Outbox Centroid Theorem

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ABSTRACT. An outbox of a given convex quadrilateral is a rectangle such that each vertex of the quadrilateral lies on one side of the rectangle and all the vertices lie on different sides, with all the sides of the rectangle external to the quadrilateral. This paper reports on a new geometrical result concerning outboxes of convex quadrilateral – the *Outbox Centroid Theorem*, and gives a new proof of an existing of M. F. Mammana. Interestingly, the investigation that leads to this new result comes from dynamic-geometry explorations.

1. Introduction

An *outbox* of a given convex quadrilateral $ABCD$ is a rectangle $PQRS$ such that each vertex of the given convex quadrilateral lies on one side of the rectangle and all vertices lie on different sides, with all sides of the rectangle being external to the quadrilateral. An example is shown in Figure 1 and two non-examples in Figure 2. Sometimes, non-examples such as those in Figure 2 are termed as 'illegal' outboxes.

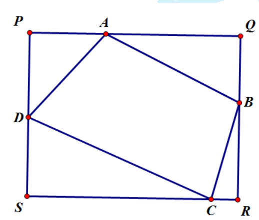

FIGURE 1. An example of an outbox

The term 'outbox' was coined by D. Zhao in [**Zha10**] but the notion itself is not new. Already in the Book IV of "Elements", according to [**Fai93**], Euclid recorded the following definitions:

DEFINITION 1.1 (Inscribing rectilinear figure). A rectilinear figure is said to be *inscribed* in a rectilinear figure when the respective (vertices) angles of the inscribed figure lie on the respective sides of the one in which it is inscribed.

DEFINITION 1.2 (Circumscribing rectilinear figure). A rectilinear figure is said to be *circumscribed* about a rectilinear figure when the respective sides of the circumscribed figure pass through the respective (vertices) angles of the one about which it is circumscribed.

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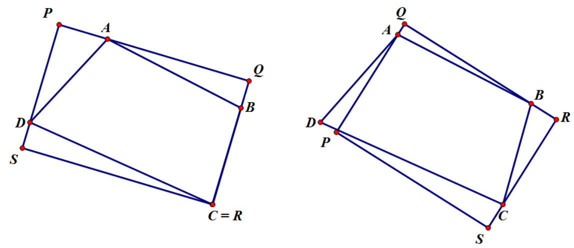


FIGURE 2. Two ‘illegal’ outboxes

In this ancient lingo, the ‘modern’ definition of an outbox can be given as follows: An outbox of a convex quadrilateral $ABCD$ is a rectangle $PQRS$ circumscribed to it, i.e., each side of $PQRS$ contains one and only one vertex of $ABCD$. Another way of saying the same thing is that a quadrilateral is *rectangle-inscribable* (*r-inscribable*, for short) if there exists a rectangle that is circumscribed to it, i.e., it has an outbox. For convenience, we shall make use of all the above terminologies in our ensuing discussion.

A *maximal outbox*, if it exists, is one with the largest area. D. Zhao in [**Zha10**] posed the maximal outbox problem which asked for the area of the maximal outbox of a given convex quadrilateral. Therein, he provided a ‘solution’ using calculus.

However, there were some issues with Zhao’s solution. Firstly, Zhao assumed implicitly that every convex quadrilateral is r-inscribable which is simply false. A deeper analysis carried out by Mammana in [**Mam08**] already characterized r-inscribable convex quadrilateral to be those for which the sum of every pair of consecutive angles (i.e., angles adjacent to each other) is less than three right angles. Even without having to study the formal proof, this characterisation is visually compelling: for instance, it is impossible for the corner (right angle) of a rectangle to ‘fit in’ so as to circumscribe an isosceles trapezium with each base angle measuring 40° each (see Figure 3) because the angle at which the two non-parallel sides (extended) meet already exceeds a right angle.

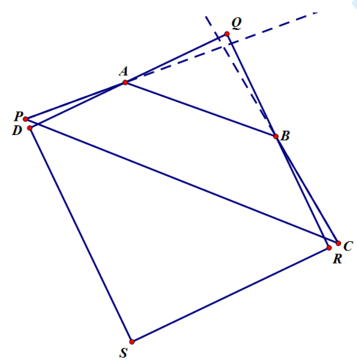


FIGURE 3. A non-r-inscribable convex quadrilateral

Secondly, although the calculus approach in [**Zha10**] yields a configuration that is supposed to give the maximum possible area of an outbox for a fixed convex quadrilateral, it was never verified whether such a configuration is actually achievable in the given situation. Furthemore, the maximal outbox problem arises from a geometrical situation and so it is natural to ask for a purely geometrical solution. Note that the earlier work of Mammana, [**Mam08**], gave a complete solution by characterising those convex quadrilaterals whose maximal outbox exists based on geometry and trigonometry.

This paper exploits a DGS (Dynamic Geometry System) for geometrical exploration into the geometrical properties of outboxes. In the course of this exploration, we discover and prove a novel theorem in geometry we term as the *Outbox Centroid Theorem*. Using this new theorem, not only do we obtain the formula for the area of the maximal outbox provided it exists but also re-establish Mammana's characterisation theorem for r-inscribable convex quadrilaterals to admit maximal outboxes [**Mam08**, Theorem 4].

Throughout this paper, we adopt a sign convention for angles: counterclockwise angles are positive. To illustrate this convention, we make use of Figure 3. The angle $\angle DAP$ has an anti-clockwise sense and is defined to be positive; the implicit reference line being AP. Using this convention, $\angle PAD$ would have a clockwise sense and is thus negative. To understand this paper, the reader requires high-school geometry, e.g., angle properties in a circle (such as Thale's theorem), and trigonometry.

2. DGS-aided discovery

Any initial attempts at the maximal outbox problem inevitably compel one to the drawing board. Traditional paper-and-pencil method in seeking any geometrical invariants proves tedious. So, we turn to *Geometer's Sketch-pad* (GSP, for short) – a Dynamic Geometry System (DGS). The application of DGS as an experimental approach to theoretical thinking has recently been given a thorough treatment in [**ABMS12**]. The potentialities of exploiting DGS in high schools have been explored by [**Mar00, Mar01, Leo03, LLT03**]. In order to consider all possible outboxes of $ABCD$, if they exist at all, we construct a dynamic prototype of an outbox. The "dragging" feature of GSP allows the user to range over all the outboxes of a fixed r-inscribable convex quadrilateral and observe the geometrical properties associated with it.

To illustrate the logical dependence of the elements involved in the construction of an outbox for a given r-inscribable convex quadrilateral, the reader is invited to follow the self-explanatory steps illustrated in Figure 4.

The segment XY controls the direction of the side PQ of the outbox by moving the point Y. Dragging Y enables us to range over all possible outboxes (including some 'illegal' ones). Using the area measuring facility of GSP, one can adjust the direction of XY via trial-and-error till the maximum area is achieved.

3. Outbox centroid theorem

To determine the position of the maximal outbox of a given r-inscribable convex quadrilateral $ABCD$, it would be ideal if one can *parameterize* the position of an outbox by a single point.

The special point we have in mind is the *centroid* of an outbox (the intersection of the medians, or equivalently, the diagonals). This point seems to be a good candidate, but one must answer two pressing questions:

A. Can we determine if a point is the centroid of some outbox?
B. Even if we know that a point is indeed the centroid of some outbox (whose position is not yet determined), can we determine the position (and hence the dimensions) of this outbox?

