The Practical Paradigm: Teaching of the Problem Solving Process in University Mathematics

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Abstract

This paper presents a pedagogical innovation for the teaching of undergraduate and postgraduate mathematics. Using the science practical paradigm, mathematics "practical" lessons emphasizing problem-solving processes via the undergraduate content knowledge were embedded within the curriculum delivered through the traditional lecture-tutorial system. The Practical Worksheet is an instructional scaffold we adopted to help our students develop problem solving dispositions alongside the learning of the subject matter. The paradigm and practical worksheet have been successfully used in a design experiment aimed at teaching problem solving in secondary schools. The same were used in our university courses which included Number Theory, Differential Equations, Computational Mathematics and Real Analysis. It was observed that the students were able to respond to the requirements of the mathematics "practical" to go through the entire process of problem solving. They were also able to carry out the "Look Back" stage of Pólya's problem solving model: checking the correctness of their solution, offering alternative solutions and expanding the given problem. Students were also able to make good conjectures in the "practical" environment. Our analysis of the students' work shows that the "practical" problem solving design holds promise for mathematics courses at the tertiary level.

Keywords: Mathematics, Problem Solving, Tertiary, Mathematics Practical worksheet, Polyá

Introduction

This paper gives a precís-style survey of collective efforts made by some mathematicians in the National Institute of Education (NIE, for short) of Nanyang Technological University to deploy a pedagogical innovation for teaching mathematics at both the undergraduate and the postgraduate level.

In NIE, mathematics content courses for pre-service mathematics teachers in an undergraduate programme, B.A.(Ed)/B.Sc.(Ed), are typically taught by mathematicians; likewise for a certain postgraduate programme, M.Sc. (Mathematics for Educators). The primary concern for mathematicians, without surprise, is to deliver the mathematical definitions, lemmas, theorems, corollaries, examples, counter-examples, and so on to the learner, while the opportunity to model pedagogical methods of the delivery itself is often viewed as secondary. With regards to the effectiveness of the traditional "chalk-and-talk" manner of teaching content, there appeared to be somewhat a shared sense of frustration among mathematician colleagues to see that mathematics students, when challenged with non-routine mathematics problems, resolved to "doing nothing and would wait for (the instructor) to provide the solutions steps" (Toh et al., 2013). This kind of frustration, as was revealed from interviews with these mathematicians, stemmed not from mere disappointment of their students' mathematical incompetence but rather a mismatch of expectations; namely, why should solving non-routine mathematical problems which is *the very heartbeat* of any honest-to-goodness mathematician be an experience completely missed out by our mathematics apprentices? This brings us to state a longstanding non-mathematical problem faced by mathematicians: *How can mathematics be taught at the university level the problem-solving way?*

Mathematicians are not the only ones struggling with this problem. In the field of mathematics education, the same concern manifests in a different form: how can mathematics problem solving be infused (i.e., implemented) and diffused (i.e., disseminated) in the actual school setting. In particular, the funded project Mathematical Problem Solving for Everyone (MProSE) which aimed at dealing with this research question had successfully deployed a design experiment that is based on a certain *mathematics practical* paradigm (Dindyal, Tay, Toh, Leong, & Quek, 2012). Three key beliefs underpin the MProSE approach: (a) Scientists perform scientific experiments, and students learn science by 'doing' science, i.e., through hands-on during science practical lessons. Mathematicians perform *mathematical* experiments, and students learn *mathematics* by 'doing' *mathematics*, i.e., through hands-on during *mathematics* practical lessons (Toh, Quek, Leong, Dindyal, & Tay, 2011). This novel idea that stretches well beyond a mere syntactic substitution[1] causes a profound shift of paradigm: problem solving is about making mathematics practical! (b) A formal assessment system complements the innovative curriculum that addresses not just the product but also the processes of problem solving. Without assessment, students and teachers are unlikely to place much emphasis to the

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[1] The original substitution of "mathematics/mathematician" for "science/scientist" over expository paragraph of Woolnough and Allsop (1985), p.32 can be found on Toh et al. (2011), p. 18.

processes of problem solving. (c) Since "the theory (of problem solving and its pedagogy) had been worked out", the MProSE team aimed at finishing the "hard and unglamorous work of following (the theory) through in practical terms" (Schoenfeld, 2007, p. 539). Practicality in an authentic classroom setting is the primary consideration in the modes of implementation of the design experiment. For issues of the implementations in schools, refer to Toh et al. (2014).

Serendipity saw to it that the problem-solving mathematicians met their problem solving math-educator~~s~~ counterparts somewhere down the corridor, and countless discussions ensued, and hence this paper.

Organization

We began by recapitulating the essential theoretical overheads of problem solving jargon, as well as the part of MProSE's work salient to our present discussion. This is followed by a brief summary of the four translations of the MProSE design experiment from the secondary school context to the university setting by three mathematicians, P. C. Toh (M1), T. L. Toh (M2) and W. K. Ho (M3) (who are respectively second, third and first authors of the present paper) in the respective courses they taught: Number Theory (N1), Differential Equations (M), Computational Mathematics and Real Analysis (N2). We then make qualitative comparisons of these translations in terms of (1) the emphases on teaching *about*, *for* and *through* problem solving within each of the content knowledge domains, (2) the didactical tools deployed in connection to teaching and learning the processes of problem solving, and (3) the learning outcomes (intended and actual) of the students. Authentic students' work will be analyzed alongside ~~the~~ certain implications which we draw out of each mathematician's teaching journey.

Problem Solving and MProSE design experiment

Theoretical overheads

For completeness, we include in this section the essential terminologies used in the discourse of problem solving concerning the aspects discussed in relation to the four translations. As it is not our purpose to be exhaustive in the literature review of research works done in the area of problem solving, we refer a reader to the theories illustriously presented in the celebrated works of Schoenfeld (1985, 2007), Lester (1994), and the contextual issues raised recently in Leong et al. (2011) and Dindyal et al. (2012). The genesis of MProSE can be traced to the foundational framework laid by George Pólya (1954), and also to a later but more significant body of research established by Alan Schoenfeld (1985). Problem solving processes, as modelled by Pólya, comprise Understanding the Problem (UP), Devising a Plan (DP), Carrying out the Plan (CP), and Look Back (which we call Check & Expand (CE)), but authentic problem solving trajectories along these components are mostly ~~a~~cyclic, looping backwards to previous stages (Kantowski, 1977). Extraneous to this model are many factors that are crucial to successful problem solving, and these salient aspects were captured succinctly in Alan

Schoenfeld’s (1985) four components: Cognitive Resources (CR), Heuristics (H), Control (C) and Belief System (BS).

MProSE: Mathematics Practical Paradigm shift

Building on the aforementioned works, MProSE focused on the practical implementation of problem solving in the classroom. Inevitably, teaching about problem solving bears upon the learner to familiarize oneself with the lingo of problem solving, e.g., the labels (UP), (CP) and so on. These terminologies are in themselves crucial to the development of good problem solving habits as learners become conscious of these ‘labelled’ processes and actively engage in them. To enact the problem solving curriculum in a real classroom, a *Mathematics Practical Worksheet* (MPW) which resembled those used in science practical lessons was developed, with original designs first emerging from Tay et al. (2007). With the practical mindset, students would treat problem-solving lessons as mathematics “practical” classes, and MPW’s as a laboratory report. Reflecting on past failures of others who attempted to introduce problem solving into the school curricula, the MProSE team was cognizant of the need to simultaneously design the worksheet together with a formal assessment rubric. Importantly, this set of rubric credits the problem solver not just for the final solution of the problem but also for the processes that took place.

Mathematics Practical Worksheet

Since the MPW is the central piece of innovation (supported by the mathematics practical paradigm) reported in this paper, it is important to look at its structure alongside its design principles. The MPW is a problem solving scaffold in the sense of Wood and Middleton (1975) that comes in the form of a template, a brief outline of which is shown in Figure 1.

The MPW contains sections explicitly guiding the students to use Pólya’s stages and problem solving heuristics to solve a mathematics problem. The worksheet is divided into four sections (I – IV) corresponding to the four Pólya stages (UP, DP, CP & CE). Each section takes up a page, and the students are allowed to go backwards to the previous pages if needed, and can fill in more of each section if required. The number of attempts, and the plan numbers may be indicated at each stage. The MPW, introduced across a few lessons, is to induct the students into the Pólya stages and Schoenfeld’s framework of problem solving, thereby cultivating in them good habits of effective problem solving.

I Understand the problem (UP)

(You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.)

(a) Write down your feelings about the problem. Does it bore you? scare you? challenge you?

(b) Write down the parts you do not understand now or that you misunderstood in your previous attempt.

(c) Write down the heuristics you used to understand the problem.

II Devise a plan (DP)

(You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.)

(a) Write down the key concepts that might be involved in solving the question.

(b) Do you think you have the required resources to implement the plan?

(c) Write out each plan concisely and clearly.

III Carry out the plan (CP)

(You may have to return to this section a few times. Number each implementation accordingly as Plan 1, Plan 2, etc., or even Plan 1.1, Plan 1.2, etc. if there are two or more attempts using Plan 1.)

