Tips on Crafting Examination & Test Questions for H2 Math

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Outline

1. Introduction

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2. Given parameters and constraints

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2. Given parameters and constraints
3. Item construction techniques

Outline

The Singapore-Cambridge effect

We begin by making an observation, which by now is folkloric:

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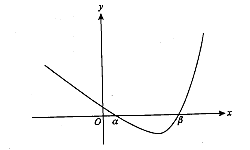
The Singapore-Cambridge Effect

Every instance of a question type in the Cambridge examination yields the reproduction of many questions of the same type in the subsequent year(s) for all the junior colleges' preliminary examination.

The Singapore-Cambridge effect

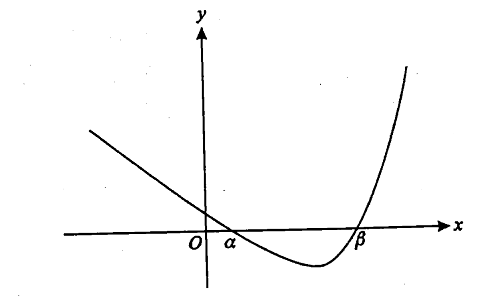
N2007/I/9



The diagram shows the graph of $y = e^x - 3x$. The two roots of the equation $e^x - 3x = 0$ are denoted by α and β, where $\alpha < \beta$.

The Singapore-Cambridge effect

N2007/I/9



(i) Find the values of α and β, each correct to 3 decimal places.

The Singapore-Cambridge effect

N2007/I/9

A sequence of real numbers $x_1, x_2, x_3, \cdots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3}e^{x_n}$$

for $n \geq 1$.

(ii) Prove algebraically that, if the sequence converges, then it converges to either α or β.

(iii) Use a calculator to determine the behaviour of the sequence for each of the cases $x_1 = 0$, $x_1 = 1$, $x_1 = 2$.

The Singapore-Cambridge effect

N2007/I/9

(iv) By considering $x_{n+1} - x_n$, prove that

$$\begin{cases} x_{n+1} < x_n & \text{if } \alpha < x_n < \beta, \\ x_{n+1} > x_n & \text{if } x_n < \alpha \text{ or } x_n > \beta. \end{cases}$$

The Singapore-Cambridge effect

N2009/I/4

It is given that

$$f(x) = \begin{cases} 7 - x^2 & \text{for } 0 < x \leq 2, \\ 2x - 1 & \text{for } 2 < x < 4, \end{cases}$$

and that $f(x) = f(x+4)$ for all real values of x.

(i) Evaluate $f(27) + f(45)$.

(ii) Sketch the graph of $y = f(x)$ for $-7 \leq x \leq 10$.

(iii) Find $\int_{-4}^{3} f(x)\, dx$.

The Singapore-Cambridge effect

N2011/I/6

(i) Using the formula for $\sin(A \pm B)$, prove that

$$\sin(r+\frac{1}{2})\theta - \sin(r-\frac{1}{2})\theta \equiv 2\cos r\theta \sin\frac{1}{2}\theta.$$

(ii) Hence find a formula for $\sum_{r=1}^{n} \cos r\theta$ in terms of $\sin(n+\frac{1}{2})\theta$ and $\sin\frac{1}{2}\theta$.

The Singapore-Cambridge effect

N2011/I/6

(iii) Prove by the method of mathematical induction that

$$\sum_{r=1}^{n} \sin r\theta = \frac{\cos \frac{1}{2}\theta - \cos\left(n + \frac{1}{2}\right)\theta}{2\sin \frac{1}{2}\theta}$$

for all positive integers n.

Points to take note

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- Rigour
- Correctness
- Elegance
- Style
- Background knowledge

Rigour

N2010/II/3(ii)

(ii) A curve has equation $y^2 = x^2(x+2)$.

(a) Find exactly the possible values of the gradient at the point where $x = 0$.

(b) Sketch the curve $y^2 = x^2(x+2)$.

Correctness

Specimen H3 paper/Q3(i)

A function f is defined by $f(x) = \frac{4}{2-x}$ for $x \neq 2$.