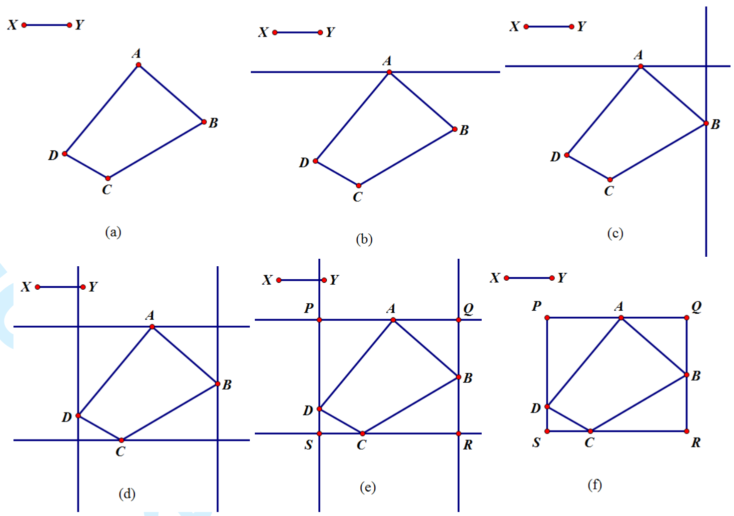


FIGURE 4. Step-by-step construction of a dynamic outbox

In order to answer these questions, one must determine the locus of the centroid of an outbox for a fixed r-inscribable convex quadrilateral. By dragging Y and tracing the centroid K, one pleasantly discovers that K traces out what seems to be circular arc:

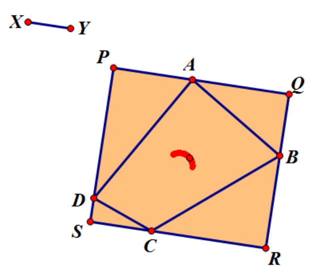


FIGURE 5. Locus of the centroid of an outbox

In order to develop the proof for the above observed phenomenon, we rely on a geometrical setup modified from Mammana [**Mam08**, p.84–86]. Given a convex quadrilateral $ABCD$, where $\angle BAD = \alpha$, $\angle CBA = \beta$, $\angle DCB = \gamma$ and $\angle ADC = \delta$, we may assume without loss of generality that α is largest interior angle, i.e., $\alpha = \max\{\alpha, \beta, \gamma, \delta\}$.

It is easy to see that $\alpha = \frac{\pi}{2}$ or α is obtuse; otherwise, $\alpha < \frac{\pi}{2}$, together with $\alpha \geq \beta, \gamma, \delta$, implies that $\alpha + \beta + \gamma + \delta < 2\pi$ which is a contradiction. If $\alpha = \frac{\pi}{2}$, then it could not be the case that one of the angles β, γ and δ is acute (i.e., strictly less than α); otherwise, $\alpha+\beta+\gamma+\delta < 2\pi$ a contradiction. So, if $\alpha = \frac{\pi}{2}$, it must be that $\beta = \gamma = \delta = \frac{\pi}{2}$ or equivalently that $ABCD$ is a rectangle. Since a rectangle clearly has an outbox, we shall only need to focus on the case where α is obtuse.

Because $\alpha \geq \beta, \gamma, \delta$, it follows that $\alpha+\beta \geq \beta+\gamma$ and $\alpha+\delta \geq \gamma+\delta$. Thus, the maximum sum of the adjacent pairs of interior angles must be equal to $\max\{\alpha + \beta, \alpha + \delta\}$. Here we deviate from the labelling convention of [**Mam08**] in that we do not specify that $\alpha + \beta$ is the largest amongst the sum of adjacent interior angles.

Suppose $PQRS$ is an outbox of $ABCD$ as shown in the Figure 6. Since $PQR = \frac{\pi}{2}$, the locus of Q is a subset of the open semi-circular arc Γ_{AB} whose diameter is AB and which lies external to $ABCD$. As to which subset this is, we shall determine it as we proceed with our discussion.

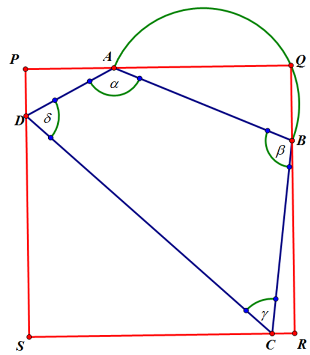


FIGURE 6. A particular outbox of an r-inscribable quadrilateral

Because PQ is a straight line segment,

$$\angle DAP + \angle BAD = \angle BQA + \angle ABQ = \frac{\pi}{2} + (\pi - \angle CBA - \angle RBC)$$

and thus, $\alpha + \beta = \angle BAD + \angle CBA = \frac{3\pi}{2} - \angle DAP - \angle RBC$. Since AQ (respectively, QR) is external to the quadrilateral $ABCD$, we have $\angle DAP > 0$ and $\angle RBC > 0$ and so,

$$\alpha + \beta < \frac{3\pi}{2}. \tag{3.1}$$

Applying the same analysis on each of the remaining sides of the quadrilateral, one arrives at the following inequalities:

$$\beta + \gamma < \frac{3\pi}{2} \tag{3.2}$$

$$\gamma + \delta < \frac{3\pi}{2} \tag{3.3}$$

$$\delta + \alpha < \frac{3\pi}{2} \tag{3.4}$$

In summary, it is necessary that the sum of any adjacent pair of interior angles is less than 3 right angles.

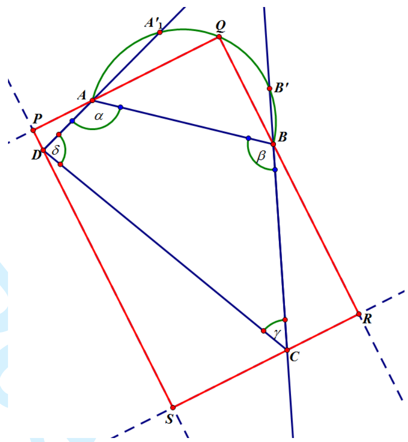
It turns out that this condition is sufficient to guarantee that $ABCD$ is r-inscribable. To see this, note that Q is the vertex of an outbox of $ABCD$ with $\angle BQA = \frac{\pi}{2}$ if and only if

(1) the straight line QA (respectively, QB) extended is external to $ABCD$, and
(2) the perpendicular to QA (respectively, QB) that passes through D (respectively, C) is external to $ABCD$.

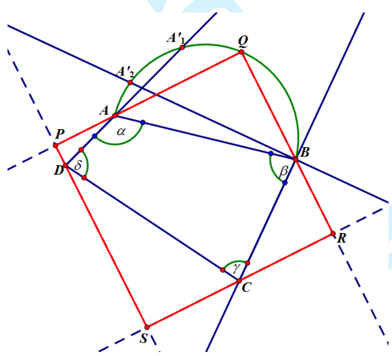
In (1), the straight line QA extended is external to $ABCD$ if and only if $\angle QAB < \pi - \alpha$. Denoting by A_1' the point of intersection of DA produced with the semicircle Γ_{AB} (see either of the diagrams in Figure 7), the upper bound described by the preceding inequality corresponds to one extreme position where $\angle A_1' AB = \pi - \alpha$.

For the necessary and sufficient condition for the straight line QB extended to be external to $ABCD$, one considers two mutually exclusive cases.

- $\beta = \frac{\pi}{2}$ or obtuse. Then, the line segment BC is tangential to Γ_{AB} or meets Γ_{AB} at some point $B' \neq B$. This extreme position B' gives rise to a lower bound for $\angle QAB$, i.e., $\angle QAB > \angle B'AB = \beta - \frac{\pi}{2}$.



- β is acute. Then, the perpendicular to BC through B meets Γ at some point A_2'. This extreme position A_2' gives rise to an upper bound for $\angle QAB$, i.e., $\angle QAB < \angle A_2'AB = \beta$.



In summary, when $\frac{\pi}{2} \leq \beta < \pi$, the condition (1) holds if and only if Q lies on the open arc $\widehat{A_1'B'}$; and when $0 < \beta < \frac{\pi}{2}$, the condition (1) holds if and only if Q lies on the open arc $\widehat{A_1'B} \cap \widehat{A_2'B}$.

As for the condition (2), one must consider the perpendicular to CD through A (respectively, through B), and label A'' (respectively, B'') the point of intersection of this perpendicular with the semicircular arc Γ. Note that AB is parallel to CD if and only if $A = A''$ and $B = B''$. Taking logical negations, this would mean that AB is not parallel to CD if and only if $A \neq A''$ or $B \neq B''$. In this case, the first possibility of $A \neq A''$ entails that $B = B''$; (see Figure 7)and by symmetry of the situation, the second possibility of $B \neq B''$ entails that $A = A''$ (see Figure 8). Note that Figures 7 and 8 illustrate only the case where $\frac{\pi}{2} \leq \beta < \pi$.