(i) Write down in the *Control* column, the key points where you make a decision or observation, for eg., go back to check, try something else, look for resources, or totally abandon the plan.

(ii) Write out each implementation in detail under the *Detailed Mathematical Steps* column.

IV Check and Expand (C/E)

(a) Write down how you checked your solution.

(b) Write down your level of satisfaction with your solution. Write down a sketch of any alternative solution(s) that you can think of.

(c) Give one or two adaptations, extensions or generalisations of the problem. Explain succinctly whether your solution structure will work on them.

Figure 1. Instructions on a Practical Worksheet

Translations of MProSE design experiment to university courses

The three mathematicians usually taught their content courses via traditional methods, i.e., delivery of content via 2 one-hour lectures, and discussion of problems set as weekly homework via one-hour tutorial, spanning over 12 weeks. It was a common observation among the three mathematicians that when students encountered problems that they were unfamiliar with, i.e., they had no known methods or procedures with which they might apply, they generally gave up and waited for solutions to be supplied by the instructors. In addition, students neither knew how to get started nor persevered to solve the problems at hand. After discussion with their colleagues specializing in mathematics education, they considered the possibility of infusing ~~the~~ Polya's and Schoenfeld's frameworks of problem solving into their courses. Essentially, the three of them translated the MProSE design experiment to a university setting. While the innovation could not entirely replace the usual lecture-cum-tutorial mode of instruction, they were faithful to three underlying principles of the origin experimental design, i.e., students were (1) shown how problem solving could be carried out using the four Pólya stages of problem solving as well as the four components of Schoenfeld's model by their course lecturers, (2) instructed to explicitly make use of these processes, together with the problem solving terminologies, heuristics, mathematics practical worksheet, etc., conscientiously in their solution of the problems set in the assignments, and (3) assessed using the problem solving rubrics.

Three of the case-studies had already been published in a journals and a conference proceedings and as a presentation (see Toh et al. (2014) for the Number Theory course, Ho (2013) for Computational Mathematics, Toh et al. (2013) for the Differential Equation course), and thus we do not replicate them. Instead, we give below a summarized form of the case-studies. The fourth case-study is on a recent translation to a graduate course in real analysis in 2013. By making qualitative comparisons among these, we hope to be able to tease out the features of the Mathematics Practical Paradigm, with particular emphasis on the Mathematics Practical Worksheet. The problem solving processes were graded and assessed for the first three undergraduate courses, but it was not the case for the fourth one, though MPWs were also used.

We organize our discussion in the following way. As a majority of those students M1 taught in the Number Theory course progressed to be taught by M3 in the Computational Mathematics course, we present these two case-studies together.

Number theory

Student profile and course description

There were 59 students teachers enrolled in this course, of whom 56 came from the Calculus I course also taught by M1. It was M1's intention to improve on the aforementioned situation gathered from his experience of teaching the Calculus I course by using the Problem Solving approach and the MPW as the essential scaffolding device.

The content covered in this Number Theory course was typical of similar courses taught elsewhere and included for example, divisibility, congruences, Diophantine equations and Euler's generalization of Fermat's Little Theorem. Aware that the Pólya model would be formally introduced to these students in method courses in the third year of the degree programs, M1's focus was to exemplify the use of the model rather than to explicitly discuss the theoretical basis and pedagogical implications.

While M1 intended to make problem solving a prominent feature in the course, he would not compromise the rigor of the mathematical content. This challenge led him to integrate problem solving into his lecture delivery in the sense that whenever there were results in the course which he reckoned were problems amenable to a demonstration of the problem solving strategy, he "seized upon the opportunity to model Pólya's stages and heuristics." Instead of presenting a theorem directly and followed by examples, M1 would apply a style that gave the audience an impression that he was genuinely engaging in the solution of the problem stated in the theorem. This was enacted by talking aloud his "problem-solving thoughts", and inviting the student teachers to do the same in their group discussions.

The table below showed how M1 weaved problem solving elements into the course:

Table 1:*Overview of How Problem Solving Elements are Weaved into the Course*

Weeks	Content Taught	Significant Problem Solving Milestones
1 – 2	Number systems and methods of proof.	Introduce and model Pólya's first three stages.
3 – 4	Prime numbers, divisibility, the Euclidean algorithm and gcd.	Actively involve students in the problem solving process.
5 – 6	Diophantine equations.	Formally introduce Pólya's model (including the fourth stage) and the Practical Worksheet.
7 – 8	The fundamental theorem of arithmetic, congruences.	Students submit the first problem solving assignment.
9 – 10	Modular number systems and cryptography.	Return the graded assignment with feedback.
11– 12	Exponents and Euler's generalization of Fermat's little theorem.	Students submit the second problem solving assignment.

Method

In line with the principle of placing formal assessment as the crucial motivation for the student teachers, they were assigned to work on two problems using the MPW. Together with another ten regular homework assignments, these two problem-solving assignments counted towards 10% of their final grades for the course. The first of these problem-solving assignments was submitted in Week 7, marked with corrections on the mathematical mistakes highlighted, graded and then returned to the students as formative feedback. The second assignment was submitted in Week 12, graded but not returned as the course had ended by then. We focus our discussion on the second assignment because they were to submit this assignment in the form of the MPWs.

The second assignment called for the student teachers to solve:

Let a, b, c be natural numbers satisfying $a + b + c = 2012$. If we have

$$a!b!c! = m\,10^n$$

for some integers m and n, where 10 does not divide m, find the smallest possible value of n.

The student teachers were allowed three weeks to work on this problem with the understanding that they could make discussion with their classmates, but the written work must reflect their own problem solving journey. This could be ensured because the MPW required them to make entries of the problem solving processes, e.g., how was the question interpreted and understood, what was the question demanding of the solver, what were the initial plans thought of, to what extent these were successful, and so on. While it is true that a solution of this problem lies in the public domain, and it is difficult to verify if a student could have copied from it, it would be deemed unlikely that an authentic problem solving experience be fabricated and be presented in the MPW. This is because a genuine problem solving trajectory is characterized by ~~a~~cyclic transversals of the Pólya stages. If fabrication in the form of MPW were really to take place, then the process of recording would have been so tedious and uninviting, if not, punishing for the student teacher that it serves as an effective deterrent for such possible mal-practices.

The MPW submitted by the student teachers form the data that was analysed formally along four categories: (1) student teachers' solution approaches, (2) application of the first three Pólya stages, (3) ability to move onto the fourth Pólya stage of Check and Expand, and (4) the use of heuristics.

Student teachers' work

Of the 55 student teachers who submitted their assignment, 46 displayed some evidence of attempting to understand the problem in their MPW. The heuristics that were mostly frequently employed included "Looking for patterns", "Use smaller numbers", and "Guess and Check". Furthermore, each student teacher employed a mixture of these heuristics.

A typical example of the use of the heuristic "Look for patterns" is shown in Figure 2. In his first attempt, the student teacher Alan (pseudonym) made a list from 0! to 25! and attempted to tease out a pattern. Alan noted the increase in the number of trailing zeros as about 1 in every 5 natural numbers, but stopped at the 25th pattern. Alan abandoned his first plan, and moved to his second plan which was to make use of prime factorization. The design of the MPW, while forcing the solver to keep record of his failed attempts, indirectly achieved the goal of instilling perseverance in the problem solver. The metacognitive aspect of problem solving is also captured here in that the solver exercised the aspect of control, making a decision to abandon the initial plan in favor of an alternative one.

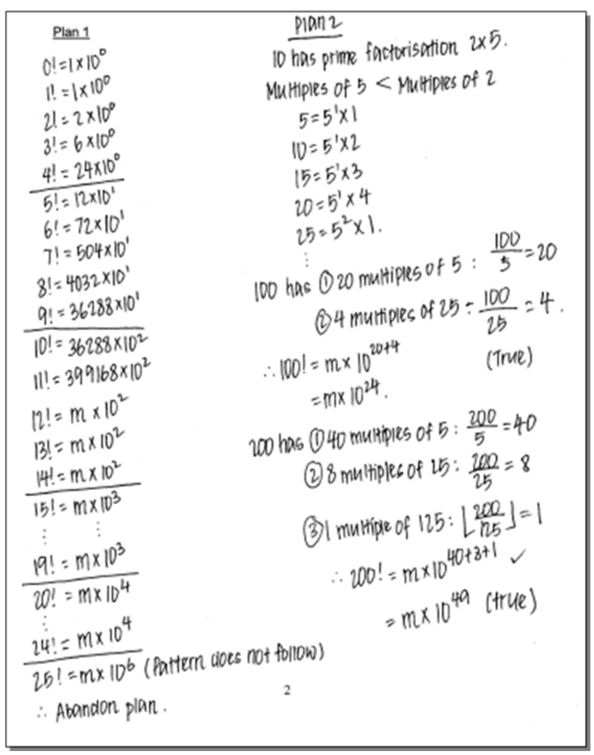


Figure 2. Use of the heuristic "look for pattern" by Alan.

