From Mathematical Misconception to Mathematical Conception

JCMTC Term 2 Sharing: Misconceptions in Pure Mathematics

Dr. Ho Weng Kin
MME, NIE
wengkin.ho@nie.edu.sg

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Outline

1. Introduction

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2. Misconceptions

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Outline

1 Introduction

2 Misconceptions

3 Research-guided strategies

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A shift of paradigm

Figure : A spooky train ride or a scenic train ride, it is up to us ...

A brief tour

Example

$$\sin^{-1}(0) = \frac{1}{\sin(0)}$$

Then you get:

A brief tour

Example

The sum of the first n terms of a geometric progression of which the first term is a and common ratio is 2 is denoted by S_n.
Find the least integer value of n such that

$$S_n > 1000a.$$

A brief tour

Example

A student's working is shown below:

$$
\begin{aligned}
S_n - 1000a &> 0 \\
\frac{a(2^n - 1)}{2 - 1} - 1000a &> 0 \\
2^n &> 1001 \\
n &< \frac{\ln(1001)}{\ln(2)}.
\end{aligned}
$$

A brief tour

Example

A student's working is shown below:

$$S_n - 1000a > 0$$

$$\frac{a(2^n - 1)}{2 - 1} - 1000a > 0$$

$$2^n > 1001$$

$$n < \frac{\ln(1001)}{\ln(2)}.$$

Reasoning

He recalls that his teacher cautioned the class for the umpteenth time to change the direction of the inequality sign at the last step.

A brief tour

Example

Inspired by a teacher's working

$$\sum_{r=0}^{n} r = \sum_{r=1}^{n} r = \frac{n}{2}(n+1),$$

a student made a simplification as follows:

$$\sum_{r=0}^{n} 2^{-r} = \sum_{r=1}^{n} 2^{-r} = \frac{2^{-1}(1-2^{-n})}{1-2^{-1}}.$$

A brief tour

Example

The function f is given by

$$\mathrm{f}(x) = x^2 + x - 1, \quad x \leq -1.$$

Find f^{-1}.

A brief tour

Example

$$y = x^2 + x - 1$$

$$x^2 + x - (y+1) = 0$$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-y-1)}}{2}$$

$$x = \frac{-1 \pm \sqrt{4y+5}}{2}$$

A brief tour

Example

Here are common errors made by JC students:

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- $f^{-1}(x) = \dfrac{-1+\sqrt{4x+5}}{2}$, $x \geq -\frac{5}{4}$
- $f^{-1}(x) = \dfrac{-1\pm\sqrt{4x+5}}{2}$, $x \geq -\frac{5}{4}$.

A brief tour

Example

Some students believe in the following integration technique:

$$\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \sin(x)\,\mathrm{d}x \cdot \int_0^{\frac{\pi}{2}} \cos(x)\,\mathrm{d}x.$$

A brief tour

Example

A line ℓ has vector equation given by

$$\ell : \ \mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

It is quite common for students to think that

$$\mathbf{r} \text{ is parallel to } \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}.$$

A brief tour

Example

A student does not know the rôle of the constant d in an equation of the plane

$$\Pi : \ \mathbf{r} \cdot \mathbf{n} = d$$

although he or she knows the meaning of the vector $\mathbf{n}$.

A brief tour

Example

The volume of the solid obtained by revolving a region R bounded between two curves $y = f_1(x)$ and $y = f_2(x)$ and the two vertical lines $x = a$ and $x = b$ is often calculated wrongly by

$$\pi \int_a^b (f_1(x) - f_2(x))^2 \, \mathrm{d}x.$$

A brief tour

Example

$$|3+2i| = \sqrt{3^2 + (2i)^2} = \sqrt{9-4} = \sqrt{5}.$$

A brief tour

Example

$$|1+\cos\theta + i\sin\theta| = |1+\cos\theta| + |\sin\theta|.$$

A brief tour

Example

To obtain the roots of the equation

$$z^7 = \frac{1}{2} - \frac{\sqrt{3}}{2} i,$$

it is not uncommon to see the following students' working:

$$z^7 = e^{-\frac{\pi}{3} i}.$$

$$z = e^{\left(2k\pi - \frac{\pi}{21}\right) i},$$

where $k = 0, \pm 1, \pm 2, \pm 3$.

