What is the problem?

A short excursion into problem posing and problem solving in mathematics classrooms

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Outline

Subtopics to be covered

1. Motivation for good problem posing
2. Some criteria for good mathematical problems
3. Strategies for posing good problems

Workshop coverage

In today's workshop, we shall together

1. find out what is the problem about problem posing,
2. discover what constitutes a good mathematics problem,
3. learn about the Brown and Walter's method,
4. create some inverse problems,
5. discover how we can make use of analogies in formulating new problems, and
6. pick up some re-packaging skills via scenario building.

Motivating examples

Let us consider (and solve) the following three problems.

Motivating examples

Question A ($1)

It is given that of three positive integers, two are the same. Furthermore, their product is 36 and their sum is 13. Find the values of these integers.

Solution to Question A

Solution.

Let x, y and z be these three positive numbers. Then we have:

$$\begin{aligned} x + 2y &= 13 \\ xy^2 &= 36 \end{aligned}$$

So this reduces to a cubic equation $(13 - 2y)y^2 = 36$. The positive integer solution is $y = 2$. Thus

$$x = 9, \ y = 2.$$

$\square$

Motivating examples

Question B ($5)

There are three positive integers x, y and z whose product is 36 such that their sum is 13. Given that at least one of these numbers is more than 6, find the values of x, y and z.

Solution to Question B

Solution.

Since the product of these numbers is given, we have:

$$xyz = 36$$

and their sum given as well,

$$x + y + z = 13.$$

One easily obtains:

$$\frac{36}{xy} = 13 - x - y$$

□

Solution to Question B

Solution.

We may suppose that $z > 6$ so that

$$13 - (x + y) > 6$$

or equivalently,

$$x + y < 7.$$

Since $\frac{36}{xy} = 13 - (x + y)$, it follows that $\frac{36}{xy} > 6$, i.e.,

$$xy < 6.$$

$\square$

Solution to Question B

Solution.

The possible triples are

$$\{1,5,7\},\ \{1,4,8\},\ \{1,3,9\},\{1,2,9\},\{1,1,11\},\{2,2,9\}$$

but this set quickly reduces to only one possibility of

$$\{2,2,9\}$$

when one considers that the product is 36. $\square$

Motivating examples

Question C ($20)

Ali and Mr. Lee have the following conversation:

Ali: I forgot how old your three kids are.

Mr. Lee: The product of their ages is 36.

Ali: I still don't know their ages.

Mr. Lee: The sum of their ages is the same as your house number.

Ali: I still don't know their ages.

Mr. Lee: The oldest one has red hair.

Ali: Now I know their ages!

How old are the three children of Mr. Lee?

Solution to Question C

Solution.

From the statement that the product of their ages is 36 the possibilities of the three individual ages are:

- $\{1, 1, 36\}$, $\{1, 2, 18\}$, $\{1, 3, 12\}$, $\{1, 4, 9\}$, $\{1, 6, 6\}$
- $\{2, 2, 9\}$
- $\{2, 3, 6\}$
- $\{3, 3, 4\}$

$\square$

Solution to Question C

Solution.

From the statement that the sum equals the house number it is possible to eliminate all but two possibilities.

The sums of the rest are unique and would allow for an immediate answer.

For example if the house number were 16 the ages must be 1, 3, and 12. The two remaining possibilities are 2, 2, and 9; or 1, 6, and 6. □

Solution to Question C

Solution.

After the clue that the oldest has red hair you can eliminate 1, 6, and 6 because the oldest two have the same age thus there is no oldest son. The only remaining possibility is 2, 2, and 9. □

Which is the best?

Activity 1 (10 min)

In your small groups of 3,

1. Decide which question appeals to you as a good question?
2. Which, if any, is the best question?
3. Give your reasons.

Why do we need good problem posing?

- Good problems engage students actively in learning mathematics
- Asking and solving good problems pushes research frontiers
- Ensures the very survival of mathematics as a subject

What does good problem posing involve?

Problem posing ...

- is NOT an easy task.
- requires one to understand thoroughly the related content materials.
- requires one to analyze and reprocess the existing information or problems.

Good problem posing

We set out to systematically examine various methods of posing mathematical problems.