In his second attempt, Alan noticed that there were many more multiples of 2 compared to the multiples of 5 and eventually figured out the correct formula for computing the trailing zeros of *a*!, albeit this was not explicitly reflected in his MPW. Figure 3 showed another middle-ability student teacher, Beatrice (pseudonym), who provided more details on how she arrived at the formula for $a < 3125$, using similar reasoning.

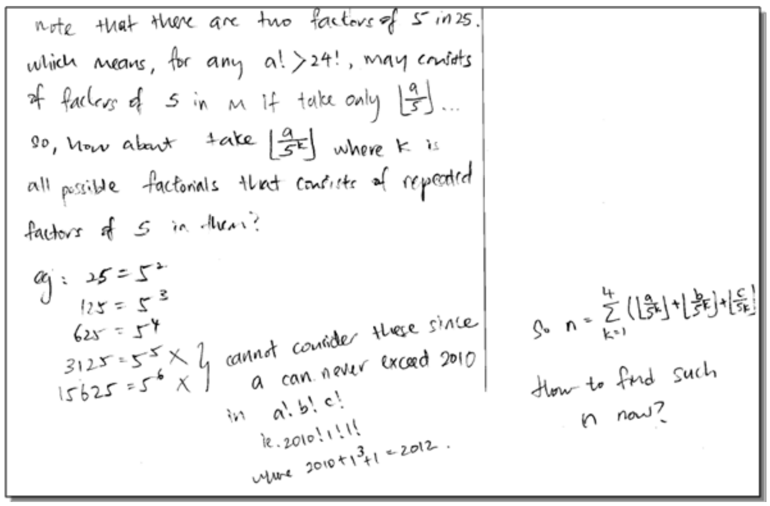


Figure 3. How Beatrice arrived at the correct formula.

The control column in Stage III (Carry out the Plan) was heavily utilized by the student teachers. In Beatrice's work (see Figure 3), she jotted down the question "How to find such *n* now?" after she had actually written a summation formula for it. Desmond, another student teacher, used the control column (see Figure 4) to make important remarks, detailing his reasoning that picking $a = b = 624$ would result in reducing the number of trailing zeros.

Stage IV (Check and Expand) encouraged students to move beyond their comfort zone, especially after they had already reached a solution to the problem. In Stage IV of the MPW, 48 out of 55 student teachers attempted to check their solutions. This helped those who had quickly jumped to the conclusion that $a = 670$, $b = c = 671$ would give the minimal, to revise their answers. Geraldine (pseudonym) wrote the following in her MPW: "I am quite happy that I checked my solution as I discovered careless mistakes in my discovery of the powers of 10." Checking is not a common practice of students – an observation gathered by M1 in his Calculus I course with this same group of student teachers. The explicit stipulation in the last stage of the MPW provided the impetus for the problem solvers to perform the required checks for their solutions.

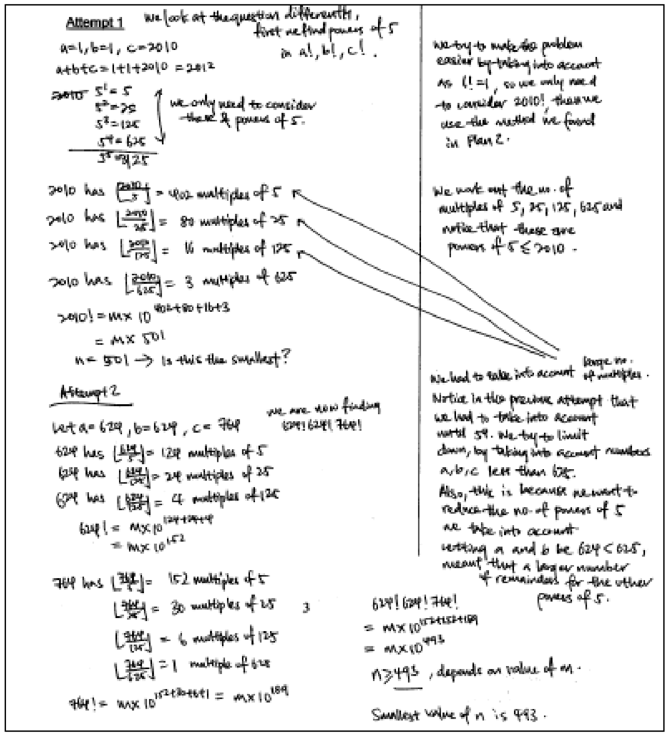


Figure 4. Desmond's reasoning that picking $a = b = 624$ would reduce the number of trailing zeroes.

The last stage of the Pólya model for this problem also caused student teachers to look for alternative solutions. One student teacher attempting to find alternative solutions hit upon another combination $a = 1249$, $b = 624$ and $c = 139$, which gave the same value of $n = 493$. A group of seven student teachers managed to present the complete solution by proving that the value of $n = 493$ was the minimum value achievable. They calculated the lower bound for n using the crucial inequality stated in the third line of their presentation in their MPW (see Figure 5).

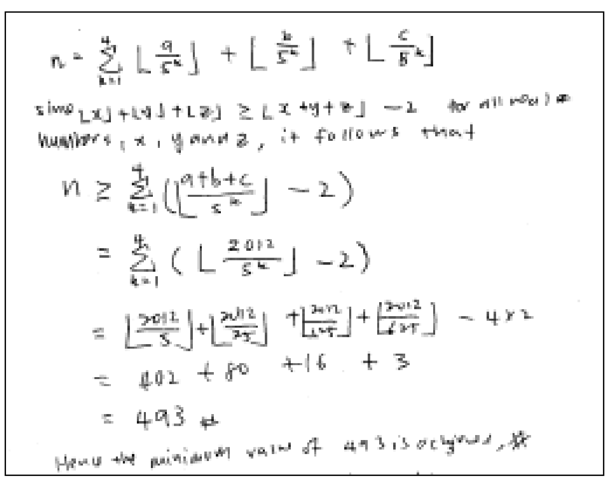


Figure 5. Fiona's work showing how the lower bound was calculated.

A total of 33 went beyond the solution of the given problem, that is, to extend the problem by either changing the values of 2012, or increasing the number of variables beyond *a*, *b* and *c*. However, none of them attempted to outline a solution strategy that could solve the more general problem.

Researcher's findings

Almost 70% of the student teachers who submitted the assignment obtained the correct answer of what M1 considered as a challenging problem at the first year of the undergraduate level. Apart from the final answer, over half of these student teachers demonstrated clear reasoning in their solutions as they communicated it via the MPW. The student teachers' responses far exceeded M1's prior conceptions of how the problem would be tackled. As part of the reflective process of an experienced teacher, M1 thus expressed in a most modest manner his satisfaction that he was "encouraged that these student teachers were not merely performing calculations and following procedures; they were largely applying suitable heuristics along the lines of Pólya's stages in making productive advances towards solutions to the problem." With the absence of M1's direct guidance, the student teachers were then able to assume a higher level of problem solving competency as portrayed by their attempts recorded in the MPWs. The student teachers' works finally presented in the MPWs overwhelmingly contrasted against the initial problem solving disposition (i.e., if any) of the same group of student teachers prior to this translational enterprise.

From this study, M1 gathered that assessment matters. Motivated by the credit they would receive in the assignment (no matter how small) for their problem solving efforts, the student teachers made significant movement from the original Zone of Proximal Development (i.e., able to use results, formulae, methods or procedures when encountered with routine problems) to the next (i.e., able to, *without the direct guidance of the instructor*, carry out productive problem solving) despite the problem's difficulty. Importantly, M1 pointed out the use of the MPW – that it is organized along the lines of Pólya stages and heuristics – provided scaffold for the student teachers to follow the overarching model and processes. Regarding the last stage of Check and Expand, there remained room for improvement as M1 observed that aside from those simple extensions mentioned earlier, insufficient consideration was given to generalize the original problem and its solution. M1 claimed that one reason for this was the inadequacy of his emphasis on this last stage of problem extensions in his lectures.

Computational mathematics

Student profile and course description

Interestingly, this study was intended to be a spin-off from the MProSE design experiment, though its theoretical principles, methodologies and practical implementations were known to the investigator M3. What was common, however, was the vision shared among M3 and some of the members of the MProSE team, i.e., to promote independent learning and investigation via a *problem solving methodology*. The subject of study was incidentally the same[2] group of students who were promoted from the first year to the second year of the undergraduate course. Computational mathematics covers numerical methods such as approximations of roots using bisection method, Newton-Raphson's method, and quadrature methods via rectangular, trapezoidal and Simpson rules. Simulation, involving Monte Carlo methods, is also taught in the course. The vehicular language of implementation is MATLAB, a common scientific programming language used by computational mathematicians, engineers and scientists alike because of its simplicity and versatility.

There was no intentional coordination between M1 and M3, and hence M3 carried out this study independently. What inspired M3 was the enthusiastic influence of the MProSE team: every of its members is an ardent believer of problem solving – problem solving is for everyone! M3 had also earned a postgraduate diploma in education and hence was conversant with the Pólya model and Schoenfeld's framework.