A brief tour

Example

One feels shaky about the validity ranges of x associated to the power series expansions of

- $(1+x)^{-1} = 1 - x + x^2 - \ldots + (-1)^k x^k + \ldots, \ \ -1 < x < 1.$

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Example

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- $(1+x)^{-1} = 1 - x + x^2 - \ldots + (-1)^k x^k + \ldots, \quad -1 < x < 1.$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^k x^{k+1}}{k+1} + \ldots, \quad -1 < x \leq 1.$

A brief tour

Example

A sequence a_n $(n = 1, 2, \ldots)$ can be defined by a recurrence relation, say:

$$a_{n+1} = a_n^2 - 1, \; a_1 = a$$

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A common superstition is to believe that

$$\lim_{n\to\infty} a_n$$

always exists and its 'limit' given by

$$\frac{1 \pm \sqrt{5}}{2}.$$

A brief tour

Example

One obtains ‘the’ binomial series expansion of the following expression

$$\frac{3}{1-x-2x^2}$$

via partial fractions

$$\frac{2}{1-2x}+\frac{1}{1+x}=2(1-2x)^{-1}+(1+x)^{-1}$$

or

$$\frac{3}{1-(x+2x^2)}=3(1-(x+2x^2))^{-1},$$

and does not think there is a difference!

A brief tour

Example

The 'formula'

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

is taken literally so that we can calculate:

$$\int_{-1}^{2} \frac{1}{x}\,\mathrm{d}x = [\ln|x|]_{-1}^{2} = \ln(2) - \ln(1) = \ln(2).$$

A summary of Smith's findings

In the rest of my talk, I bring to your attention some of the recent findings reported by Smith, diSessa and Roschelle in [1].

A summary of Smith's findings

My talk which is based on their findings uses the knowledge domain of H2 Mathematics as the platform of illustration.

Central assertions

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- Students have misconceptions.
- Misconceptions originate in prior learning.
- Misconceptions can be stable and widespread among students. They are strongly held and resistant to change.

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- Misconceptions interfere with learning.
- Misconceptions must be replaced.
- Instruction should confront misconceptions.
- Research should identify misconceptions.

Assertion 1

Students have misconceptions.

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- Novice concepts are *different* from the currently accepted disciplinary concepts presented in instruction.

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Students' conceptions are known under the following guises:

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- näive theories

Assertion 1

Example

The sigma notation $\sum_{r=1}^{n} u_r$ has a schizophrenic nature, swapping between

$$u_1 + u_2 + \ldots + u_n$$

and

$$u_1, u_2, \ldots, u_n.$$

Assertion 2

Misconceptions originate in prior learning.

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2. from their interaction with the physical or social world.

Assertion 2

Example

A student applies the 'dot product' to find the length of projection of a line segment AB onto a plane Π, e.g.,

$$p = \left|\vec{AB} \cdot \mathbf{n}\right|,$$

where $\mathbf{n}$ is a unit vector normal to the plane.
The student defends this answer by identifying projection as an application of the dot product.

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Misconceptions are strongly held and resistant to change.

- Misconceptions are not momentary.
- Misconceptions appear before and after instruction.
- Misconceptions occur in students and adults alike.
- Misconceptions still appear after the correct approach is taught.
- Misconceptions co-exist alongside the correct approach.

Assertion 3

"The initial didactical models [the misconceptions from previous instruction] seem to become so deeply rooted in the learner's mind that they continue to exert an unconscious control over mental behavior even after the learner has acquired formal mathematical notions that are solid and correct." – p. 16 of Fischbein et al. (1985)

Assertion 3

Example

The convergence or divergence of a sequence can be determined by inspecting only a finite number N of initial terms

$$u_1, u_2, \ldots, u_N.$$

Assertion 3

This misconception arises from the students' encounter with certain questions phrased as follows:

[N2007/I/9]

A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3}e^{x_n}$$

for $n \geq 1$.

