Characterising **E**-projectives via Comonads

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Contents

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Definition (Projective)

An object P of a category $\mathbf{C}$ is *projective* if for every epimorphism $e : A \longrightarrow B$ and every morphism $f : P \longrightarrow B$, there is a $\mathbf{C}$-morphism (not necessarily unique) $f' : P \longrightarrow A$ such that

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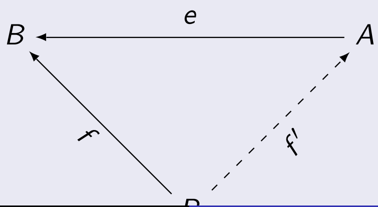

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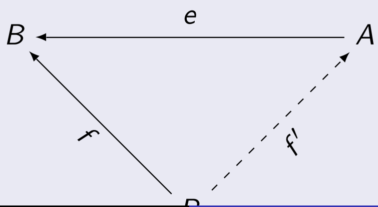

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E-projectives

In the certain categories, projectives are scarce.

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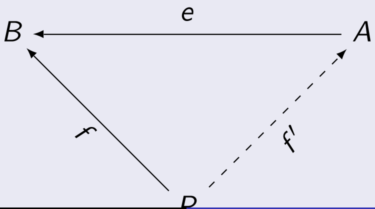

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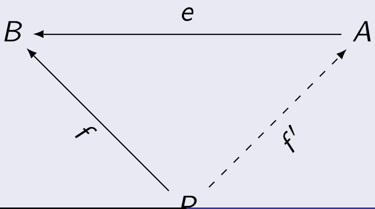

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Example (Regular-projectives)

E-projectives = regular projectives

E-projectives

Example (Banaschewski's Theorem)

The regular-projectives in the category **Frm** of frames are exactly the stably completely distributive lattices.

E-projectives

Example (Gleason's Theorem)

The regular-projectives in the category **KHausSp** of compact Hausdorff spaces are exactly the extremally disconnected spaces.

D. Zhao's result

Lemma

(Zhao, 1997)

Let $G : \mathbf{C} \longrightarrow \mathbf{D}$ and $F : \mathbf{D} \longrightarrow \mathbf{C}$ be a pair of functors such that F is left adjoint to G, with co-unit denoted by ε.
Further, let $\mathbf{E}$ denote the collection of all $\mathbf{C}$-morphisms $f : A \longrightarrow B$ such that $G(f)$ has a section, i.e., a right inverse in $\mathbf{D}$.
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(i) A *is* $\mathbf{E}$*-projective.*

(ii) $\varepsilon : FG(A) \longrightarrow A$ *has a right inverse.*

(iii) A *is a retract of some* FX *for some object* X *in* $\mathbf{D}$*.*

Sample application 1

Theorem

(Zhao, 1997)
The **E***-projective Z-frames are precisely those which are stably Z-continuous.*

Sample application 2

Theorem

(Wang & Zhao, 2010)
The **E***-projective* Z*-quantales are precisely those which are stably* Z*-continuous.*

A closer analysis

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Definition (Right KZ-monad)

Let (T, η, μ) be a monad on a poset-enriched category $\mathbf{X}$, and assume that T is a poset-functor.

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Let (T, η, μ) be a monad on a poset-enriched category $\mathbf{X}$, and assume that T is a poset-functor.
We say that T is a right *KZ-monad* if

$$\eta_{TX} \leq T\eta_X$$

for all $X \in \mathbf{X}$.

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Theorem

(Escardó, 1998)

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(i) *A is right injective over right T-embeddings.*

(ii) *A is injective over right T-embeddings.*

(iii) *A is a T-algebra.*

These conditions imply that

(iv) *A is a right Kan object over right T-arrows.*

D. S. Scott's result

Theorem

(Scott, 1972)
The injective T_0-spaces (over subspace embeddings) are exactly the continuous lattices endowed with the Scott-topology. Moreover, if $f : X \longrightarrow D$ is a continuous map into a continuous lattice and j is a subspace embedding, then f has the largest extension

$$f/j(y) = \bigvee \left\{ \bigwedge f(U \cap X) \mid U \text{ is open}, y \in U \right\}.$$

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Proof.