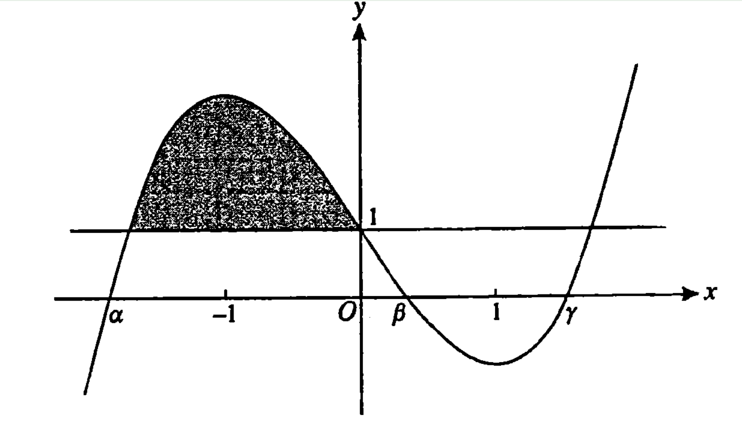
(a) Find $f^3(x)$.

(b) Find all possible integers n and real numbers x such that

$$f^n(x) = 4x.$$

Elegance

N2010/I/6



Elegance

N2010/I/6

The diagram shows the curve with equation $y = x^3 - 3x + 1$ and the line with equation $y = 1$. The curve crosses the x-axis at $x = \alpha$, $x = \beta$ and $x = \gamma$ and has turning points at $x = -1$ and $x = 1$.

(i) Find the values of β and γ, giving your answers correct to 3 decimal places.

(ii) Find the area of the region bounded by the curve and the x-axis between $x = \beta$ and $x = \gamma$.

(iii) Use a non-calculator method to find the area of the shaded region between the curve and the line.

(iv) Find the set of values of k for which the equation $x^3 - 3x + 1 = k$ has three real distinct roots.

Style

N2011/II/7

When I try to contact (by telephone) any of my friends in the evening, I know that on average the probability that I succeed is 0.7. On one evening I attempt to contact a fixed number, n, of different friends. If I do not succeed with a particular friend, I do not attempt to contact that friend again that evening. The number of friends whom I succeed in contacting is the random variable R.

Style

N2011/II/7

(i) State, in the context of this question, two assumptions needed to model R by a binomial distribution.

(ii) Explain why one of the assumptions stated in part **(i)** may not hold in this context. Assume now that these assumptions do in fact hold.

(iii) Given that $n = 8$, find the probability that R is at least 6.

(iv) Given that $n = 40$, use an appropriate approximation to find $P(R < 25)$. State the parameters of the distribution you use.

Background knowledge

N2011/II/8(ii)

A stone is dropped from a stationary balloon. It leaves the balloon with zero speed, and t seconds later its speed v metres per second satisfied the differential equation

$$\frac{dv}{dt} = 10 - 0.1v^2.$$

(a) Find t in terms of v. Hence find the exact time the stone takes to reach a speed of 5 metres per second.

(b) Find the speed of the stone after 1 second.

(c) What happens to the speed of the stone for large values of t?

Assessment Objectives (AO)

There are three level of assessment objectives for the examination.

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The assessment will test candidates' abilities to:

AO1 understand and apply mathematical concepts and skills in a variety of contexts, including the manipulation of mathematical expressions and use of graphic calculators;

AO2 reason and communicate mathematically through writing mathematical explanation, arguments and proofs, and inferences;

AO3 solve unfamiliar problems; translate common realistic contexts into mathematics; interpret and evaluate mathematical results, and use the results to make predictions, or comment on the context.

Main focus

Our main focus for today's talk will be centred around [AO3], with passing remarks on the other objectives made as an when appropriate.

Context sourcing

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Principle

Keep looking around you, and be imaginative!