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for $n \geq 1$.

(ii) Prove algebraically that, if the sequence converges, then it converges to either α or β.

(iii) Use a calculator to determine the behaviour of the sequence for each of the cases $x_1 = 0$, $x_1 = 1$, $x_1 = 2$.

Assertion 4

Misconceptions interfere with learning.

Due to their persistence and flawed nature, misconceptions interfere with learning expert concepts.

Assertion 4

Example (Gambler's fallacy)

It is common to believe that if an experiment is repeated and one outcome occurs several times in a row, then it is less probable to appear upon the next completion of that same experiment.

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Heads and tails

If I flip a fair coin ten times in a row and each time I obtain "tails", then a commonly held intuition is that the eleventh coin toss is more likely to come up with "heads".

Assertion 4

Example (Law of small numbers)

It refers to the tendency to assign a greater probability of occurrence to finite sets of independent outcomes whose relative frequencies more closely resemble the respective probabilities of each outcome.

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Red and green

If we roll a fair six-sided die with two faces painted green (G) and four faces painted red (R), then many consider obtaining the sequence "RGRGGG" to be more likely than obtaining the sequence "GRGGG".

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- More adequate expert knowledge must be developed, and these *replace* existing misconceptions.
- Learning involves both the acquisition of expert concepts and the dispelling of misconceptions.
- Removing misconceptions has no negative consequences because they play no productive rôle in expertise.

Assertion 5

Example

A teacher, upon seeing a student's mistake below,

$$\int_{-1}^{2} |x(1-x)|\ \mathrm{d}x = \left| \int_{-1}^{2} x(1-x)\ \mathrm{d}x \right| = \ldots$$

puts a big cross and asks the student to do corrections.

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- To neutralize the interference of misconceptions, instruction must *confront* students with the disparity between their misconceptions and expert concepts.
- Confrontation begins with external, social interactions, and then the competition between misconceptions and expert concepts must be internalized by students.
- Misconceptions are hindrances and must be *overcome* by confrontation and replaced by expert conceptions.

Assertion 6

Example

A student solves the following inequality

$$|x+1| < 4$$

by writing

$$x+1 < \pm 4.$$

Assertion 6

Example

The teacher helps the student confront this misconception by looking at the graphical solution:

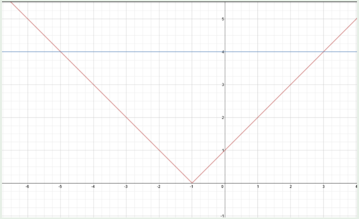

Figure : Graphs of $y = |x + 1|$ and $y = 4$

The solution $-5 < x < 3$ is not the same as $x + 1 < \pm 4$.

Assertion 7

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- A major task for research in mathematics and science learning is to *document* misconceptions in as many subject-matter domains as possible.
- Much less emphasis was given to model the learning of successful students in those domains, to characterize how misconceptions (and the cognitive structures that embed them) evolve, or to describing the nature of instruction that successfully promotes such learning.

Assertion 7

Example

What has earlier been presented is clearly part of the effort of classification of errors and misconceptions in the domain of H2 Mathematics.

Constructivism

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Constructivism characterizes the process of learning as the gradual re-crafting of existing knowledge that, despite many intermediate difficulties, is eventually successful.

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One can infer from the prevalent misconception research that

- novice misconceptions and expert conceptions are very different
- novice misconceptions and expert conceptions are unitary, independent and separable entities.

Constructivism

Replacement model seems to be an over-simplified explanation to justify the removal of misconceptions.

Constructivism

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- In focusing only on how student ideas conflict with expert concepts, the misconception perspective offers no account of productive ideas that might serve as resources for learning.