In summary, the condition (2) is equivalent to the statement that Q lies on the open arc $\widehat{A''B''}$ of Γ_{AB}. We use the following two notations:

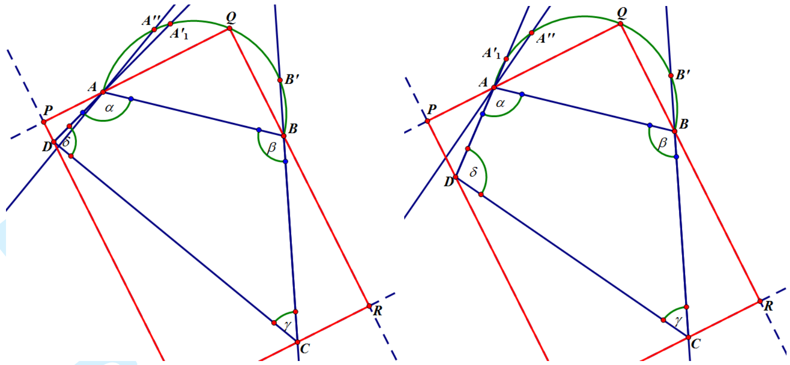


FIGURE 7. $A \neq A''$: $\angle A'_1AB < \angle A''AB$ (left), $\angle A'_1AB \geq \angle A''AB$ (right)

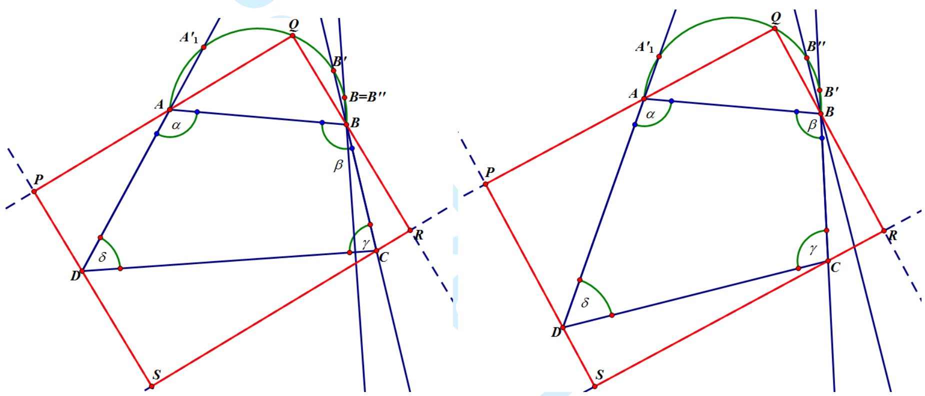


FIGURE 8. $B \neq B''$: $\angle B'AB > \angle B''AB$ (left), $\angle B'AB \leq B''AB$ (right)

(i) $x \dot{-} y$ denotes the truncated difference of x and y, i.e.,

$$x \dot{-} y := \begin{cases} x - y & \text{if } x > y; \\ 0 & \text{otherwise.} \end{cases}$$

(ii) $\hat{\beta} := \frac{\pi}{2} - \left(\frac{\pi}{2} \dot{-} \beta\right)$, i.e.,

$$\hat{\beta} := \begin{cases} \beta \text{ if } 0 < \beta < \frac{\pi}{2}; \\ \frac{\pi}{2} \text{ otherwise.} \end{cases}$$

(iii) $$\phi := \begin{cases} \frac{3\pi}{2} - \alpha - \delta & \text{if } \alpha + \delta > \pi; \\ \frac{\pi}{2} & \text{otherwise.} \end{cases}$$

Thus, we have:

PROPOSITION 3.1. *The following are equivalent for a given convex quadrilateral $ABCD$ and a point Q on Γ_{AB}:*

(1) *Q is the vertex of an outbox for $ABCD$ with $\angle BQA = \frac{\pi}{2}$.*
(2) *Q lies on the non-empty intersection of the open arcs $\widehat{A'_1B'}$, $\widehat{A'_1B}$, $\widehat{A'_2B}$ and $\widehat{A''B''}$ of Γ_{AB}.*

(3) *The acute angle* $\angle QAB$ *satisfies the inequalities:*

$$\max\{\angle B'AB, \angle B''AB\} < \angle QAB < \min\{\angle A'_1AB, \angle A'_2AB, \angle A''AB\}.$$

(4) *The acute angle* $\angle QAB$ *satisfies the inequalities:*

$$\max\{\beta \dot{-} \frac{\pi}{2}, \beta + \gamma \dot{-} \pi\} < \angle QAB < \min\{\pi - \alpha, \hat{\beta}, \phi\}.$$

In particular, when $\alpha + \beta$ *is the largest amongst possible sums of adjacent interior angles, the condition* $\alpha + \beta < \frac{3\pi}{2}$ *is equivalent to (4), and hence (1), i.e.,* $ABCD$ *has an outbox.*

Note that $ABCD$ has an outbox if and only if (2) holds, i.e., the intersection of the open arcs $\widehat{A'_1B'}$, $\widehat{A'_1B}$, $\widehat{A'_2B}$ and $\widehat{A''B''}$ is non-empty. We denote this non-empty intersection of these open arcs by the open arc $\widehat{A'''B'''}$, where A''' is either A'_1, A'_2 or A'', depending on the comparison between $\angle A'_1AB$, $\angle A'_2AB$ and $\angle A''AB$, and B''' is either B' or B'' depending on the comparison between $\angle B'AB$ and $\angle B''AB$.

From this point till the end of Section 5, we assume that the convex quadrilateral $ABCD$ is not a parallelogram. The case of parallelograms is taken up in Section 6. Let W be the intersection of the diagonals AC and BD of the given r-inscribable convex quadrilateral. In addition to that labelling convention of $\angle DAB := \alpha$ being the largest interior angles, from this point onwards we also adopt the orientation of $\angle AWB := \vartheta$ is either $\frac{\pi}{2}$ or obtuse. This labelling convention is the same as that in Section 3 of [**Mam08**].

LEMMA 3.2. *Let ABCD be an r-inscribable convex quadrilateral.*

(1) *For each point* Q *of* Γ_{AB} *(as described above), the centroid* K_Q *of the rectangle* $\mathscr{R}_Q := PQRS$*, formed by the extensions of* QA *and* QB*, and the perpendiculars to* QA *through* D *and to* QB *through* C*, lies on the circle* Λ *with diameter* LN*.*
(2) *The assignment* $Q \mapsto K_Q$ *in (i) defines an injective function* $K : \Gamma_{AB} \longrightarrow \Lambda$*.*

PROOF. (1) Given Q on Γ_{AB}, let P, R and S be the rest of the vertices of the rectangle $\mathscr{R}_Q$ formed by the extensions of QA and QB, and the perpendiculars to QA through D and to QB through C. Note that $\mathscr{R}_Q := PQRS$ may not be an outbox of $ABCD$ since it might well be the case that Q lies outside the open arc $\widehat{A'''B'''}$. For the ensuing argument, refer to Figure 9. Let U be the midpoint of PQ, V that of RS, X that of PS and Y that of QR. It is clear that the line segments XY and UV are respectively parallel to PQ and QR. Moreover, XY is the perpendicular bisector of PQ and SR, while XY is the perpendicular bisector of PS and QR. Clearly, the centroid K_Q of the rectangle $\mathscr{R}_Q := PQRS$ is the intersection of UV and XY.