A. Day's filter monad $\mathcal{U}$ on the category **Top** of T_0-spaces is a right KZ-monad; the right $\mathcal{U}$-arrows are exactly the continuous

Poset adjunctions

Recall that for posets P and Q, a *monotone map* $f : P \longrightarrow Q$ is a function which preserves order, i.e., $x \leq_P y$ implies $f(x) \leq_Q f(y)$.

Poset adjunctions

If a pair of monotone maps $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ is such that

$$f(p) \leq q \iff p \leq g(q)$$

for all $p \in P$ and $q \in Q$, then we say that f is *left adjoint* to g, or equivalently g is *right adjoint* to f.
An adjunction pair as described above is denoted by $f \dashv g$.

Poset adjunctions

For monotone maps, $f \dashv g$ if and only if

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If in addition $f \circ g = \mathrm{id}_Q$, we say that $f \dashv g$ is *reflective*; and dually, *coreflective* if $g \circ f = \mathrm{id}_P$.

Comonads

A *comonad* in a category $\mathbf{D}$ consists of a functor $U : \mathbf{D} \longrightarrow \mathbf{D}$ together with two natural transformations $\varepsilon : U \longrightarrow \mathrm{id}_{\mathbf{D}}$ (the *counit*) and $\nu : U \longrightarrow U^2$ (the *comultiplication*), subject to the following conditions:

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1. (Associativity) $U\nu_X \circ \nu_X = \nu_{UX} \circ \nu_X$, and
2. (Unit laws) $\varepsilon_{UX} \circ \nu_X = U\varepsilon_X \circ \nu_X = id_{UX}$

for any object X of $\mathbf{D}$.

Co-algebra

Let $U = (U, \varepsilon, \nu)$ be a comonad.

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Let $\mathbb{U} = (U, \varepsilon, \nu)$ be a comonad.
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A *poset functor* $U : \mathbf{C} \longrightarrow \mathbf{D}$ is *poset-faithful* if all A and B in $\mathbf{C}$ and all $\mathbf{C}$-morphisms $f,\ g : A \longrightarrow B$,

$$f \leq g \iff Uf \leq Ug.$$

KZ comonads

Lemma

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(KZ$_1$) *For all* $X \in \mathbf{D}$, *an arrow* $\beta : X \longrightarrow UX$ *is a co-structure map if and only if* $\beta \dashv \varepsilon_X$ *is a coreflective adjunction (i.e.,* $\varepsilon_X \circ \beta = \mathrm{id}_X$*).*

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(KZ_2) $\nu_X \dashv \varepsilon_{UX}$ *for all* $X \in \mathbf{D}$.

(KZ_3) $U\varepsilon_X \dashv \nu_X$ *for all* $X \in \mathbf{D}$.

KZ comonads

Definition (KZ comonads)

Let **D** be a poset-enriched category. A *left KZ-comonad* in **D** is a comonad (U, ε, ν) in **D** with U a poset functor, subject to the equivalent conditions of the preceding lemma. Poset-dually, one defines *right KZ-comonads*.

KZ terminology

Whenever there is no confusion, we just write 'KZ-comonad' for 'left KZ-comonad'. Here "KZ" abbreviates "Kock-Zöberlein".

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The triple $(M, \cdot, \leq)$ is an *ordered monoid* if M is a monoid with identity 1_M, together with a partial order $\leq$ on it which is compatible with the monoid operation, i.e.,

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- Every monoid is a trivial ordered monoid with the discrete order.
- The set of natural numbers has two different well-known ordered monoid structures, namely, $(\mathbb{N}, +, \leq)$ and $(\mathbb{N}, \max, \leq)$.

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OrdMon: the category of ordered monoids and ordered monoid morphisms.

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(iv) For all $r \in R$, $0 \cdot r = 0 = r \cdot 0$.

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(i) $(R, +)$ is a commutative monoid with identity element 0.

(ii) $(R, \cdot)$ is a monoid with identity element 1_R.

(iii) For all $a,\ b$ and $c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$.