Daily activities

Daily activities

1. Moving things

Daily activities

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2. Communication

Daily activities

1. Moving things
2. Communication
3. Traffic flow

Daily activities

1. Moving things
2. Communication
3. Traffic flow
4. Playing games

Daily activities

1. Moving things
2. Communication
3. Traffic flow
4. Playing games
5. Money matters

Daily activities

1. Moving things
2. Communication
3. Traffic flow
4. Playing games
5. Money matters
6. Logistic problems

Daily activities

1. Moving things
2. Communication
3. Traffic flow
4. Playing games
5. Money matters
6. Logistic problems
7. Security procedures

Daily activities

1. Moving things
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3. Traffic flow
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7. Security procedures
8. Time-related activities

Daily activities

1. Moving things
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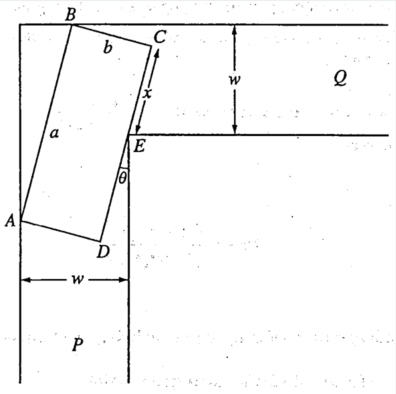
6. Logistic problems
7. Security procedures
8. Time-related activities
9. Political elections

Moving things

Specimen paper



Moving things

Specimen paper

Two straight corridors, P and Q, each of width w, meet at right angles. $ABCD$ is a rectangular crate of length a and breadth b. In the position shown in the diagram, the angle between DC and the wall of the corridor P is θ, where $\theta < 45°$. The crate touches the outer walls at A and B, and touches the inside corner at E, where $CE = x$.

Moving things

Specimen paper

(i) Show that $x\cos\theta + b\sin\theta = w$, and find an equation relating a, b, x, w and θ.

(ii) By eliminating x from the equations in part (i), or otherwise, show that

$$\frac{1}{2}a\sin 2\theta + b = w(\sin\theta + \cos\theta).$$

(iii) Let $\theta = 45^\circ - \phi$. Show that the equation in part (ii) may be expressed as

$$a\cos^2\phi - (w\sqrt{2})\cos\phi + (b - \frac{1}{2}a) = 0.$$

(iv) Find θ in the case when $a = 4$, $b = 1$ and $w = 2$.

Security procedures

Sample resource

The key to a certain bank vault is a secret code, consisting of five integers $(a_4, a_3, a_2, a_1, a_0)$. For security reasons, only partial information concerning this key is distributed to each of the 5 high-ranking officers (numbered 1, 2, 3, 4 and 5) in the bank. The i-th officer is assigned an ordered pair $K_i = (i, P(i))$, where

$$P(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

Security procedures

Sample resource

(a) The ordered pairs the 5 high-ranking officers are holding are given below:

i	1	2	3	4	5
K_i	$(1, -15)$	$(2, -56)$	$(3, -73)$	$(4, 54)$	$(5, 517)$

Prove that $P(x) = 3x^4 - 10^3 - 3x^2 - 7x + 2$ and recover the key to the bank vault.

Sample resource

(b) Based on mathematical considerations, describe

  (i) one strength, and
  (ii) one weakness

of the secret-key assignment system described above.

For the weakness you mention, suggest one way to overcome it.

Common items

Common items

1. Physical objects/situations

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2. Daily items

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3. Familiar fixtures and landscape-features

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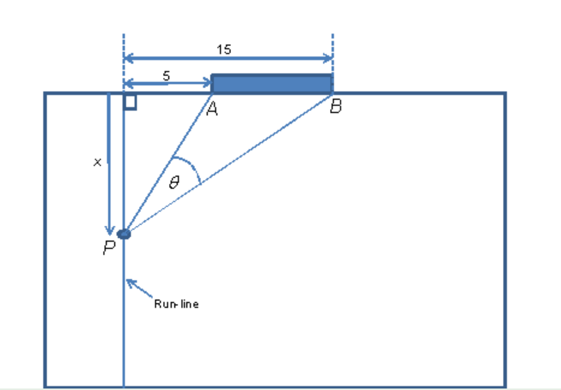
4. Food items
5. Sporting items

Physical situations

Sample resource



Physical situations

Sample resource

A soccer player P dribbles a football along a run-line which is perpendicular to the base line of a field shown in the figure. The goalposts A and B are respectively of distance 5 metres and 15 metres from the run-line. At a certain point along the run-line of distance x metres away from the base line, P kicks the ball towards the goal. Let $\angle APB$ be θ.