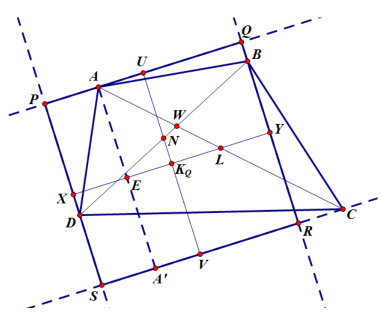


FIGURE 9. Construction for proof of (1)

We first show that XY meets AC at L the midpoint of AC. To this end, construct the perpendicular AA' to SR that passes through A. Since XY bisects PS and QR, it must also bisect AA' at E. Since LE is parallel to CA', it follows that the triangle LAE is similar to the triangle CAA'. Moreover, because E bisects AA', one has that L bisects AC as desired. Similarly, one can conclude that UV meets BD at N, the midpoint of BD. Since XY and UV are perpendicular, together with L lying on XY and N on UV, it follows that $\angle NK_QL$ is a right angle. Because Q is an arbitrary point on Γ_{AB}, it then follows from Thale's theorem that the locus of K_Q lies on the circle Λ with diameter LN.

(2) Since every rectangle $\mathscr{R}_Q$ possesses only one centroid K_Q and by (1) $K_Q \in \Lambda$, it follows that the assignment $K : Q \mapsto K_Q$ is a function from Γ_{AB} to Λ. It remains to show that it is injective. Let l_A be the perpendicular to AB through L and l_B be the parallel to AB through L (see Figure 10). Denote by K_A (respectively, K_B) the intersection of l_A (respectively, l_B) with the circle Λ (other than the point L whenever there are two points of intersection). Since l_A and l_B are perpendicular by construction, it follows that $\angle K_BLK_A = \frac{\pi}{2}$ so that by Thale's theorem K_A and K_B are diametrically opposite.

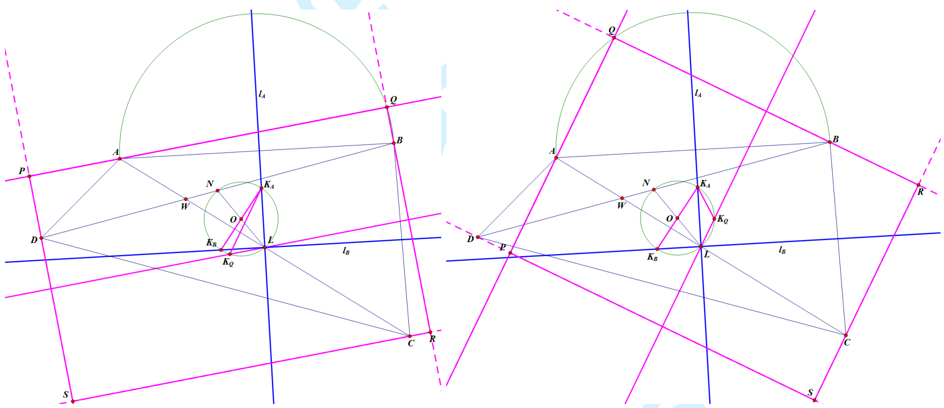


FIGURE 10. K_A and K_B

We claim that $\angle QAB = \angle K_QK_AK_B$ for all $Q \in \Gamma_{AB}$. There are two possible cases to prove:

- K_A and L are on the same side with respect to the chord K_BK_Q (see Figure 10(top)).
  By the inscribed angle theorem, $\angle K_QK_AK_B = \angle K_QLK_B$. Notice that the latter angle is that made between the lines l_B and K_QL produced. Since these two lines are respectively parallel to AB and AQ, it follows by the virtue of corresponding angles that $\angle QAB = \angle K_QLK_B$ and hence $\angle QAB = \angle K_QK_AK_B$.
- K_A and L are on opposite sides of the chord K_BK_A (see Figure 10(bottom)).
  By the cyclic quadrilateral theorem, $\angle K_QK_AK_B = \pi - \angle K_BLK_Q$. But the latter angle is equal to the acute angle between the line K_QL and l_B, and in turn this equal to $\angle QAB$ due to corresponding angles. Hence $\angle QAB = \angle K_QK_AK_B$.

Suppose that $K_Q = K'_Q$ for some $Q,\ Q' \in \Gamma_{AB}$. Then, by the preceding result, $\angle QAB = \angle K_Q K_A K_B = \angle K_{Q'} K_A K_B = \angle Q'AB$. Since Q and $Q' \in \Gamma_{AB}$, we must have $Q = Q'$. This then completes the proof that the function $K : Q \mapsto K_Q$ is injective.

□

We now turn to the problem of determining the locus of the centroid K_Q of an outbox $PQRS$ of $ABCD$, where $Q \in \Gamma_{AB}$. This is equivalent to locating all the possible positions of K_Q on the circle Λ as Q moves on the open arc $\widehat{A'''B'''}$ as described earlier. In what follows, we continue to denote by $\mathscr{R}_Q$ the rectangle formed by the extensions of QA and QB, and the perpendiculars to QA through D and to QB through C (where Q is a point on the open semi-circular arc Γ_{AB}). We also use the notations K_A and K_B as described above.

Define the open semi-circular arc $\Lambda_{K_A K_B}$ of Λ to be that with diameter $K_A K_B$ such that for every point M on it, $0 < \angle M K_A K_B < \frac{\pi}{2}$. Note that by our sign convention of angles, $\angle M K_A K_B$ has a counterclockwise sense and so the open semi-circular arc $\Lambda_{K_A K_B}$ is uniquely determined.

PROPOSITION 3.3. *Let $ABCD$ be an r-inscribable convex quadrilateral. Then, the co-restriction of the function K on the open semi-circular arc $\Lambda_{K_A K_B}$, i.e.,*

$$K : \Gamma_{AB} \longrightarrow \Lambda_{K_A K_B},\ Q \mapsto K_Q$$

is a bijection between Γ_{AB} and $\Lambda_{K_A K_B}$. Indeed, the locus of K_Q as Q varies on the open semi-circular arc Γ_{AB} is exactly the open semi-circular arc $\Lambda_{K_A K_B}$.

PROOF. By Lemma 3.2, it suffices to show that this co-restriction of K on $\Lambda_{K_A K_B}$ is surjective. To this end, let M be any point on $\Lambda_{K_A K_B}$. Construct a line k parallel to LM through A. Since $\angle M K_A K_B$ is equal to the acute angle between $K_B L$ and LM, it follows that the angle between k and AB which is equal to $\angle M K_A K_B$ is acute. Thus, k meets Γ_{AB} non-emptily at some point Q_M. We now show that $K_{Q_M} = M$. First construct the rectangle $\mathscr{R}_Q := PQRS$ which is formed by the lines AQ and QB produced and the perpendiculars to AQ and QB through D and C respectively. It follows that PQ is parallel to LM and to SR. Since L is the midpoint of AC, it follows from arguments involving similar triangles that LM extended bisects QR at a point Y, and PS at a point X so that XY is a median of the rectangle $\mathscr{R}_Q$. Similarly, the line MN extended bisects PQ at a point U and SR at a point V since N is the midpoint of BD. So, UV is the other median of the rectangle $\mathscr{R}_Q$. Since M is the intersection of the medians XY and UV by construction, it follows that M is the centroid of the rectangle $\mathscr{R}_Q$. This proves that $K_{Q_M} = M$, and thus the co-restriction $K : \Gamma_{AB} \longrightarrow \Lambda_{K_A K_B}$ is surjective, as desired. □

Since $Q \in \Gamma_{AB}$ is completely determined by $\angle QAB$ and $K_Q \in \Lambda_{K_A K_B}$ by $\angle K_Q K_A K_B$, the equality $\angle QAB = \angle K_Q K_A K_B$ then allows one to perceive the bijection K as the identity map on the open interval $\left(0, \frac{\pi}{2}\right)$. Proposition 3.1 asserts that $Q \in \Gamma_{AB}$ is the vertex of an outbox $PQRS$ (with $\angle AQB = \frac{\pi}{2}$) if and only if

$$\max\{\angle B'AB, \angle B''AB\} < \angle QAB < \min\{\angle A'_1 AB, \angle A'_2 AB, \angle A'' AB\}$$

Applying the bijection K, it follows that

$$\max\{\angle K_{B'} K_A K_B, \angle K_{B''} K_A K_B\} < \angle K_Q K_A K_B < \min\{\angle K_{A'_1} K_A K_B, \angle K_{A'_2} K_A K_B, \angle K_{A''} K_A K_B\}$$

or equivalently, $\angle K_{B'''} K_A K_B < \angle K_Q K_A K_B < \angle K_{A'''} K_A K_B$, i.e.,

$$\angle B''' AB < \angle K_Q K_A K_B < \angle A''' AB.$$

Thus, by Proposition 3.3, we have established the main theorem of this paper:

THEOREM 3.4 (Outbox Centroid Theorem). *Let $ABCD$ be an r-inscribable convex quadrilateral. The locus of the centroid K_Q of an outbox $PQRS$ of $ABCD$ is an open arc $\widehat{K_{A'''}K_{B'''}}$ of the semi-circular arc $\Lambda_{K_AK_B}$ (as described above) such that*

$$\angle K_{B'''}K_AK_B < \angle K_QK_AK_B < \angle K_{A'''}K_AK_B.$$

REMARK 3.5. (1) Every parallelogram has an outbox since the sum of adjacent angles is always π (which is less than $\frac{3\pi}{2}$). In the case where $ABCD$ is a parallelogram, the points L and N coincide with K. Thus, the locus of the centroid of an outbox reduces to a point (i.e., the radius of the circle Λ is zero). Thus, the case of a parallelogram can be seen as a limiting case of what we are considering in this section. For completeness, we have included a separate treatment for the maximal outbox for the case of parallelograms for readers who prefer a 'limit-free' approach to the present one (see Section 6).

(2) Notice that Theorem 3.4 gives the complete answer to both questions (A) and (B) raised at the beginning of this section.

(3) Our DGS-aided discovery made in the preceding theorem exploits the *wandering dragging* approach - a method described in [**ABMS12**] as "moving the basic point(s) on the screen randomly, without plan, in order to discover interesting configurations or regularities in the figures". In our case, the basic point is Y and the regularity is the locus of the centroid of the outbox. For the use of dragging in dynamic geometry environment, the reader is referred to [**Leu08, LRL06**].

4. Characteristic triangles

Amongst all the possible outboxes of a given r-inscribable convex quadrilateral $ABCD$, which, if it exists, is the one with the maximum area? Further experimentation using GSP reveals more.

Let I (respectively, J) be the foot of the perpendicular from L (respectively, N) to the diagonal BD (respectively, AC). See Figure 11 (left). Recall also that $\angle DAB := \alpha$ is the largest of the interior angles of $ABCD$ and $\angle AWB := \vartheta$ is either $\frac{\pi}{2}$ or obtuse.

By construction $\angle LIB = \angle LIN = \frac{\pi}{2}$ and LN is the diameter of the circle Λ (as defined in the preceding section), it follows that I coincides with the point of intersection of the diagonal BD with the circle Λ. Likewise, J coincides with the point of intersection of the diagonal AC with the circle Λ.

In the process of our experimentation, we observe something very special about the triangle IKJ that corresponds to a given outbox $PQRS$ (we call this the characteristic triangle of $PQRS$). Here K is the centroid of the outbox $PQRS$. Whenever the area of $PQRS$ (denoted by $[PQRS]$) collapses to 0 (in which case this is an 'illegal' outbox), the area of IKJ (denoted by $[IKJ]$) is 0. This leads us to conjecture that the ratio of the area of an outbox to that of its characteristic triangle is a *constant*. We are convinced by compelling evidence using DGS (see Figure 11).

Our observations made in Figure 11 using DGS show clearly that as K moves along the circle Λ, the angle IKJ is constant by virtue of the inscribed angle theorem. This indicates that the lengths of IK and JK are the only measurements which completely determine the area of the triangle IKJ by virtue of the sine rule. This train of thought leads us to the following lemma.

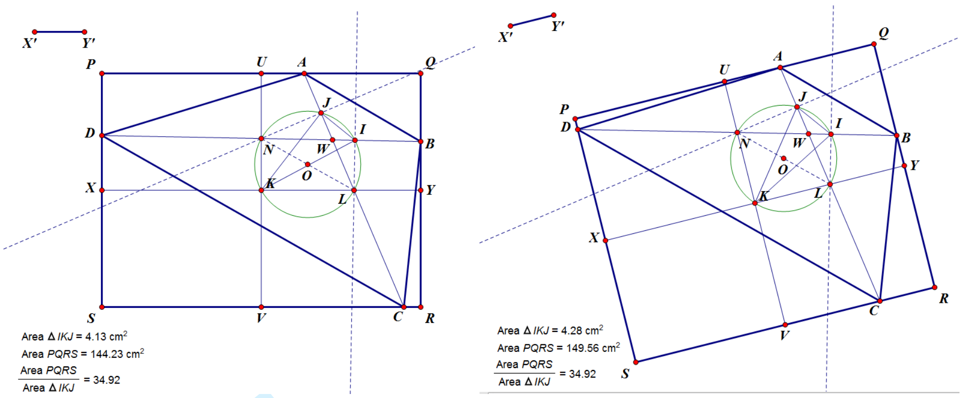


FIGURE 11. Two characteristic triangles

LEMMA 4.1. *Let $ABCD$ be a fixed r-inscribable convex quadrilateral as shown in Figure 13. The point U (respectively, V) is the midpoint PQ (respectively, SR) while the point X (respectively, Y) is the midpoint of PS (respectively, QR). Then for any outbox*

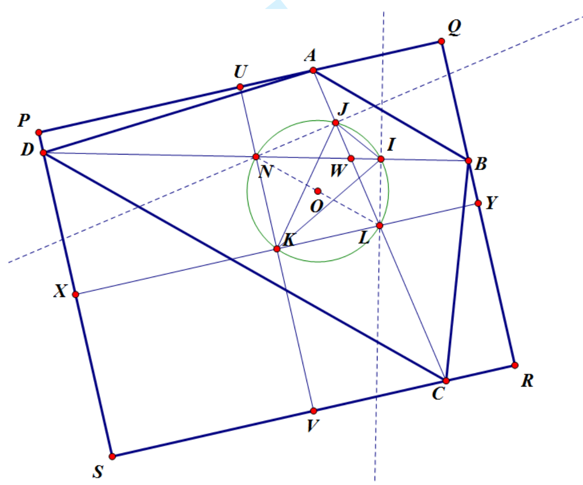


FIGURE 12. Characteristic triangle of $PQRS$

$PQRS$ of $ABCD$,

$$\frac{IK}{XK} = \frac{LN}{DN} \text{ and } \frac{JK}{VK} = \frac{NL}{CL},$$

where IKJ denotes the characteristic triangle of $PQRS$. In particular, these ratios are invariants over all possible outboxes $PQRS$ of $ABCD$.

PROOF. Since the segment IL subtends both $\angle IKL$ and $\angle INL$, by the inscribed angle theorem it follows that $\angle IKL = \angle INL$. So, $\angle XKI = \angle DNL$. Next we show that $DXLI$ is a cyclic quadrilateral. To this end, note that $\angle LXD = \frac{\pi}{2}$ by construction. Also, $\angle LIN = \frac{\pi}{2}$ because I is the foot of perpendicular to BD from L. Since D, N and I (irrespective of the order) are collinear, it follows that $\angle LID = \angle LIN = \frac{\pi}{2}$. By the cyclic quadrilateral theorem, $DXLI$ is a cyclic quadrilateral and so, by the inscribed angle theorem, $\angle LDI = \angle LXI$. So, $\angle LDN = \angle KXI$. Hence $\triangle IKX$ is similar to $\triangle LND$. Consequently, $\frac{IK}{XK} = \frac{LN}{DN}$. Since the points D, L and N are fixed for a given r-inscribable convex quadrilateral $ABCD$, it follows that the ratio $\frac{IK}{XK}$ is invariant over all outboxes $PQRS$ of $ABCD$. Similarly, one can show that $\frac{JK}{VK} = \frac{NL}{CL}$ is also invariant over all possible outboxes $PQRS$ of $ABCD$. □

THEOREM 4.2. *Let $ABCD$ be a fixed r-inscribable convex quadrilateral. Then the ratio of the area of any outbox $PQRS$ to that of its characteristics triangle, i.e., $[PQRS] : [IKJ]$, is an invariant over all possible outboxes $PQRS$.*