(iv) For all $r \in R$, $0 \cdot r = 0 = r \cdot 0$.

(v) $1 \neq 0$.

Semi-rings

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(iii) Let R be a semi-ring. The set of ideals of R, denoted by $\mathrm{Id}(R)$, with the usual addition $I + J := \{i + j \mid i \in I,\ j \in J\}$ and multiplication of ideals $I \cdot J := \{i \cdot j \mid i \in I,\ j \in J\}$, is a semi-ring.

SRng

If R and S are semi-rings, then a function $\gamma : R \longrightarrow S$ is a *semi-ring morphism* if the following conditions hold:

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(iii) $\gamma(r + r') = \gamma(r) + \gamma(r')$ and $\gamma(r \cdot r') = \gamma(r) \cdot \gamma(r')$ for all $r,\ r' \in R$.

Normal semi-rings

Definition

A semi-ring R is *normal* if

$$x \cdot y + x = x = y \cdot x + x$$

for all x and $y \in R$.

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SRng: the category of semi-rings and semiring morphisms.
NSRng: subcategory of **SRng** whose objects are normal semi-rings.

Normal semi-rings

Example

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1. A bounded distributive lattice $(L, \vee, \wedge)$ is a normal semi-ring since for any $x,\ y \in L$, one has

$$(x \wedge y) \vee x = (x \wedge y) \vee (x \wedge 1) = x \wedge (y \vee 1) = x \wedge 1 = x.$$

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$$I \subseteq IJ + I \subseteq IR + I = I + I \subseteq I.$$

OrdMon ⊣ NSRng

Let $(M, \cdot, \leq)$ be an ordered monoid. Define

$$D_0(M) := \{\downarrow A \mid A \in \mathcal{P}_{\mathrm{fin}}(M)\}.$$

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On $D_0(M)$, define the semi-ring addition as binary union.
As for the semi-ring multiplication $\otimes$, define as follows: for any $\downarrow A, \ \downarrow B \in D_0(M)$,

$$\downarrow A \otimes \downarrow B := \downarrow \{a \cdot b \mid a \in A, \ b \in B\}.$$

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Crucially, for an ordered monoid $(M, \bigcup, \otimes)$, the triple $(D_0(M), \bigcup, \otimes)$ is a normal semi-ring.

OrdMon $\dashv$ NSRng

For an arbitrarily given **OrdMon**-morphism $f : M \longrightarrow N$ a **NSRng**-morphism

$$F(f) : D_0(M) \longrightarrow D_0(N), \;\; Ff(\downarrow A) = \downarrow f(A),$$

for any $A \in D_0(M)$.

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for any $A \in D_0(M)$.
Then F : **OrdMon** $\longrightarrow$ **NSRng** is a functor.

OrdMon $\dashv$ **NSRng**

In the opposite direction, any normal semi-ring $(S, +, \cdot)$ can be given an ordered monoid structure, namely, $(S, \cdot, \leq)$, where

$$a \leq b \iff a + b = b$$

for any $a, \ b \in S$.

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In the opposite direction, any normal semi-ring $(S, +, \cdot)$ can be given an ordered monoid structure, namely, $(S, \cdot, \leq)$, where

$$a \leq b \iff a + b = b$$

for any $a, \ b \in S$.
Thus, we have the forgetful functor $G : \mathbf{NSRng} \longrightarrow \mathbf{OrdMon}$.

Normal semi-rings

Proposition

Let $(R, +, \cdot)$ *be a normal semi-ring and* $A \subseteq_{\text{fin}} R$*. Then*

$$\bigvee A = \sum_{a \in A} a.$$

Comonad arising from **OrdMon** ⊣ **NSRng**

Specializing the above result to the adjunction $F \dashv G$ between the categories **OrdMon** and **NSRng**, $U : \mathbf{NSRng} \longrightarrow \mathbf{NSRng}$ is defined by $U = FG$. The counit ε is given by

$$\varepsilon_R : UR \longrightarrow R, \ \ \downarrow A \mapsto \sum_{a \in A} a = \bigvee A \quad (A \subseteq_{\text{fin}} R \in (\mathbf{NSRng})),$$

while the comultiplication $\nu = F\eta_G$ is explicitly given by

$$\nu_R : \downarrow A \mapsto \{\downarrow B \in FGR \mid \exists a \in A. \ \downarrow B \subseteq \downarrow a\}.$$

Comonad arising from **OrdMon** $\dashv$ **NSRng**

For the poset-enriched category **NSRng**, the following property holds:

Proposition

For any normal semi-ring R, it holds that

$$\varepsilon_{UR} \leq U\varepsilon_R.$$

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Remark

The above (U, ε, ν) is a KZ-comonad.