Method

The main research question that M3 had in mind then was to study how problem solving in mathematics as modelled by Pólya could be transferred to the area of MATLAB programming in computational mathematics. The key idea was that the processes involved in programming (also known as scripting or coding), irrespective of the vehicular programming language, are very similar to those in mathematical problem solving. The first step to scripting a program inevitably requires the programmer to understand the requirements and specifications of the programming job at hand: what does the user want to achieve via this program? This corresponds to the first Pólya stage of Understanding the Problem (UP). The second step is to think of ways to write the codes of the program to meet the specifications. Simply put, one takes note of what the input arguments, the given parameters as well as the output arguments are. The scripting plan may involve several scripting techniques, for instance, the use of declaration of variables, placement of for-loops and while loops, or use of input-output commands. Planning stages often require the programmer to sketch the flow chart that outlines the logical flow of the program. This stage tallies with the second Pólya stage of Devising a Plan (DP). The third step in scripting, of course, is the scripting itself. More often than not, for a sufficiently complex (and previously unseen) programming job, one's first attempt fails. So this would then be followed by a sequence of de-bugging (i.e., fixing troubles with exceptions thrown to the programmer by the compiler or nonsensical outputs when the program is run on trial data sets), as well as re-writing of subroutines that worked partially. This stage matches up with the third Pólya stage of Carrying out the Plan (CP). The last stage of programming which is reminiscent of Check and Expand involves the programmer to look back at the final product, i.e., the completed scripts, and to

[2] This is essentially the same group with 6 additional students from the Diploma program crossing over to the B.A./B. Sc. (Ed) into Year 2 level. These students have been removed from the case study.

improve on the robustness and performance capabilities (in terms of program correctness and run-time efficiency).

Unlike the mathematical counterpart, the programming world needs no further justification in terms of its semblance to science practical sessions. This is clearly so since every programming problem requires the programmer to create the codes, experiment with them, run them and re-code if needed. The practical aspect of programming is even more intrinsic and conspicuous than its theoretical underpinnings. Though M3 was aware of the scaffolding device, he chose *not* to enforce its use in the case study for two reasons: (1) MPW is only a scaffold, and thus at some point of learning, the learner should be able to perform at the higher zone of proximal development *without explicitly depending on it*. (2) In assessing the MATLAB programs scripted by the students, there are many aspects to examine besides the problem solving components. In that sense, the MPW may not be the best platform on which the computer program should be presented. Already, there are many other existing and well-accepted ways of documentation used by software developers.

The student teachers were given 12 weeks to tackle the following problem:

Script a MATLAB program which meets the following specifications:

- *The program expects an input of a two-variable real-valued continuous function*

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$

- *The program is to calculate and output a numerical estimate for the coordinates* (x_1^*, x_2^*) *of a point P in* $[0,1]^2$ *for which the input function f attains its minimum value. The program should, of course, output this minimum value.*

Again, M3 is aware that the numerical solution of minimizing such two-variable real-valued functions is available as ready-to-use MATLAB programs. But these are not acceptable as submissions because the student teacher needs to explain how the codes work in the final report. As expected, since problem solving is the main object of study, the submission format the student teacher is required to report comprises a chronicle of the problem solving journey, making explicit~~ly~~ use of the problem solving terminologies, e.g., UP, DP, CP and CE. The use of any heuristics should be spelt out, e.g., the problem was first simplified to a one-variable function and then solved. The assignment explicitly points to the works of Pólya and Schoenfeld with regards to the problem solving strategies and heuristics. Sample runs and outputs on different input functions of the student's choice were to be included in the report as part of the checking and program validation process. It was phrased in the assignment that "a good report should also talk about possible extensions and generalizations of the proposed problem", which encouraged the student to improve on the computing capabilities of the final program.

It is important to note that very little guidance and advice was given by M3. In particular, M1 who happened to be one of the tutors for this course was told not to help the students with regards to this assignment in any way.

The data collected through the submission of the final written report (not exceeding 30 pages) and the MATLAB scripts was the only source of analysis. The Problem Solving component of this assignment was 25% of the total score. The assignment was also graded (in equal weightage of 25% each) for suggestions of how the program can be applied in real-life contexts, the program correctness and overall style (labelled as MATLAB component), as well as the overall quality of the written report. The student teachers were informed that they would be assessed on their problem solving processes, but did not know the exact percentage breakdown described above.

For the problem solving component, the reports submitted by the student teachers were also *coincidentally* analysed along four lines similar to those stated by M1 in the preceding subsection, i.e., (1) student teachers' programming strategies and approaches, (2) application of the first three Pólya stages, (3) ability to move onto the fourth Pólya stage of Check and Expand by considering how their present programs can be improved upon, and (4) the use of heuristics.

Students' work

Over 75% of the student teachers showed evidence in their reports that they engaged the Pólya stages of problem solving. This was easily detected in the way they structured their reports, i.e., unambiguously with headings labelled as UP, DP, CP and CE. Pólya's work was mentioned explicitly in about 20% of these reports. A handful of reports (less than 10) showed traces of evidence that they were following the format of the MPW; for instance, they wrote responses such as:

> "Initially when I first look at the questions, I was stunned. I am not sure whether I was competent and fully prepared to come up with a program to solve this problem."
> [Authors' note: This addresses the sub-question (a) What is your feeling about the problem?]

Of those that showed evidence of using the Pólya model, over 60% displayed their problem solving trajectory by detailing multiple attempts and plans. A middle ability student, Jack (pseudonym), recounted that in the second stage of problem solving, Pólya suggested the questioning of possible connections between data and unknowns. Jack wrote:

> "He [Polya] recommended the consideration of auxiliary problems when an immediate connection cannot be found. The following 3 plans documented are based on Polya's recommendations."

M3 classified the different approaches roughly under these categories: (1) Path method: using a set of initial (fixed or randomized) guesses (as seeds), and in most reports such was a singleton, students teachers proceed to move a small (discrete) step in a finite number of assigned directions (e.g., north, south, east or west) in the *x*-*y* plane, and obtain that number of new values of $z = f(x,y)$. This new set of values, together with the initial z-values of the seed(s), is then compared for the minimum value. The next approximation is then set at the new position at which the value of z is smallest. The process was then repeated till it stopped, with suitable stopping criteria. (2) Evaluate the function f on a fixed set of points (evenly spaced on the unit square in some fashion). The point(s) which gives the minimum value(s) will stay, and the region in its neighbourhood was 'magnified' for the same random process to iterate (with different scaling factors). The process was then repeated till a suitable stopping criterion was reached. (3) A random variant of (2) was implemented in the sense that the initial set of points was randomly generated on the unit square.

Incorporated within the written form of the plans they devised, many student teachers made use of ~~the~~ flow charts to explain the execution of their plans (see Figure 6). This approach was used in the lectures by M3 to illustrate how the plans could then be translated into MATLAB command lines.

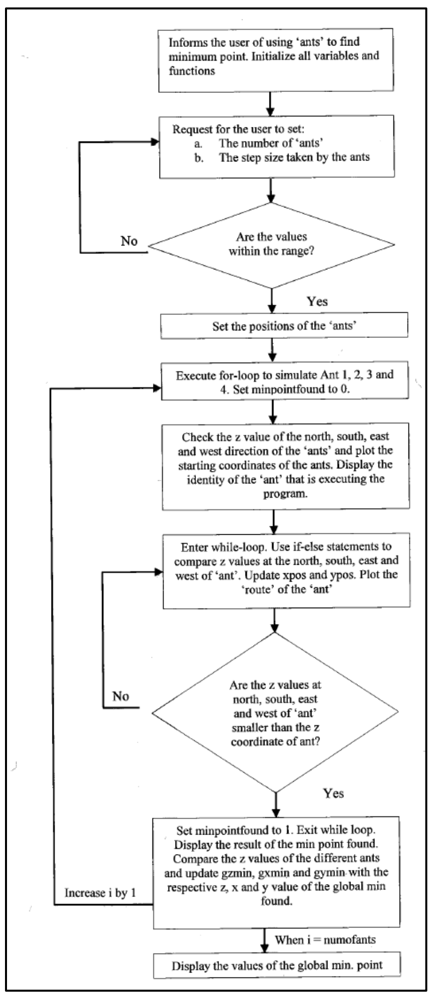
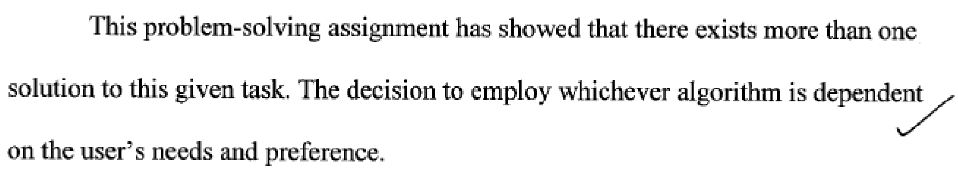


Figure 6. Flowchart guided by the stage "Devising a Plan".