PROOF. Suppose $PQRS$ is an outbox of $ABCD$ and K is the centroid of $PQRS$. By the sine rule, the area of $\triangle IKJ$ is $\frac{1}{2} IM \cdot JM \sin\angle IKJ$. Because IJ is fixed chord of the circle Λ, $\angle IKJ$ and hence $\sin\angle IKJ$ is a constant over all possible outboxed by the inscribed angle theorem. Applying Lemma 4.1, it follows that

$$[IKJ] = \frac{1}{2}\left(XM \cdot \frac{LN}{DN}\right)\left(VM \cdot \frac{NL}{CL}\right)\sin\angle IKJ = 2\left(\frac{LN^2}{CL \cdot DN}\right)\sin\angle IKJ \cdot [PQRS].$$

Since $k = 2\left(\frac{LN^2}{CL \cdot DN}\right)\sin\angle IKJ$ is an invariant over all possible outboxes $PQRS$, the desired result follows. □

COROLLARY 4.3. *Let $ABCD$ be a fixed r-inscribable convex quadrilateral, and the points I and J are defined as above. Then the maximal outbox of $ABCD$, if it exists, is achieved when its centroid K is at the point M on Λ which is furthest away from the chord IJ.*

PROOF. Assume the existence of some maximal outbox of $ABCD$. By Theorem 4.2, a characteristic triangle with the maximum area yields a maximal outbox. In turn, a characteristic triangle (with a fixed base IJ) attains maximum area when the vertex K is the point on Λ which is furthest away from IJ. □

Assuming for the moment the given r-inscribable convex quadrilateral has a maximal outbox, we derive the formula for its area.

Denote the point of intersection of the diagonals of the given r-inscribable convex quadrilateral by W. For the purpose of analyzing the position of the centroid of the maximal outbox (assuming it exists), we zoom into the circle Λ and a characteristic triangle IJK, where K lies on the major arc subtended by the chord IJ. Since $\angle AWB := \vartheta$ is $\frac{\pi}{2}$ or obtuse, we have only two possible situations: (1) I and N are on the same side with respect to W along the diagonal BD, or (2) I and N are on opposite side with respect to W along the diagonal BD.

Assume first that I and N are on the same side with respect to W, as shown in the two situations of Figure 13.

In first situation as shown in Figure 13(left), I and N lie on the same side of W along the diagonal BD. By the inscribed angle theorem, $\angle IKJ = \angle INJ$. So $\angle IKJ = \angle INJ = \angle JWI - \angle WJN = \vartheta - \frac{\pi}{2}$. In the second situation as shown in Figure 13(right), I and N lie on opposite sides of W along the diagonal BD. So, $\angle IKJ = \pi - \angle INJ = \pi - (\angle NWJ + \angle NJW) = \pi - \left(\pi - \angle AWB + \frac{\pi}{2}\right) = \vartheta - \frac{\pi}{2}$.

Thus, we have:

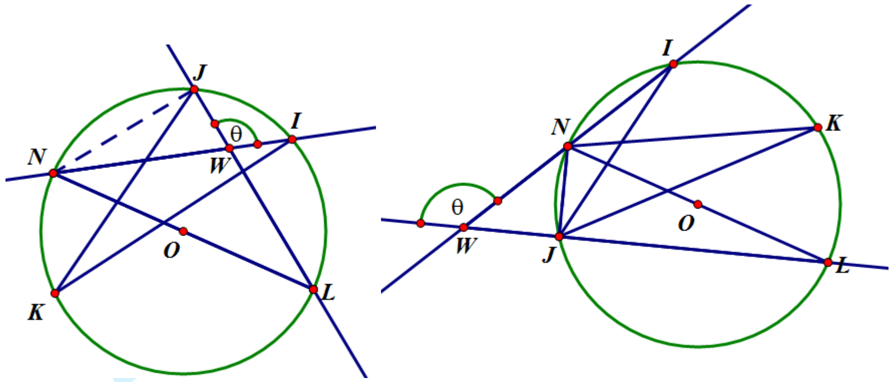


FIGURE 13. Zoom-in

LEMMA 4.4. *Let $ABCD$ be a given r-inscribable convex quadrilateral and the points W, I and J as defined above. Then for any outbox $PQRS$ whose centroid is K, we have*

$$\angle JKI = \angle AWB - \frac{\pi}{2}.$$

THEOREM 4.5. *Let $ABCD$ be an r-inscribable convex quadrilateral whose diagonals AC and BD are of length d_1 and d_2 respectively, and make an angle of $\angle AWB := \vartheta$ (where $\frac{\pi}{2} \leq \vartheta < \pi$). Then, the area of the maximal outbox of $ABCD$, if it exists, is given by the formula*

$$\frac{1}{2} d_1 d_2 (1 + \sin\vartheta).$$

PROOF. From Lemma 4.1, we have $\frac{JK}{VK} = \frac{LN}{LC}$ and $\frac{IK}{XK} = \frac{LN}{DN}$. Because $LC = \frac{1}{2}d_1$ and $DN = \frac{1}{2}d_2$, we have $VK = \frac{JK}{LN} \cdot \frac{1}{2}d_1$ and $XK = \frac{IK}{LN} \cdot \frac{1}{2}d_2$. Denoting the radius of the circle Λ by r, we have $LN = 2r = 2 \cdot OK$. When K represents the centroid of the maximal outbox, K is the furthest point on the circle Λ away from IJ by Corollary 4.3 as shown in the figure below:

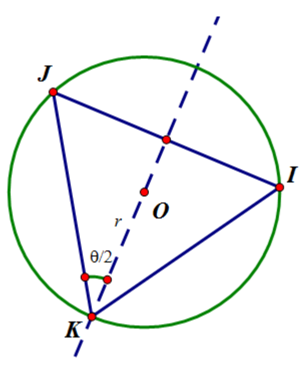


FIGURE 14. Maximal outbox achieved when K is furthest from IJ

Thus, by Lemma 4.4, $\angle OKJ = \angle IKO = \frac{1}{2}\left(\vartheta - \frac{\pi}{2}\right)$. So, at this position where K is the centroid of the maximal outbox, $VK = \frac{1}{2}d_1 \cdot \cos\frac{1}{2}\left(\vartheta - \frac{\pi}{2}\right)$ and and $XK = \frac{1}{2}d_2 \cdot \cos\frac{1}{2}\left(\vartheta - \frac{\pi}{2}\right)$.

Finally, by the sine rule and the double angle formula, the area of the maximal outbox, if it exists, is given by $4\cdot\frac{1}{2}d_1\cdot\cos\frac{1}{2}\left(\vartheta-\frac{\pi}{2}\right)\cdot\frac{1}{2}d_2\cdot\cos\frac{1}{2}\left(\vartheta-\frac{\pi}{2}\right)=\frac{1}{2}d_1d_2(1+\sin\vartheta)$. □

REMARK 4.6. The formula for the area of the maximal outbox derived in [**Zha10**] by D. Zhao was $d_1d_2|\cos(\frac{\pi}{4}-\frac{\vartheta}{2})\sin(\frac{\pi}{4}+\frac{\vartheta}{2})|$. One can show easily that our formula above is equivalent to his using the factor formula.

5. Existence of maximal outbox

We finally turn to the problem of characterizing those r-inscribable convex quadrilaterals that admit maximal outboxes. By Proposition 3.1 and Corollary 4.3, it suffices to find the necessary and sufficient condition for the following inequalities

$$\max\{\beta\dot{-}\frac{\pi}{2},\beta+\gamma\dot{-}\pi\}<\angle QAB<\min\{\pi-\alpha,\hat{\beta},\phi\}$$

to hold at the instant when K_Q is the furthest point on Λ from the chord IJ.

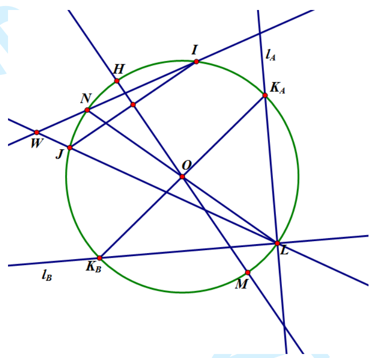


FIGURE 15. Analysis of the size of $\angle K_QK_AK_B$

For this purpose, it is important to relate the size of $\angle K_QK_AK_B$ (which is equal to $\angle QAB$ in size) with the geometrical structure of $ABCD$. Let H be the diametrically opposite of M with respect to O (see Figure 15).

