).$$

The mathematician himself

The normal distribution is often credited to

Figure: C. F. Gauss

and the version he considered is the one we derived, i.e.,

$$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}.$$

Applications of normal distribution

- Sampling theory

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- Hypothesis testing

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- Regression analysis
- Thermodynamics: velocity of particles
- Diffusion theory
- Quantum oscillator

References

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