Regarding the proportion of students who engaged in the last Pólya stage of Check and Expand, there was about 78% of student teachers who attempted to construct programs that could handle functions of type $\mathbb{R}^n \longrightarrow \mathbb{R}$, where n is higher than 2. A number of students employed their knowledge of Calculus II (a course taught in the same time frame as Computational Mathematics) to check that the output produced by their MATLAB program matched the one they obtained via theoretical means (e.g., using partial derivatives).

Overall, a majority of 80% of the students expressed that the problem solving approach to programming allowed them to realize that a programming problem can be achieved in a myriad of ways. Another student, Adrian (pseudonym) came to this conclusion after he decided which of the three programs (generated by different plans) was the best for the present task:



This problem-solving assignment has showed that there exists more than one solution to this given task. The decision to employ whichever algorithm is dependent on the user's needs and preference.

Figure 7. Problem solving approach teased out the students' appreciation of multiple solutions.

Researcher's findings

Via a correlational analysis between the scores obtained for the Problem Solving component and the MATLAB component for each student, M3 reported that ~~with~~ the active engagement in the Pólya stages significantly increased the likelihood of scripting a fully functional MATLAB program which meets the given specification. In Ho (2012), the students' reports were also analyzed for the problem solving trajectories, i.e., a directed multi-graph whose vertices are the stages and edges the transition from one stage to another. It appeared that ~~a~~cyclical looping around the four stages took place for student teachers who used conscientiously the MPW, and this somewhat determined whether the problem solving experience was authentic.

Differential equations

Student profile and Course description

The re-design applied to the Differential Equation course occurred in Year 3 of the degree program. The chosen cohort of the 51 Year 3 student teachers was comparable with the early cohorts in terms of the entry level, i.e., performance in Years 1 and 2 mathematics courses. These students were resistant to following any model of problem solving in solving non-routine problems. Even for those who were mathematically stronger, there was reluctance to give the solutions a check for correctness, and little effort was made to find alternative solutions. M2 aimed to enhance the student teacher's skills of problem solving with emphasis on Stage IV Check and Expand, as well as their metacognitive awareness in problem solving.

Cognizant of the constraints of time and the need to cover a core curriculum of topics, M2 modified the traditional lecture-tutorial mode of delivery (24 hours of lectures and 12 hours of tutorial lessons) to accommodate a series of practical lessons (16 hours of lectures, 8 hours practical lessons, and 12 hours of tutorial). The 8-hour practical lessons adopted the mode of teaching *about* problem solving, making use of “resources” taught in the 16-hour lecture segment of the module. The instructor, in each practical lesson, started by introducing a particular aspect of problem solving (see Toh et al. (2011) for the detailed lesson plan) and engaged the students to solve a particular problem on differential equations, based on the content of the lesson material. Student teachers were allotted 40 minutes to solve a given problem, after which the instructor would explain the solution of the problem while the student teachers performed peer marking after exchanging their scripts with their peers. Below are some of these problems:

> **Problem One.** Sketch the slope field for the differential equation $\frac{dy}{dx} = \sin(x)$.
>
> **Problem Three.** Must a first order separable equation always be exact? Must an exact differential equation always be separable?
>
> **Problem Five.** Find the general solution of the differential equation $\frac{dy}{dx} = 2(2x - y)$.

Again, the student teachers were asked to treat the problem solving classes as a mathematics practical lesson, to solve the given problems with the MPW. In this way, it was hoped that there would be a paradigm shift in the way student teachers looked at these ‘difficult, unrelated’ problems which had to be done in a special class (see Toh et al, 2011). Assessment of the mathematics practical lessons was a critical part of the re-design, and student teachers were informed that their performance in these sessions constitute a significant part of their continual assessment of the entire module.

Method

Prior to this course, student teachers were already exposed to the Pólya model of problem solving. M2 carried out a session prior to the first practical lesson to revise with them the Pólya stages of problem solving. At the start of the first practical lesson, M2 introduced the assessment rubrics (see Toh et al., 2011). Each practical session would be centered about a “Problem of the Day”. Students teachers used the assessment rubrics to peer assess the solutions of the peers. The MPWs were then collected and re-assessed by the instructor who acted as moderator to this peer assessment system. This choice of peer assessment system was informed by the research findings of McTighe and Wiggins (2004).

The following description of the scoring rubric is taken from Toh et al., (2011).

The scoring rubric focuses on the problem-solving processes highlighted in the Practical Worksheet. There are four main components to the rubric, each of which would draw the students' (and teachers') attention to the crucial aspects of as authentic as possible an attempt to solve a mathematical problem:

- Applying Pólya's 4-phase approach to solving mathematics problems
- Making use of heuristics
- Exhibiting 'control' during problem solving
- Checking and expanding the problem solved

In establishing the criteria for each of these facets of problem solving, we ask the question, "What must students do or show to suggest that they have used Pólya's approach to solve the given mathematics problems, that they have made use of heuristics, that they have exhibited 'control' over the problem-solving process, and that they have checked the solution and extended the problem solved (learnt from it)?"

The rubric is presented below.

- *Pólya's Stages* [0-7] – this criterion looks for evidence of the use of cycles of Pólya's stages (Understand Problem, etc.).
- *Heuristics* [0-7] – this criterion looks for evidence of the application of heuristics to understand the problem, and devise/carry out plans.
- *Checking and Expanding* [0-6] – this criterion is further divided into three sub-criteria:
  - Evidence of checking the correctness of solutions [1 mark]
  - Providing for alternative solutions [2]

Adapting, extending and generalizing the problem [3] – full marks for this are awarded for one who is able to provide (a) two or more generalizations of the given problem with solutions or suggestions to solution, or (b) one significant extension with comments on its solvability.

Students' work

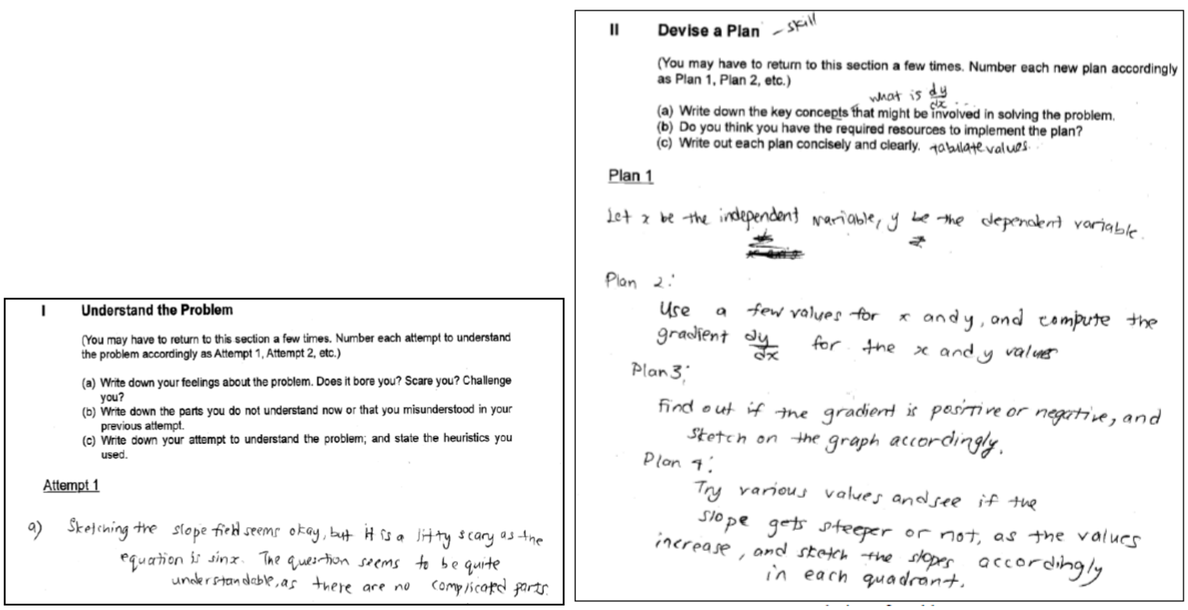
The data collected by M2 were the student teachers' working on the MPWs, and these were analyzed according to the categories: (1) Correctness of Solution, and (2) Checking and Expanding. The table below summarizes the frequency of each occurrence over the first six problems carried out in the practical lessons.

Table 2: *Frequencies of occurrences for correctness of solution & Stage IV (Checking and Expanding).*

Categories based on Correctness of Solution	**No. of students for Problem**					
	One	**Two**	**Three**	**Four**	**Five**	**Six**
Completely Correct Solution	35	41	24	45	45	22
Partially Correct with appropriate Use of Heuristics	15	9	13	5	6	25
Incorrect Solution with appropriate use of Heuristics	1	0	13	1	0	4
Incorrect Solution without use of appropriate heuristics	0	0	1	0	0	0
Categories on Stage IV: Checking and Expanding	**No. of students for Problem**					
	One	**Two**	**Three**	**Four**	**Five**	**Six**
No attempt in Stage IV	27	0	31	3	1	11
Attempt to check reasonableness of answer	21	8	4	14	7	16
Attempt to check answer + either alternative solution or generalize the problem	3	11	13	13	7	15
Attempt to check answer + alternative solution + generalize the problem	0	11	3	21	36	9

As a pre-post comparison, we look at the work of two student teachers, Amy and Eric (pseudonyms), on Problem One and on the test. Because the structure of the MPW allows the problem solver to pen down his or her thoughts and feelings about the problem, both students made use of the scaffold. It was evident through their writing that they recalled, inspected and reflected on their entire problem solving process. The MPW therefore provided the two students a chronological record of what happened during their problem solving journey; this allowed them to remember what they 'did right' to reach the final solutions. Because the MPW raised the student teachers' awareness to record their metacognitive control under the 'Control' column, the students actively recorded their self-regulatory thinking.