Left U-quotients

Definition

Let $F : \mathbf{D} \longrightarrow \mathbf{D}$ be a poset-functor on a poset-enriched category $\mathbf{D}$.

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A *left F-arrow* is a morphism $f : X \longrightarrow Y$ in $\mathbf{D}$ such that $Ff : FX \longrightarrow FY$ has a right adjoint denoted by $\hat{f} : FY \longrightarrow FX$.

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A *left F-arrow* is a morphism $f : X \longrightarrow Y$ in $\mathbf{D}$ such that $Ff : FX \longrightarrow FY$ has a right adjoint denoted by $\hat{f} : FY \longrightarrow FX$.
If the adjunction is reflective (i.e., $Ff \dashv \hat{f}$ and $Ff \circ \hat{f} = \mathrm{id}_{FY}$), we say that f is a *left F-quotient*.

Main theorem

Theorem

The following statements are equivalent for a left KZ-comonad (U, ε, ν) in a poset-enriched category $\mathbf{D}$ *and any object* $A \in \mathbf{D}$:

(1) A *is a left projective over left* U*-quotients.*

(2) A *is projective over left* U*-quotients.*

(3) A *is a* U*-coalgebra.*

Main theorem

Theorem (Continued)

These conditions imply

(4) *A is a left Kan object over the left U-arrows.*

Moreover, assuming that any one of the equivalent conditions (1) - (3) holds, if $p : Y \longrightarrow X$ is a left U-arrow and $f : A \longrightarrow X$ is any arrow in $\mathbf{D}$, then

$$f/p = \varepsilon_Y \circ \hat{p} \circ Uf \circ m_A,$$

where $m_A : U \longrightarrow UA$ is the co-structure map of the coalgebra A.

Perfect semi-ring morphisms

Definition

A semi-ring morphism $f : R \longrightarrow S$ between normal semi-rings R and S is said to be *perfect* if Uf has a reflective right adjoint, i.e., a semi-ring morphism $s : US \longrightarrow UR$ such that $Uf \dashv s$ and $Uf \circ s = \mathrm{id}_{US}$.

Perfect semi-ring morphisms

For any normal semi-ring R, the co-unit map

$$\varepsilon : UR \longrightarrow R, \ \ {\downarrow}A \mapsto \bigvee A$$

is such that $U\varepsilon_R : U^2R \longrightarrow UR$ has a reflective right adjoint $\nu_R : UR \longrightarrow U^2R$ given by

$${\downarrow}A \mapsto \{{\downarrow}B \mid \exists a \in A. \ {\downarrow}B \subseteq {\downarrow}a\}.$$

This provides a natural example of a perfect semi-ring morphism.

E-projective normal semi-rings

Theorem

Let **E** *be the class of perfect semi-ring morphisms.*
The following are equivalent for any normal semi-ring R:

(i) R *is* **E**-*projective.*

(ii) R *is the underlying object of a U-coalgebra.*

(iii) R *is ???.*

E-projective normal semi-rings

Lemma

A normal semi-ring $(R, +, \cdot)$ is the underlying object of a U-coalgebra if and only if it is stably F-continuous.

E-projective normal semi-rings

Proof.

By KZ-Lemma, R is the underlying object of a U-coalgebra if and only if its co-unit $\varepsilon_R : UR \longrightarrow R$ has a coreflective right adjoint $\beta : R \longrightarrow UR$.

E-projective normal semi-rings

Proof.