(i) Show that

$$\theta = \tan^{-1}\frac{15}{x} - \tan^{-1}\frac{5}{x}.$$

(ii) Determine the value of x for which θ is maximized.

(iii) Discuss the relevance of part (ii) to the soccer player P.

Natural phenomenon

Natural phenomenon

1. Population dynamics

Natural phenomenon

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2. Diseases

Natural phenomenon

1. Population dynamics
2. Diseases
3. Measurements/attributes of organisms, plants and animals.

Natural phenomenon

SAJC/2011/II/11

The variation of the population of Komomo dragons in a particular region with time is modelled by the first order differential equation

$$\frac{dy}{dx} + y(y-1) = 0,$$

given that y (in thousands) denotes the size of the population at time t (in years).
Find the general solution of the differential equation and express y in terms of x.

Physical phenomenon

Physical phenomenon

1. Rate of chemical reactions

Physical phenomenon

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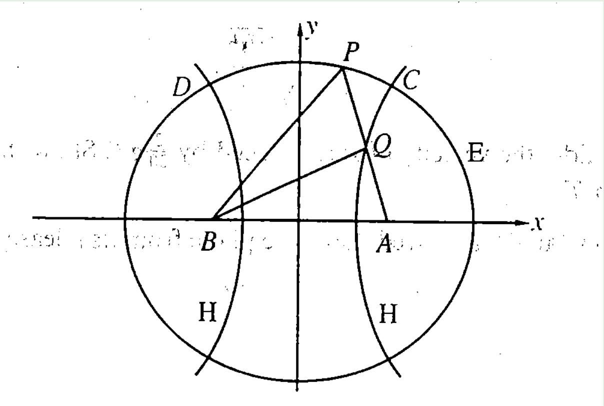
2. Rate of flow, diffusion and dilution

Physical phenomenon

1. Rate of chemical reactions
2. Rate of flow, diffusion and dilution
3. Movement of objects, e.g., celestial bodies, cars, points, shadow, etc.

Physical phenomenon

J93/S/5 (modified)



Physical phenomenon

J93/S/5 (modified)

Celestial bodies move along respectively the elliptical orbit E and the hyperbolic orbit H, whose equations are given by the parametric equations

$$E : x = 4\cos\theta,\ \ y = 2\sqrt{3}\sin\theta,$$

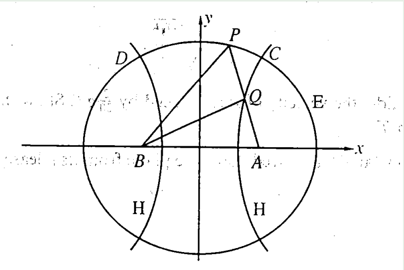
and

$$H : x = \sec\phi, y = \sqrt{3}\tan\phi.$$

Find the coordinates of C and D, the common points of E and H with positive y-coordinates.

Physical phenomenon

J93/S/5 (modified)



The points A and B have coordinates $(2, 0)$ and $(-2, 0)$ respectively. Prove that $PA + PB$ is constant, where P is any point on E.

Context sourcing

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Context sourcing

- Watch out for interesting phenomenon to mathematize
- Make a consistent and conscientious collection of contexts
- Read more on mathematical modelling

Deeper into theory

Certain mathematical techniques and results beyond the H2 syllabus can be tested using appropriate scaffolding.

Deeper into theory

N98/S/4

$T_n(x)$ is a polynomial of degree n in x defined by

$$T_n(x) = \cos(n \cos^{-1}(x)),$$

so that $T_n(\cos\theta) = \cos(n\theta)$.

(i) Show that $T_2(x) = 2x^2 - 1$ and $T_3(x) = 4x^3 - 3x$.