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- In focusing only on how student ideas conflict with expert concepts, the misconception perspective offers no account of productive ideas that might serve as resources for learning.
- Since they are fundamentally flawed, misconceptions themselves must be replaced.
- What additional relevant ideas students might have available become a mystery.

Constructivism

Let's return to the example of the erroneous way of calculating the modulus of a complex number:

Example

$$|3+2i| = \sqrt{3^2+(2i)^2} = \sqrt{9-4} = \sqrt{5}.$$

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As teachers, we may think that we can confront the misconception in our instruction.

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Consider the following confrontation.

Confronting a misconception

T: Seems like a nice way to find $|3+2i|$.

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T: Seems like a nice way to find $|3+2i|$.
S: Yup, during the lecture that's the way to do it.
T: Is that right?
S: By Pythagoras Theorem, you know: a^2+b^2, then take square root.

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T: ...

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- In order to replace the novice concept with the expert concept, there must be a set of judging criteria to be constructed and used by the learner to decide that one concept is more favourable over the other.

Constructivism

As shown in the above confrontation, for the cognitive competition (of novice versus expert conceptions), there is no reason why the expert conception must win over misconceptions.

Pause and think

- In order to replace the novice concept with the expert concept, there must be a set of judging criteria to be constructed and used by the learner to decide that one concept is more favourable over the other.
- In this scenario, it clearly shows that the student has no such set of criteria.

Constructivism

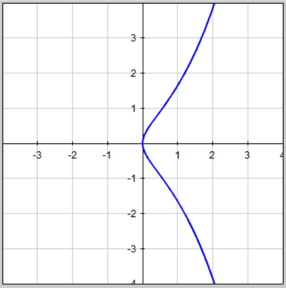
Good news!!

In fact, the perceived gap between how novice and expert handles a problem, and the way they invoke their knowledge resources is NOT that big.

Constructivism

Example (A small experiment)

The following is a sketch of the graph whose equation is $f(y) = g(x)$ for some functions f and g:



[Assume that the domains of f and g are $\mathbb{R}$.]

Constructivism

Example (A small experiment)

Based on the given sketch, sketch the graph whose equation is

$$f(y-1) = g(x+1).$$

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Review

Share among yourselves how you make use of your mathematical resource to handle the new situation.

Constructivism

Experiment

In your classroom, you may wish to find out how your students go about working out the problem (without your intervention).
Document the manner of reasoning and the language they make use of.
To what extent do their (novice) conceptions differ from your (expert) conceptions?

Three aspects

Three aspects

- Knowledge in pieces

Three aspects

- Knowledge in pieces
- Continuity

Three aspects

- Knowledge in pieces
- Continuity
- Functionality

Knowledge in pieces

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Knowledge in pieces

- Mathematical knowledge is distributed across a far greater number of interrelated general and context-specific components than those presented in current analyses of expertise or textbooks.
- We need a more fine-grained model for mathematical knowledge acquisition.

Continuity

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Continuity

- The more advanced states of knowledge is continuous with respect to prior states.
- Students' ideas (misconceptions) can be productive in some contexts if there are more opportunities for them to play direct roles in expert reasoning.

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- Functionality is about evaluating the degree of success of conceptions and misconceptions.

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productive/unproductive.

Here, productive refers to generative of new knowledge.

Refinement instead of replacement

Because of continuity and functionality, we begin to think of refinement of concepts rather than replacing wrong concepts by right ones.

Discussion instead of confrontation

Instead of opposing those ideas to be relevant expert view, instruction should help students reflect on their present commitments, i.e., finding new productive contexts for specific mathematical purposes.

Role of misconceptions

Misconceptions may be faulty in many contexts, but they have a role to play.

Role of misconceptions

Example (Notion of commutativity)

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Many times we enjoy them:

1. $x + y = y + x$
2. $|zw| = |z||w|$

But there are times we hope they are true but they aren't:

1. $fg = gf$
2. $\lim_{n\to\infty} \int_0^1 f_n(x)\ \mathrm{d}x = \int_0^1 \lim_{n\to\infty} f_n(x)\ \mathrm{d}x$

Role of misconceptions

Example (Notion of commutativity)

The notion of swapping order of doing things is prevalent in mathematics.