Since H and M are diametrically opposites (respectively, K_A and K_B), it follows that the chords HK_A and MK_B are of the same length. Thus, the angles subtended by these chords are equal in size, i.e., $\angle MK_AK_B=\angle HMK_A$. But we have $\angle HMK_A=\angle HMI+\angle IMK_A$. Now, $\angle HMI=\frac{1}{2}\angle JMI$, and $\angle JMI=\angle JLI=\vartheta-\frac{\pi}{2}$. Also, $\angle IMK_A=\angle ILK_A=\angle DBA$. Thus, $\angle HMK_A=\frac{1}{2}(\vartheta-\frac{\pi}{2})+\angle DBA=\frac{1}{2}(\vartheta-\frac{\pi}{2})+\pi-\angle BAC-\vartheta=\frac{3\pi}{4}-\frac{1}{2}\vartheta-\angle BAC$.

Hence, $ABCD$ has a maximal outbox if and only if each of the six inequalities are satisfied for the acute angle $\angle HMK_A$:

$$\max\{\beta\dot{-}\frac{\pi}{2},\beta+\gamma\dot{-}\pi\}<\angle QAB=\angle HMK_A<\min\{\pi-\alpha,\hat{\beta},\phi\}$$

These inequalities can be presented as follows:

(1) $0<\angle HMK_A<\frac{\pi}{2}$. This is equivalent to

$$0<\frac{3\pi}{4}-\frac{1}{2}\vartheta-\angle BAC<\frac{\pi}{2}\iff\frac{\pi}{4}-\frac{1}{2}\vartheta<\angle BAC<\frac{3\pi}{4}-\frac{1}{2}\vartheta.$$

But $\vartheta\geq\frac{\pi}{2}$ so that $\frac{1}{2}\vartheta\geq\frac{\pi}{4}$. Since $\angle BAC>0$, it follows that $\angle BAC>\frac{\pi}{4}-\frac{1}{2}\vartheta$ is already satisfied. Thus, $\angle BAC<\frac{3\pi}{4}-\frac{1}{2}\vartheta$.

(2) $\beta\dot{-}\frac{\pi}{2}<\angle HMK_A$.

(i) If $\beta > \frac{\pi}{2}$, we have $\beta - \frac{\pi}{2} < \frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC$ if and only if

$$\angle BAC + \beta - \pi + \frac{1}{2}\vartheta < \frac{\pi}{4} \iff \frac{1}{2}\vartheta - \angle ACB < \frac{\pi}{4} \iff \angle ACB > \frac{1}{2}\vartheta - \frac{\pi}{4}.$$

(ii) If $\beta \le \frac{\pi}{2}$, we have $0 < \frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC \iff \angle BAC < \frac{3\pi}{4} - \frac{1}{2}\vartheta$, which is equivalent to (1). Notice also that if this condition holds, then one also has $\angle ACB > \pi - \beta - \frac{3\pi}{4} + \frac{1}{2}\vartheta \iff \angle ACB > \frac{1}{2}\vartheta - \frac{\pi}{4} + \frac{\pi}{2} - \beta \ge \frac{1}{2}\vartheta - \frac{\pi}{4}$ so that the inequality in (2)(i) is also true.

(3) $\beta + \gamma \dot{-} \pi < \angle HMK_A$.

(i) If $\beta + \gamma > \pi$, we have $\beta + \gamma - \pi < \frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC$ if and only if

$$\angle BAC + \beta - \pi + \gamma < \frac{3\pi}{4} - \frac{1}{2}\vartheta \iff \angle DCA < \frac{3\pi}{4} - \frac{1}{2}\vartheta.$$

(ii) If $\beta + \gamma \le \pi$, we have $0 < \frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC \iff \angle BAC < \frac{3\pi}{4} - \frac{1}{2}\vartheta$, which is equivalent to (1). Furthermore, this condition also implies that

$$\pi + \angle DCA - \gamma - \beta < \frac{3\pi}{4} - \frac{1}{2}\vartheta \iff \angle DCA < \frac{3\pi}{4} - \frac{1}{2}\vartheta - (\pi - \gamma - \beta)$$

which implies that $\angle DCA < \frac{3\pi}{4} - \frac{1}{2}\vartheta$, i.e., the inequality in (3)(i) also holds.

(4) $\angle HMK_A < \pi - \alpha$. We have $\frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \pi - \alpha$ if and only if

$$\alpha - \angle BAC < \frac{\pi}{4} + \frac{1}{2}\vartheta \iff \angle DAC < \frac{\pi}{4} + \frac{1}{2}\vartheta.$$

Since $\angle DAC + \angle ADB = \vartheta$, the preceding inequality is equivalent to $\angle ADB > \frac{1}{2}\vartheta - \frac{\pi}{4}$.

(5) $\angle HMK_A < \hat{\beta}$.

(i) If $\beta < \frac{\pi}{2}$, we have $\frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \beta$ if and only if

$$\frac{3\pi}{4} - \frac{1}{2}\vartheta < \pi - \angle ACB \iff \angle ACB < \frac{\pi}{4} + \frac{1}{2}\vartheta.$$

(ii) If $\beta \ge \frac{\pi}{2}$, we have $\frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \frac{\pi}{2} \iff \angle BAC > \frac{\pi}{4} - \frac{1}{2}\vartheta$, which is equivalent to 1(i). Moreover, this inequality also implies 5(i) because

$$\angle ACB < \pi - \beta - \frac{\pi}{4} + \frac{1}{2}\vartheta \implies \angle ACB < \frac{\pi}{4} + \frac{1}{2}\vartheta - \left(\beta - \frac{\pi}{2}\right) < \frac{\pi}{4} + \frac{1}{2}\vartheta \implies \angle ACB < \frac{\pi}{4} + \frac{1}{2}\vartheta.$$

(6) $\angle HMK_A < \phi$.

(i) If $\alpha + \delta > \pi$, we have $\frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \frac{3\pi}{2} - \alpha - \delta$ if and only if

$$\alpha + \delta - \angle BAC < \frac{3\pi}{4} + \frac{1}{2}\vartheta \iff \angle DAC + \delta < \frac{3\pi}{4} + \frac{1}{2}\vartheta \iff \angle BDC < \frac{3\pi}{4} - \frac{1}{2}\vartheta.$$

(ii) If $\alpha + \delta \le \pi$, then $\frac{3\pi}{4} - \frac{1}{2}\vartheta - \angle BAC < \frac{\pi}{2} \iff \angle BAC > \frac{\pi}{4} - \frac{1}{2}\vartheta$, which is just (1). Also, one has $\angle BDC < \pi - \left(\frac{\pi}{4} - \frac{1}{2}\vartheta\right) - \angle DAC - \angle ADB \implies \angle BDC < \frac{3\pi}{4} - \frac{1}{2}\vartheta$ since $\angle DAC + \angle ADB = \pi - \vartheta$. So, the inequality in 6(i) holds.

Note that since $\vartheta \ge \frac{\pi}{2}$ holds and $\angle DCB > 0$, it holds that $\angle DCB > 0 > \frac{\pi}{4} - \frac{1}{2}\vartheta$. So, $\angle DCB + \angle DCA + \vartheta = \pi$ then guarantees that $\angle DCA = \pi - \vartheta - \angle DCB$, and thus $\angle DCA < \pi - \vartheta - \frac{\pi}{4} + \frac{1}{2}\vartheta$ if and only if $\angle DCA < \frac{3\pi}{4} - \frac{1}{2}\vartheta$. Similarly, since $\angle DCA > 0$, it holds that $\angle DCA > 0 > \frac{\pi}{4} - \frac{1}{2}\vartheta$. Hence $\angle DCB + \angle DCA + \theta = \pi$ so that $\angle DCB < \pi - \vartheta - \frac{\pi}{4} + \frac{1}{2}\vartheta$ if and only if $\angle DCB < \frac{3\pi}{4} - \frac{1}{2}\vartheta$. Thus, the configuration that $\vartheta \ge \frac{\pi}{2}$ we assume guarantees that the inequalities in 3(i) and 6(i) to hold automatically.