The lecture examples on the sketching of the slope fields involved simple polynomials, but not trigonometric functions as in Problem One. Thus, this presented a non-routine problem to the student teachers. Amy found this problem a little daunting as recorded in Stage I of the MPW (see Figure 8). In Stage II, Amy identified the independent and dependent variables *x* and *y* and planned to use the heuristic of substitution of different values of (*x*,*y*) to obtain the slopes. She carried out the plan in Stage III and analyzed the slope of the equation in the first quadrant under the 'Control' column. Her solution was not completely correct, and in particular, the translations from the table of values (slope) to the graph were wrong. Her solution was assessed as "Partially Correct with Appropriate Use of Heuristics". However, she did not reach Stage IV; otherwise, she would have detected the mistakes made in Stage III. As such she was awarded 12 out of 20 marks using the assessment rubrics.



I Understand the Problem

(You may have to return to this section a few times. Number each attempt to understand the problem accordingly as Attempt 1, Attempt 2, etc.)

(a) Write down your feelings about the problem. Does it bore you? Scare you? Challenge you?
(b) Write down the parts you do not understand now or that you misunderstood in your previous attempt.
(c) Write down your attempt to understand the problem; and state the heuristics you used.

Attempt 1

a) Sketching the slope field seems okay, but it is a litty scary as the equation is sinx. The question seems to be quite understandable, as there are no complicated parts.

II Devise a Plan – skill

(You may have to return to this section a few times. Number each new plan accordingly as Plan 1, Plan 2, etc.)

(a) Write down the key concepts that might be involved in solving the problem. what is dy/dx
(b) Do you think you have the required resources to implement the plan?
(c) Write out each plan concisely and clearly. tabulate values

Plan 1

Let x be the independent variable, y be the dependent variable.

Plan 2:

Use a few values for x and y, and compute the gradient dy/dx for the x and y values

Plan 3:

Find out if the gradient is positive or negative, and sketch on the graph accordingly.

Plan 4:

Try various values and see if the slope gets steeper or not, as the values increase, and sketch the slopes accordingly in each quadrant.

Figure 8. Amy's solution of Problem One.

Eric's solution to the same problem is shown in Figure 9.

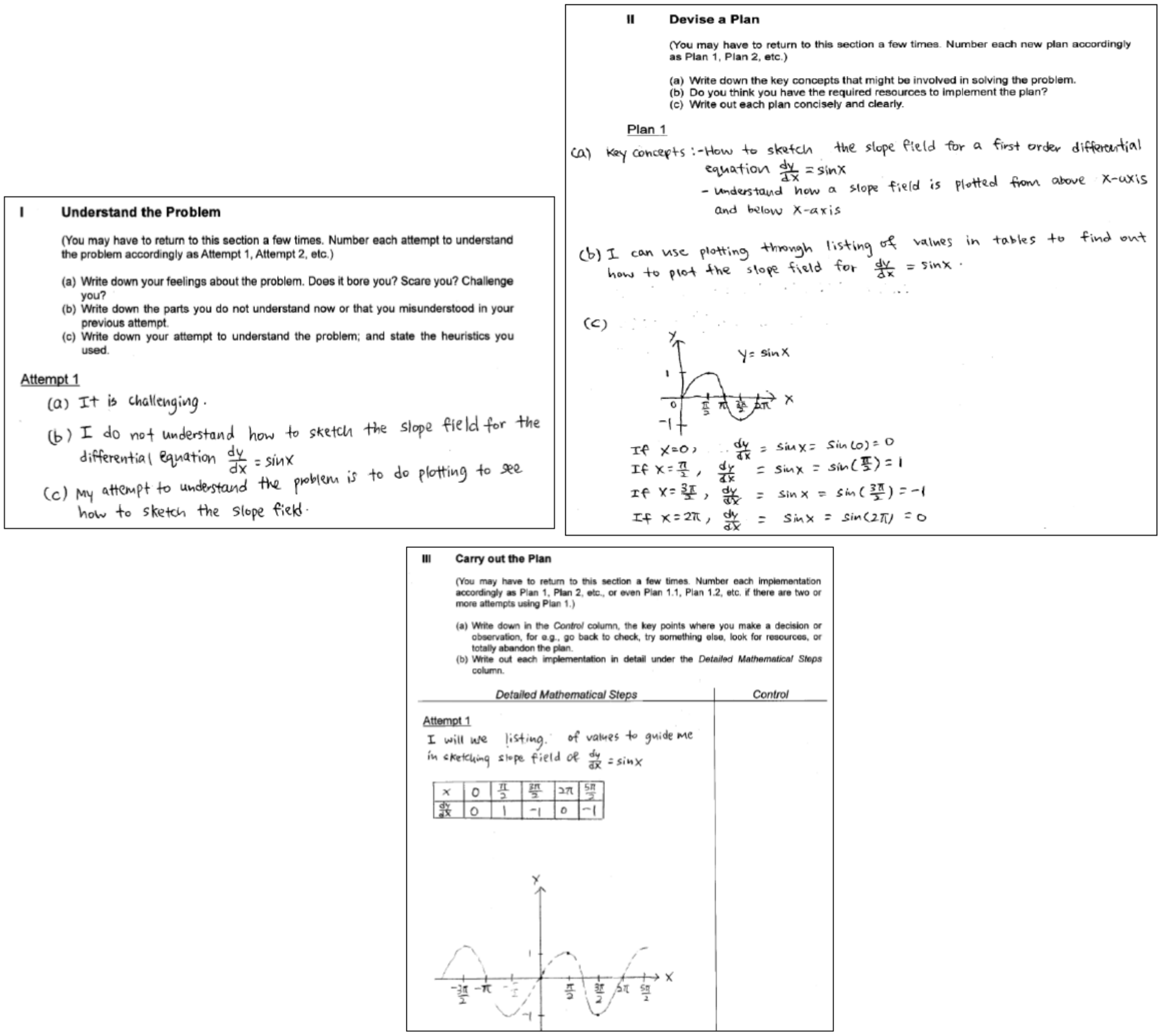


Figure 9. Eric's solution of Problem One.

Notice that Eric was struggling to understand the problem in Stage 1, even though he knew that he was supposed to plot out the points on a graph. Stage II revealed that he made an attempt to obtain the values of the slopes for several points. Eric clearly, in Stage III (Figure), demonstrated workings which revealed his misconception: he could not distinguish between the slope field given by $\frac{dy}{dx} = f(x)$ and $y = f(x)$. Eric's problem solving disposition had not allowed him to proceed to Stage IV, just like Amy. He was awarded 8 out of 20 marks.

After another four weeks, the students sat for a mathematics practical test (in Week 5 of the course). The test problem was

Problem. Find the general solution of the differential equation $\frac{dy}{dx} = x + y + 1$.

By this time, student teachers had sufficient exposure and training to the mathematical problem solving approach, and had a better mastery of the Pólya stages structured in the MPWs. It was encouraging to observe that *both* the student teachers attempted all the four stages of problem solving including Stage IV (Check and Expand). Below showed the attempts of Amy and Eric in Stage IV:

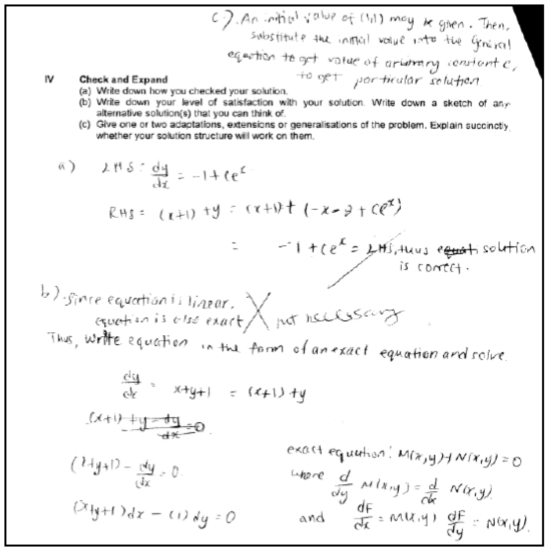


Figure 10. Amy's Stage IV of the Mathematics Practical Test

Figure 11. Eric's Stage IV of the Mathematics Practical Tests.

In their workings, it was evident that both of them managed to check the correctness of their solutions. Amy, though failed to give an alternative solution, managed to make a trivial adaptation of the problem by converting the problem of finding a general solution to an initial value problem. She was awarded 15 out of 20 marks. As for Eric, he gave an alternative solution to the first order differential equation via substitution. Eric was also able to generalize his solution method to cope with the general first order linear differential equation $\frac{dy}{dx} = nx + my + a$ using a linear substitution, and this was impressive given that this problem solving process took place under pressures of 'test' conditions.