Many times we enjoy them:

1. $x + y = y + x$
2. $|zw| = |z||w|$

But there are times we hope they are true but they aren't:

1. $fg = gf$
2. $\lim_{n\to\infty} \int_0^1 f_n(x)\, \mathrm{d}x = \int_0^1 \lim_{n\to\infty} f_n(x)\, \mathrm{d}x$
3. $\int_a^b |f(x)|\, \mathrm{d}x = \left|\int_a^b f(x)\, \mathrm{d}x\right|$

What are misconceptions then?

Definition (Misconceptions)

Misconceptions are simply faulty extensions of productive prior knowledge.

What are misconceptions then?

Example (Closure)

Binary operations describe the phenomenon of closure, i.e.,

$$\otimes : S \times S \longrightarrow S, \ (x, y) \mapsto x \otimes y$$

What are misconceptions then?

Example (Closure)

Binary operations describe the phenomenon of closure, i.e.,

$$\otimes : S \times S \longrightarrow S, \ (x, y) \mapsto x \otimes y$$

Because of the great number of situations for which this holds, it is easy to have misconceptions such as the following:

Myth

For any $a, \ b \notin \mathbb{Q}$, it always holds that $ab \notin \mathbb{Q}$.

Misconceptions are not always resistant to change

Teachers can design interventions appropriately that can result in rapid and deep conceptual change in relatively short periods of time.

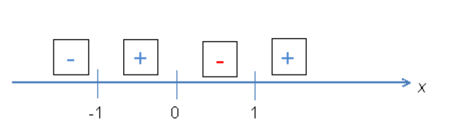
Misconceptions are not always resistant to change

Example

When students solve inequalities using number line, it is quite common for them to assume "alternating signs" prevail over different critical regions. For instance, when solving the inequality

$$(x+1)x(x-1) > 0$$

it is easy to believe that testing the sign of one interval would suffice since the rest 'alternate'.



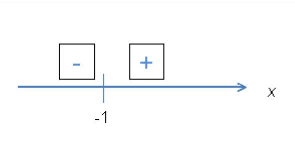
Misconceptions are not always resistant to change

Example

Invite the students to focus on the case of a single critical point, e.g., looking at a function such as

$$f(x) = x + 1$$

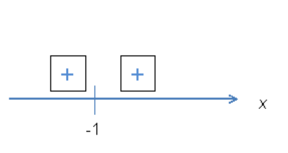
whose critical point is $x = -1$. Clearly, the signs about this critical point "alternate", i.e., changes about $x = -1$:



Misconceptions are not always resistant to change

Example

Suggest a new function g whose critical point is $x = -1$, and the signs about it are given by



The way ahead

It is time to move beyond the identification and classification of misconceptions.

The way ahead

We now need research that

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We now need research that

- focuses on the evolution of expert understandings in specific conceptual domains

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- builds on and explains the existing empirical record of students' conceptions

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- focuses on the evolution of expert understandings in specific conceptual domains
- builds on and explains the existing empirical record of students' conceptions
- provides detailed descriptions of the evolution of knowledge systems over longer durations
- includes rich case studies and a sound theoretical framework of models.

The way ahead

We now need implementation that

The way ahead

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- is research-guided

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- harnesses the productive aspect of misconceptions

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- encourages knowledge generation and problem solving

The way ahead

We now need implementation that

- is research-guided
- harnesses the productive aspect of misconceptions
- encourages knowledge generation and problem solving
- refines rather than replace novice conceptions.

References I

SMITH, J. P., D. A. A. R. J.
Misconceptions reconceived: A constructivist analysis of knowledge in transition.
The Journal of the Learning Sciences, Vol. 3, No. 2 (1993), 115–163.