Criteria for good math problems

A good mathematics problem possesses one or more of the following characteristics:

Criteria for good math problems

1. It is clearly stated and easy to understand.

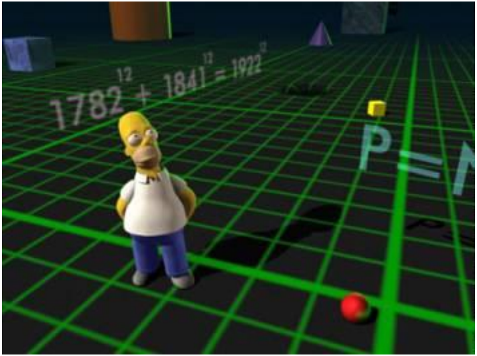
Criteria for good math problems

A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street. – David Hilbert (1862 – 1943)

Criteria for good math problems

2. It is difficult but not completely inaccessible.

Criteria for good math problems



Criteria for good math problems

Problem

For integers $n > 2$ the equation

$$a^n + b^n = c^n$$

cannot be solved with positive integers $a,\ b$ and c.

Criteria for good math problems

3. Its solution not only enhances our understanding of the existing structures, but also leads us to solutions of other problems and to the discovery of new tools or theories.

Criteria for good math problems

Of course, the above list is far from exhaustive.

Activity 2 (10 min)

In your small groups of 3,

1. can you add at least two more items to this?
2. share with us these additional criteria?

Brown and Walter's method

In Brown and Walter's textbook 'The Art of Problem Posing', one main strategy is called the

What-If-Not scheme

Brown and Walter's method

Let us do some googling.

Brown and Walter's method

The WIN scheme constitutes of the following sequence of actions:

1. Observing an existing result.
2. Listing some attributes involved.
3. Breaking these attributes down into more subtle ones (if necessary).
4. Select some of these attributes and ask the important question:

   What might the outcome be if these selected attributes were replaced by different ones?

5. Repeat this above procedure as many times as possible.

What-If-Not

Question

Let's make pairs and let's try to find ways to win the following game:

Game 1: Twenty units of yellow coloured tubes are connected with one unit of black tube. Two students on the rock-scissors-paper method determine order. Then, students take turns to take from one to three cubes. The student who takes the last cube is the winner.

What-If-Not

Activity 3 (10 min)

Let's change the game in whatever way you'd like (e.g., to modify the game after seeing the original game) and try to investigate a corresponding strategy.

What-If-Not

Sample modification

1. Place a black-coloured cube at the center with 7 yellow cubes connected on its left side, and 13 red cubes on its right side.
2. Two students will take turn to take at least one and up to three cube of he same colours. They take cubes from either right or left side.
3. The student who takes the black cube will be the loser.

What-If-Not

Activity 4 (10 min)

It is well known that

$$1 + 2 + 3 + 4 + \cdots + n = \frac{n}{2}(n+1).$$

Use the 'What-If-Not' method to generate new and challenging questions. Try to provide the solution to your problem.

Break

We now take a break.

Inverse problems

Another general method for posing mathematics problems is to formulate the inverse problem of a given one.

Inverse problems

Roughly speaking,

if problem A is ...

Given $p_1,\ p_2, \ldots, p_n$, find $q_1,\ q_2, \ldots, q_m$,

then the problem B

Given $q_1,\ q_2, \ldots, q_m$ and $p_1,\ p_2, \ldots, p_k$, find $p_{k+1},\ p_{k+2}, \ldots, p_n$.

may be regarded as an inverse problem of **A**.

Inverse problems

Problem

Verify that for 2×2 matrices with real entries, matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

commutes with every square matrix of the same order with respect to the matrix multiplication.

Inverse problems

Problem

Find the set of all those square matrices that commute with every square matrix with respect to the matrix multiplication.

Inverse problems

Solution.

We start with a matrix which can commute with every other square matrix, and let us say we denote it by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\square$

Inverse problems

Solution.

Since every matrix commutes with this special one, we can be sure that it works for the special matrix:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

so that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}$$

□

Inverse problems

Solution.