Researcher's findings

With the re-design of the mathematics content course based on the problem solving approach, the student teachers were able not only to acquire the same mathematics content knowledge through the traditional way of teaching (despite some lectures being replaced by practical lessons), they managed to demonstrate the ability to use problem solving skills to acquire university knowledge. M2 was especially encouraged to see that the student teachers were able to respond to the change of expectation in relation to mathematical problem solving. There were instances where student teachers could not completely solve a problem; nevertheless, they were able to apply problem solving heuristics to their best efforts in attempting a problem. This contrasted against their initial inertia displayed when they were asked to follow the problem solving model. Most significantly, the student teachers were able to proceed to Stage IV Check and Expand after they had obtained their solutions to a given problem.

Real Analysis

Student profile and course description

The participants of this study were a class of 7 students from the Master of Science (Mathematics for Educators) programme – a coursework based graduate program targeted for mathematics teachers. In this respect, this case study was different from the preceding ones. This course offered at Level Two was intended for the students to engage in deeper theoretical issues related to content mathematics. In this case, real analysis deepens their knowledge of calculus in a highly rigorous way. The topics covered were considered relatively advanced, compared with courses of similar titles taught at a graduate level – the topics were chosen to build towards a general theory of integration called Henstock integration. In this course, the students were introduced to the Riemann integral, the Darboux integral, Lebesgue integral and the Henstock integral, as well as measure theory. The main results are the Fundamental Theorem of Calculus and the equivalence of the Riemann and Darboux integrals. The research question was to find out if the problem solving approach could be used in a graduate course to aid the students in constructing complex proofs involving all these abstract concepts of integration theory.

Out of the seven students, only one student had taken real analysis in their undergraduate education. Interviewing these students at a later point of time revealed that the concepts taught in this course were very abstract and hard to grasp; the problem solving approach helped alleviate, to some extent, the abstractness they were grappling with.

In the first two weeks of the course, the Pólya model and Schoenfeld framework were explicitly introduced to the students. Like M1, some of the elementary results in real analysis were established via the problem solving approach, i.e., M3 presented the theorems as problems, and proceeded to talk aloud the problem-solving processes. To illustrate this point, the following elementary result was posed as a problem:

> **Problem**: Prove that for any real number a, it holds that $-|a| \leq a \leq |a|$. Argue how this inequality can be used to prove the triangle inequality of real numbers.

As M3 guided the students to understand the problem, the meaning of |*a*| was sought for. During the group discussions, the students realized that there were a number of equivalent definitions for |*a*|, and in particular, they chose to use:

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{otherwise} \end{cases}$$

Using the heuristics of drawing a diagram, i.e., a graph of the modulus function, M3 guided the students to derive the required inequality. A middle-ability student, Arnold (pseudonym), gave the following attempt using MPW to scaffold his problem solving endeavor (see Figure 12).

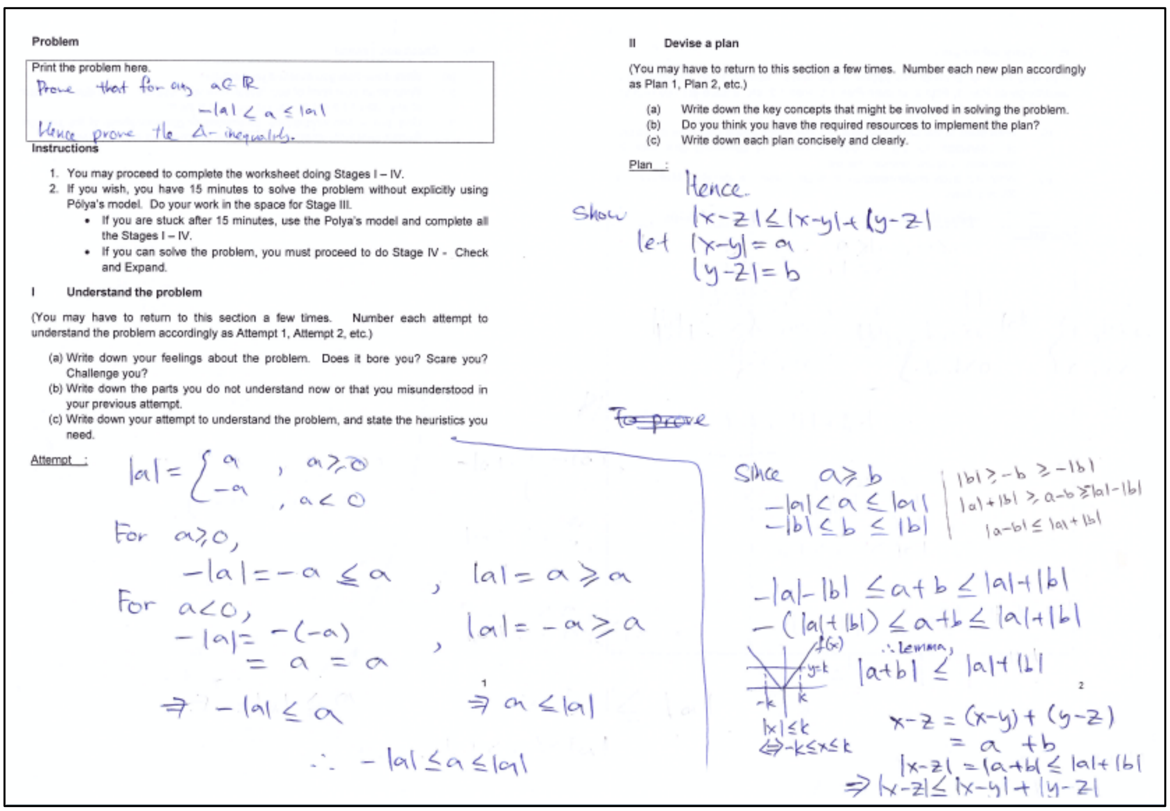


Figure 12. Stages I and II used to prove the basic inequality $-|a| \leq a \leq |a|$

Method

Throughout the 12 week-course, a total of 50 exercises were set. Some of these were seat work done in the class, while the rest were take-home work. The frequency of usage of the MPW by each student was noted. MPWs were always available for the students' usage, i.e., a large number of MPWs were provided during each lesson and the students were free to use them whenever they have something to prove or solve. Because free helpings of MPW meant ready access to the scaffolding device, students viewed the MPWs as part and parcel of "doing mathematics". The practical aspect of solving mathematics problems became second nature to them. This was maintained as the routine feature of the lessons throughout the first 6 weeks. In subsequent lessons for the next 6 weeks, MPWs were never mentioned explicitly but remained to be the "paper for rough working" when it came to constructing proofs for any results that needed to be proven. Students were told to file up all these MPWs so that they could submit their files for assessment at the end of the course – this took up 10% of the total assessment. In this way, M3 was faithful to the original design of the design experiment in that assessment is an important extrinsic motivation for using the problem solving approach via the MPWs. Data collected from the MPWs were analyzed for the frequencies of usage by the students in this class, as well as the success rates for generating correct proofs when the MPWs were used.

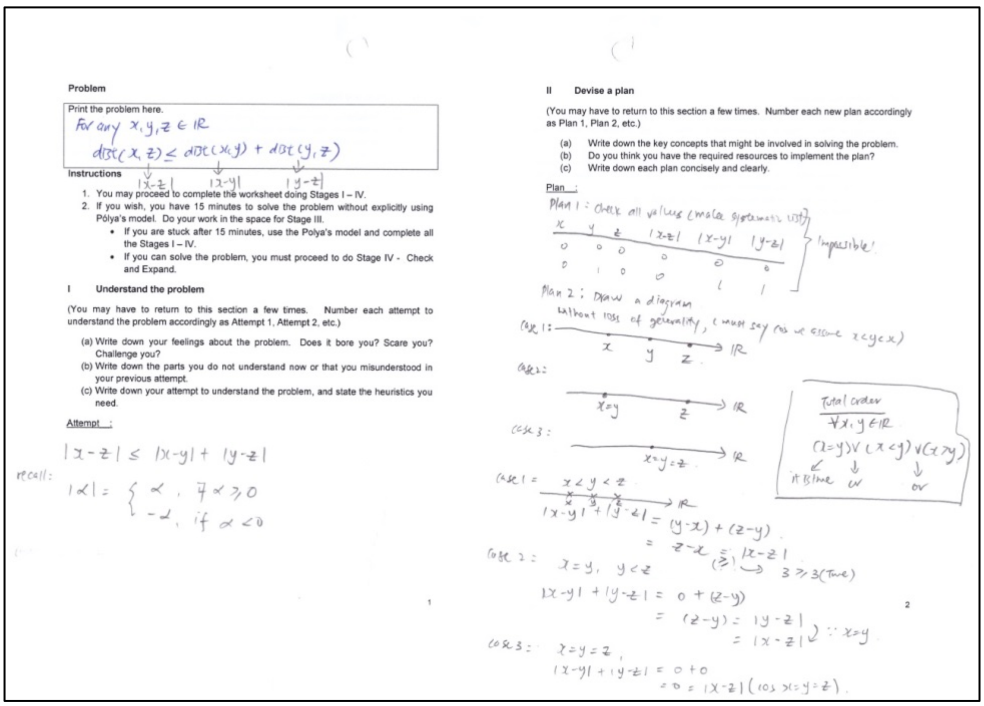


Figure 13. Stages I and II used to prove the triangle inequality of the real line

Notice that Fiona's first attempt was to try listing the different combinations for *x*, *y* and *z* and to check that for each these possibilities the inequality $|x - y| + |y - z| \geq |x - z|$ holds. Soon she realized that it was an impossible task checking that the inequality holds for an infinitude of choices of real values *x*, *y* and *z*. She gave up Plan 1, and moved on to Plan 2. She used the heuristics of "Drawing a diagram" by using the number line to record the relative positions of the three numbers, and "Making suppositions" by assuming without loss of generality that *x* < *y* < z. Somewhere in her execution of Plan 2, she considered how in each case the triangle inequality is satisfied. It was recorded that on the average each student used the MPW to help them with problem solving and proof writing about 60% of the time.