(ii) Prove that

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Deeper into theory

Special functions

There are an entire sea of special functions, together with interesting identities, whose properties can be proven by H2 Mathematics.

- Chebyshev polynomials
- Legendre polynomials
- Fourier series
- Riemann ζ-function
- Logistic map
- Sigmoid function

Modification

Perform interesting modifications on existing question items.

Natural phenomenon

N2000/S/8 (Modified)

A botanist collects leaves from rose bushes whose leaves are infected with blackspots, and counts the number of spots on each infected leaf, ignoring uninfected leaves (i.e., those that have no spots). She models the distribution of spots on the infected leaves in her collection by assuming that the probability that a randomly chosen leaf has r spots is

$$p_r = k\frac{\lambda^r e^{-\lambda}}{r!} \text{ for } r = 1, 2, 3, \cdots ,$$

where k and λ are positive constants.

Natural phenomenon

N2000/S/8 (Modified)

(i) Explain, using probabilistic considerations, why

$$\sum_{r=1}^{\infty} p_r = 1.$$

(ii) By considering (i) and the fact that

$$e^{\lambda} = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!},$$

show that

$$k = \frac{1}{1 - e^{-\lambda}}.$$

Nostalgia

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- Digging into sufficiently old past year questions can prove rewarding.
- This also helps counter surprises.

Nostalgia

D77/II/4(b) (modified)

The position vector of a fixed point A relative to an origin O is given by $\mathbf{a}$. Describe the locus of a point P whose position vector $\mathbf{r}$ relative to O satisfies the equation

$$\mathbf{r} \cdot (\mathbf{r} - \mathbf{a}) = 0.$$

Exploiting historical contexts

History of mathematics offer many interesting contexts for formulating questions.

Exploiting historical contexts

Sample resources

(i) Find, up to the term in x^2, the binomial expansion of

$$(1+x)^{\frac{1}{2}},$$

and state the range of values of x for which the expansion is valid.

Exploiting historical contexts

Sample resources

An ancient algorithm for finding the square root of a positive integer S is given in the *Bakhshali manuscript*, which is transcribed as follows:

1. Let the perfect square nearest to S be N^2.
2. Calculate $d = S - N^2$.
3. Calculate $P = \frac{d}{2N}$.
4. Calculate $A = N + P$.
5. Approximate $\sqrt{S}$ by the formula:

$$\sqrt{S} \approx A - \frac{P^2}{2A}.$$

Exploiting historical contexts

Sample resources

In summary, this amounts to

$$\sqrt{S} \approx N + \frac{d}{2N} - \frac{d^2}{8N^2}\left(N + \frac{d}{2N}\right)^{-1}.$$

(ii) Using (i) and assuming that $\left|\frac{d}{N^2}\right| < 1$, show that

$$(N^2 + d)^{\frac{1}{2}} \approx N + \frac{d}{2N} - \frac{d^2}{8N^3}.$$

(iii) Hence explain how the ancient algorithm works.

Resources

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- Question banks

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- Question banks
- Journal resources

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- Web resources

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- Mathematical Olympiads

Journal resources

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- The American Mathematical Monthly

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- The College Mathematics
- The American Mathematical Monthly
- International Journal of Mathematical Education in Science and Technology (esp., Classroom notes)

Web resources

There is a rich resource on mathematical modelling:

1. Plus teacher and student package: Mathematical Modelling at http: //plus.maths.org/content/os/issue44/package/index
2. McLaughlin, Michael P. (1999) ‘A Tutorial on Mathematical Modeling’.
3. Patrone, F. Introduction to modeling via differential equations, with critical remarks.

Web resources

I encourage you to visit and read the mathematical vignettes at http://blog.kleinproject.org/

Sister-lecture

I have given a sister-lecture on how to set good examination questions at HCI some years ago, and the slides are available at

http://math.nie.edu.sg/wkho/Talks_files/setexam.pdf

where I gave more generic guidelines related to setting good examination question items.

Professional upgrading

Now I take some time to advertise the following course:

Master of Science (Mathematics for Educators)

offered at National Institute of Education.