On the other hand, we have:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$$

$\square$

Inverse problems

Solution.

This means that

$$a = d \ \& \ b = -c.$$

Thus the kind of matrix that one is looking for is of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

$\square$

Inverse problems

Solution.

In a similar manner, we demand that this matrix commutes with the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

This way, one can deduce that such a matrix must be of the form:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

□

Inverse problems

Solution.

Thus we conclude that a matrix which commutes with every other square matrix has to be of the form

$$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

where a is a real number. □

Inverse problems

Activity 5 (10 min)

It is common to ask students for partial fractions decomposition. For instance, express

$$\frac{2}{x(x-1)(x+1)}.$$

as partial fractions.

Can we pose another related problem by using the inverse problems approach?

Analogy problems

When we have a problem in one context, we may ask the corresponding problem in another context.
This strategy is known as forming problems via analogy.

Analogy problems

We are quite familiar with quadratic functions and the theory centred about it. For instance, given that a quadratic equation has roots α and β, one can recover the expression up to the coefficient of x^2:

$$f(x) = k(x - \alpha)(x - \beta).$$

Analogy problems

Can we say something analogous in the realm of 2×2 real matrices?

Problem

Given that a quadratic function $Q(\mathbf{A})$ of a 2×2 matrix $\mathbf{A}$ has roots $\mathbf{A_1}$ and $\mathbf{A_2}$. Can you recover $Q(\mathbf{A})$ up to a certain extent? If that cannot be done generally for every $\mathbf{A_1}$ and $\mathbf{A_2}$, can it be done for special ones?

Analogy problems

Problem

There is a well-known procedure for completing the square for quadratic functions of x.

- Is there a completing the square procedure for quadratic functions of a square matrix $\mathbf{A}$?
- How can one solve a matrix form of quadratic equations?

Analogy problems

Those who are interested in this problem may consult the article "Quadratic Matrix Equations" by G.L. Shurbet, T.O. Lewis and T.L. Boullion, The Ohio Journal of Science, Vol. 74, No. 5, pages 273 – 277, September 1974.

Analogy problems

Activity 6 (10 min)

Consider the usual problem in calculus:

Find the local maxima and minima of the function

$$f(x) = x^3 - 8x + 1.$$

Form an analogy problem for a corresponding sequence of real numbers.

Scenario building

In creating interesting questions, it is sometimes important to know how to recast existing problems in a completely fresh setting. This is called scenario building.

Scenario building

Question

Boat 1 and boat 2, which travel at constant speeds, not necessarily the same, depart at the same time from docks A and C, respectively, on the banks of a circular lake. If they go straight to docks D and B, respectively, they collide. Prove that if boat 1 goes instead straight to dock B and boat 2 goes straight to dock D, they arrive at their destinations simultaneously.

Scenario building

Activity 7 (10 min)

By solving the above question, discover what is the topic that is being tested?

Scenario building

Question

There is a 6 foot wide alley. Both walls of the alley are perpendicular to the ground. Two ladders, one 10 feet long, the other 12 feet, are propped up from opposite corners to the adjacent wall, forming an X shape. All four feet of each ladder are firmly touching either the corner or the wall. The two ladders are also touching each other at the intersection of the X shape. What is the distance from the point of intersection to the ground?

Scenario building

Activity 8 (10 min)

By solving the above problem, uncover the topic that is tested.

Conclusion

In this workshop, we have encountered the following set of strategies:

1. Brown and Walter's method
2. Inverse problems
3. Analogy problems
4. Scenario building

Problem resources

The following is quite a good web-based resource for challenging mathematics problems suitable for high-school and college students:

http://mathproblems.info/working.php#17

References

1. Brown, S. and Walter, M., *The art of problem solving*, Lawrence Erlbaum Associates Publishers, 1990.
2. Zhao, D. and Lee, C.S., *Some strategies of posing mathematics problems*. Menemui Matematik (Discovering Mathematics), Vol. 28, No. 1 (2006): pages 9 – 14.
3. Shurbet, G.L., Lewis, T.O. and Boullion, T.L., *Quadratic Matrix Equations*, The Ohio Journal of Science, Vol. 74, No. 5 (1974): pages 273 – 277.