All in all, a new proof has been found for:

THEOREM 5.1. ([**Mam08**, Theorem 4]) *An r-inscribable quadrilateral $ABCD$ has a maximal outbox if and only if $\vartheta = \frac{\pi}{2}$ or obtuse and the following inequalities are simultaneously satisfied:*

$$\angle BAC < \frac{3\pi}{4} - \frac{1}{2}\vartheta \tag{5.1}$$

$$\angle ACB > \frac{1}{2}\vartheta - \frac{\pi}{4} \tag{5.2}$$

$$\angle ADB > \frac{1}{2}\vartheta - \frac{\pi}{4} \tag{5.3}$$

$$\angle ACB < \frac{1}{2}\vartheta + \frac{\pi}{4} \tag{5.4}$$

6. Maximal outbox of parallelograms

We treat the special case where the convex quadrilateral $ABCD$ is a parallelogram. Suppose $ABCD$ is a parallelogram whose diagonals AC and BD (of lengths d_1 and d_2 respectively) make an angle $\angle AWB := \vartheta$, where W is the intersection of the two diagonals. Again, we may assume, without loss of generality, that $\frac{\pi}{2} \le \vartheta < \pi$ (see Figure 16). Since the sum of any adjacent pairs of interior angles of a parallelogram is $\pi < \frac{3\pi}{2}$, it follows that all parallelograms are r-inscribable. Let $PQRS$ be an outbox of $ABCD$, and W their common centroid. Denote by X (respectively, U) the foot of perpendicular from W to PS (respectively, PQ) and by χ the angle $\angle BWU$. Since $\angle BWA = \vartheta$, we have $\angle AWX = \frac{\pi}{2} - (\vartheta - \chi)$. It follows easily that $PQRS$ has an area of

$$4 \cdot BW \cos\vartheta \cdot AW \cos\left(\frac{\pi}{2} - (\vartheta - \chi)\right) = \frac{1}{2} d_1 d_2 (\sin\vartheta + \sin(\vartheta - 2\chi)).$$

Thus, the area of the maximal outbox, if it exists, is achieved at $\chi = \frac{\vartheta}{2} - \frac{\pi}{4}$ is also given by $\frac{1}{4} d_1 d_2 (1 + \sin\vartheta)$. By the rotational symmetry of the situation, it is clear that the condition

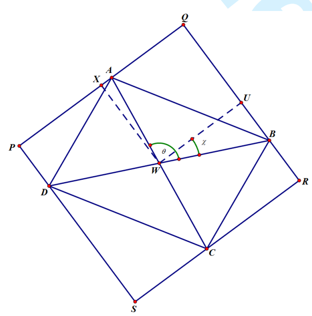


FIGURE 16. Maximal outbox of a parallelogram

$\chi = \frac{1}{2}\vartheta - \frac{\pi}{4}$ is achievable if and only if PQ is external to the parallelogram if and only if $0 < \angle PAD < \frac{\pi}{2}$. But when $\chi = \frac{1}{2}\vartheta - \frac{\pi}{4}$, $\angle PAD = \vartheta - \chi - \angle DAC = \frac{1}{2}\vartheta + \frac{\pi}{4} - \angle DAC$. Thus,

$$0 < \angle PAD < \frac{\pi}{2} \iff 0 < \frac{1}{2}\vartheta + \frac{\pi}{4} - \angle DAC < \frac{\pi}{2} \iff \frac{1}{2}\vartheta - \frac{\pi}{4} < \angle DAC < \frac{1}{2}\vartheta + \frac{\pi}{4}.$$

Note that for parallelogram $ABCD$, we have $\angle ACB = \angle DAC$ and $\angle ADB = \vartheta - \angle DAC$, it is easy to show that $\frac{1}{2}\vartheta - \frac{\pi}{4} < \angle DAC < \frac{1}{2}\vartheta + \frac{\pi}{4}$ is equivalent to the set of four simultaneous inequalities in Theorem 5.1. So, for the special case when $ABCD$ is a parallelogram, Theorem 5.1 holds.

7. Conclusion

In this paper, we make use of Dynamic Geometry Software to discover a new theorem in geometry, Outbox Centroid Theorem, concerning outboxes. This theorem provides a direct connection between the locus of a specified vertex of an outbox of a fixed r-inscribable quadrilateral and that of its centroid. This connection, which is a certain one-to-one correspondence, yields a new proof of Mammana's theorem ([**Mam08**, Theorem 4]) for characterisation of r-inscribable convex quadrilaterals which admit maximal outboxes. The maximal outbox problem can be seen as a generalization of the maximal 'out-triangle' problem. The older 'out-triangle' problem proposed in [**Fai93**] and [**Pet46**] has also been earlier studied and solved completely in [**MM05**]. This out-triangle problem was again solved independently in [**FDH14**]. It turns out that for any given triangle $\mathscr{T}$ the set F of equilateral triangles circumscribed to $\mathscr{T}$ is non-empty. Furthermore, if A, B and C are vertices of the triangle $\mathscr{T}$, such that $AB \geq AC \geq BC$, among the triangles of the set F there exists one of maximum area (i.e., a maximal 'out-triangle') if and only if the median of the side AB with the side BC forms an angle smaller than $\frac{5\pi}{6}$. It is natural to conjecture that a similar kind of centroid theorem exists for the case of triangles (or even more generally any convex polygon), and can be used to give an alternative proof of the aforementioned result. Again, preliminary investigation using DGS has already revealed compelling evidence that seems to support this conjecture.

References

[ABMS12] F. Arzarello, M. G. B. Bussi, M. A. Mariotti, and I. Stevenson, *Proof and Proving in Mathematics Education*, New ICMI Study Series, vol. 15, ch. Experimental approaches to theoretical thinking: Artefacts and proofs, pp. 97–143, Springer, 2012.

[Fai93] A. Faifofer, *Elementi di geometrica*, 9th ed., Tipografia Emiliana, 1893.

[FDH14] D. Zhao F. Dong and W. K. Ho, *On the largest outscribed equilateral triangle*, The Mathematical Gazette (2014), To appear.

[Leo03] Y. H. Leong, *Use of the Geometer's Sketchpad in secondary schools*, The Mathematics Educator **7** (2003), no. 2, 86–95.

[Leu08] A. Leung, *Dragging in a dynamic geometry environment through the lens of variation*, International Journal of Computers for Mathematical Learning **13** (2008), 135–157.

[LLT03] Y. H. Leong and S. K. Lim-Teo, *Effects of Geometer's Sketchpad on spatial ability and achievement in transformation geometry among Secondary Two students in Singapore*, The Mathematics Educator **7** (2003), no. 1, 32–48.

[LRL06] F. Lopez-Real and A. Leung, *Dragging as a conceptual tool in dynamic geometry*, International Journal of Mathematical Education in Science and Technology **37** (2006), no. 6, 665–679.

[Mam08] M. F. Mammana, *r-inscribable quadrilaterals*, Osjecki Matematicki List **8** (2008), 83–92.

[Mar00] M. A. Mariotti, *Introduction to proof: The mediation of a dynamic software environment*, Educational Studies in Mathematics (Special Issue) **44** (2000), no. 1 & 2, 25–53.

[Mar01] ______, *Justifying and proving in Cabri environment*, International Jounral of Computers for Mathematical Learning **6** (2001), no. 3, 257–281.

[MM05] M. F. Mammana and F. Milazzo, *Sui triangoli equilateri circoscritti ad un dato triangolo*, Archimede 1 (2005), 31–37.

[Pet46] J. Petersen, *Constructions gomtriques*, 5th ed., Gauthier-Villars, 1946.

[Zha10] D. Zhao, *Maximal outboxes of quadrilaterals*, International Journal of Mathematical Education in Science and Technology **42** (2010), no. 4, 534–540.