In the last six weeks, M3 did not emphasize on the use of the MPW. This decision was intentional as M3 wanted to find out if the students had internalized the habits of a problem solver. It was observed that with this lack of emphasis, the frequency of usage of MPW dropped to 30% in two weeks. At first, this was a little worrying but a careful look into the students' file revealed that most of the key problem solving processes could still be found in their written work (not done on MPWs). For instance, Ari (pseudonym) who managed to bound the quantity $S(h;P) \dot{-} L$ using an ambiguously defined function h in the first 'version' of her proof (see Figure 14 (left)) continued to offer another bound for the same quantity using the method of difference in the "Improved version" (see Figure 14 (right)). In particular, this could be interpreted that the student was engaging in Stage IV of Check and Expand, where an alternative solution was being meted out. It was noted that there were two students out of seven who finally reverted back to their 'old ways' as they expressed their belief that the mathematics practical approach did not make a big difference.

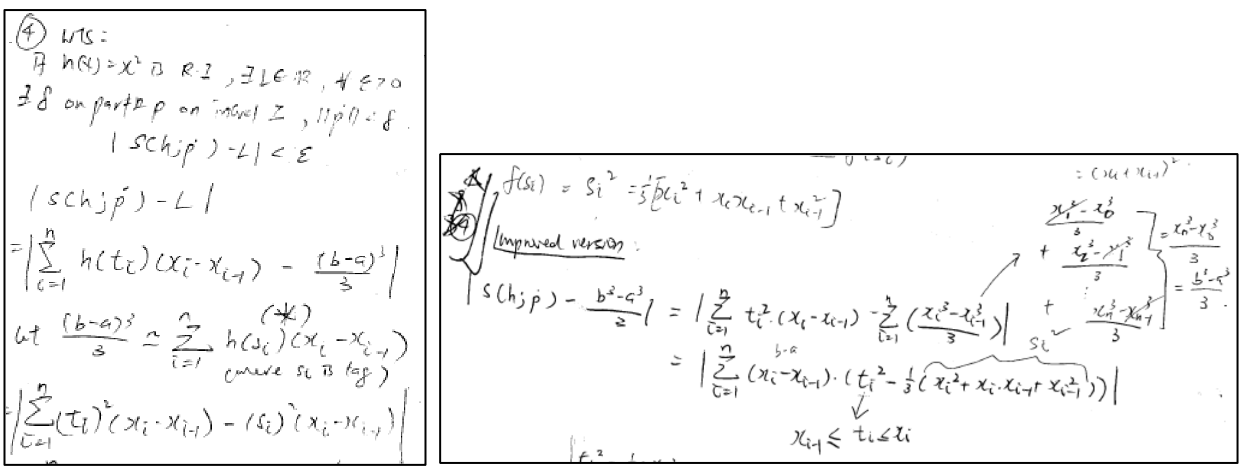


Figure 14. Non-MPW data showing the presence of problem solving at Stage IV.

Somewhat akin to the nature of integration theory, the heuristic of "Drawing a Diagram" was most commonly used among all the heuristics mentioned in the first lecture. In Figure 15 below, Ling (pseudonym) produced a diagram associated to the quadrature of a quadratic curve and the corresponding calculation of the total area A of the trapezoids. Notice somewhere in the control column, an uncompleted attempt to evaluate A was made.

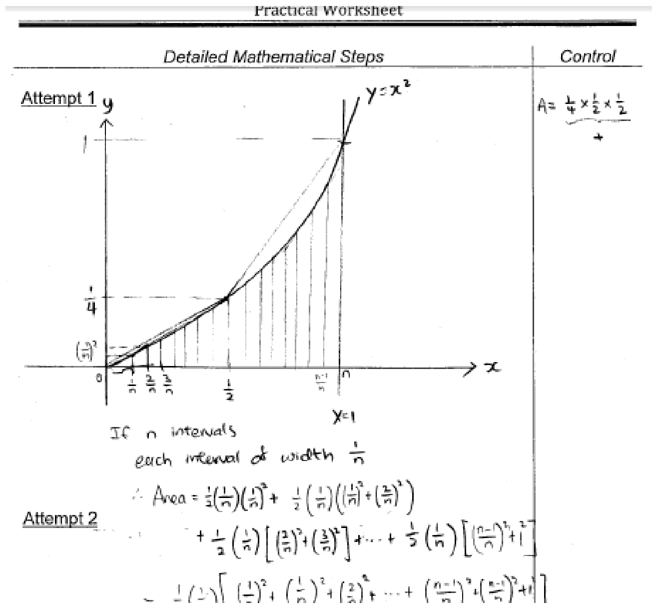


Figure 15. "Drawing a Diagram" was the most frequently used heuristic.

Researcher's findings

The investigator M3 was encouraged that the problem solving approach worked even in the graduate level course. This seems to indicate that the Pólya model for problem solving is universal among all levels of mathematical problem solving activities. A majority of the students began to see the benefits of using the problem solving approach in the first six weeks: beginning from the initial state of not being able to start with a given problem to the final state of being able to come up with sound proofs involving abstract notions was something quite remarkable. Not everything was ideal with the mathematics practical approach. That two students did not believe that the problem solving approach made any difference affirmed Schoenfeld's findings reported in Schoenfeld (1985) that the belief system of an individual could not be easily changed.

Qualitative comparison and analysis

Driven by a common belief that mathematics in practice is essentially problem solving, the three mathematicians M1, M2 and M3 aimed to infuse problem solving element into their university courses. Basing their practice on the design experiment outlined by MProSE which was successfully used in some secondary schools, the three mathematicians proceeded to translate the design to the aforementioned courses: Number theory, Computational mathematics, Differential equations and Real analysis. Understandably, difference in foci, style of delivery and nature of the disciplines resulted in varying degrees of faithfulness to the original design. However, common to all three was an insistence for their students to make explicit use of the Pólya model of problem solving. Except for Computational Mathematics, the problem solving stages in this model were demonstrated during the lesson delivery, whether lectures or mathematics practical lessons.

Crucially, the common scaffolding device in the form of Mathematics Practical Worksheet (MPW) was used explicitly in three courses, and implicitly in the Computational Mathematics assignment. The entire problem solving journey of a student teacher who was attempting to solve problems presented in these courses would be chronicled in these MPWs.

Meeting the challenge of teaching mathematical problem solving to students calls for a curriculum that emphasizes the process (while not ignoring the product) of problem solving and an assessment strategy to match it, to drive teaching and learning. Effective assessment practice begins with and enacts a vision of the kinds of learning we most value for students and strive to help them achieve. The three mathematicians recognized that if the student teachers were not assessed of their problem solving processes, then the translations would be doomed to failure. Adopting the common assessment rubrics developed in Toh et al. (2011), the student teachers were awarded marks in recognition of their problem solving efforts: correct product (the solution) is important, but the process (problem solving journey) more so.

Despite differences, there were common outcomes. Through these translations, student teachers developed good habits of an effective problem solver via the Pólya model. These mathematics apprentices were starting to behave like their masters, i.e., they were, independent of their instructors, beginning to make conjectures, carry out some solution ideas, check on the correctness of solutions by themselves, and extend their solutions. Evidently, the MPW allowed room for the student teachers to express their metacognitive processes in an explicit manner during problem solving.

Implications and concluding remarks

The pedagogical innovation of teaching undergraduate and postgraduate mathematics via the mathematics practical approach yielded positive effects on the problem solving disposition of the student teachers in the aforementioned courses. Our analysis of the students' work shows that the "practical" problem solving design holds promise for mathematics courses at the tertiary level. It would be a worthwhile enterprise, with a substantial investment of time and resources notwithstanding. It is highly satisfactory – as any instructor of a university mathematics course would – to see students show signs of developing good problem solving dispositions. This, however, should not be a one-off experience for the students; rather it would be a worthwhile goal to engage students in problem solving throughout the course of their entire university experience, be it undergraduate or postgraduate.

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