By KZ-Lemma, R is the underlying object of a U-coalgebra if and only if its co-unit $\varepsilon_R : UR \longrightarrow R$ has a coreflective right adjoint $\beta : R \longrightarrow UR$.

This is equivalent to

$$\beta \circ \varepsilon_R \leq \mathrm{id}_{UR} \quad \text{and} \quad \varepsilon_R \circ \beta = \mathrm{id}_R.$$

E-projective normal semi-rings

Proof.

Since $\varepsilon_R(\downarrow A) = \sum_{a \in A} a = \bigvee A$, the adjoint situation forces the preceding inequalities to be equivalent to

$$\beta(r) = \bigcap\{\downarrow A \in FR \mid r \leq \bigvee A\}.$$

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Thus, $s \in \beta(r)$ if and only if $s \ll_F r$.

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Thus, $s \in \beta(r)$ if and only if $s \ll_F r$.
Hence R is a F-continuous normal semi-ring.

E-projective normal semi-rings

Proof.

Since β preserves the multiplication $\cdot$, it follows that for any y and $z \in R$,

$$\beta(y \cdot z) = \beta(y) \otimes \beta(z).$$

E-projective normal semi-rings

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Since β preserves the multiplication $\cdot$, it follows that for any y and $z \in R$,

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Thus, this second condition is equivalent to:

$$\begin{aligned} x \ll_F (y \cdot z) &\iff x \in \beta(y) \otimes \beta(z) \\ &\iff (\exists y' \ll_F y) \wedge (\exists z' \ll_F z).\ (x \leq y' \cdot z'), \end{aligned}$$

which is just the condition that R is stably F-continuous. □

New notions in semi-ring theory

Definition

Let $(R, +, \cdot)$ be a semi-ring. Define an auxiliary relation $\ll_F$ (read as finitely way-below) on R as follows:

$$x \ll_F y \iff \forall A \subseteq_{\text{fin}} R. \left(\sum_{a \in A} a \geq y \implies \exists a \in A. x \geq a \right).$$

Viewing a normal semi-ring as an ordered monoid, the sum $\sum_{a \in A} a$ may be seen as $\bigvee A$.

New notions in semi-ring theory

Definition

A normal semi-ring $(R, +, \cdot)$ is said to be *F-continuous* if for any $x \in R$,

(i) $\downarrow x := \{r \in R \mid r \ll_F x\}$ is the lower closure (w.r.t. the induced partial order) of a finite subset of R, and

(ii) $x = \bigvee \downarrow x$.

A semi-ring R is said to be *stably F-continuous* if in addition to (i) and (ii) it satisfies the following condition:

(iii) $x \ll_F y \cdot z$ if and only if there exist y' and z' in R such that $y' \ll_F y$, $z' \ll_F z$ and $x \leq y' \cdot z'$.

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Theorem

Let **E** *be the class of perfect semi-ring morphisms.*
The following are equivalent for any normal semi-ring R*:*

(i) R *is* **E***-projective.*

(ii) R *is the underlying object of a* U*-coalgebra.*

(iii) R *is stably* F*-continuous.*

E-projective normal semi-rings

It is clear that

$$\mathrm{lcm}(\gcd(x, y), x) = x$$

so that the semi-ring $R = (\mathbb{N}, \mathrm{lcm}, \gcd)$ with lowest common multiple as addition and greatest common divisor as multiplication is normal.

E-projective normal semi-rings

Viewed as an ordered monoid, the partial order $\leq$ on R is just the divisibility relation, i.e., $a \leq b \iff a \mid b$.

Clearly, $0 \ll_F 0$ and $1 \ll_F 1$ in R, and crucially, if $y \neq 0, 1$, then $x \ll_F y \iff x = p^k$ for some prime p and $x \mid y$.

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It follows immediately by Euclid's lemma that R is a stably continuous normal semi-ring. This normal semi-ring is thus **E**-projective.

Concluding remarks

1. We have successfully applied the dualization of Escardó's result to characterize the **E**-projective objects of certain categories.

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1. We have successfully applied the dualization of Escardó's result to characterize the **E**-projective objects of certain categories.
2. We aim to sharpen our results by characterising those **E**-morphisms in each of the categories considered.

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