On the largest outscribed equilateral triangle

Fengming Dong, Dongsheng Zhao, and Weng Kin Ho

Abstract. An outscribed triangle of a triangle $\triangle ABC$ is a triangle $\triangle DEF$ such that each side of $\triangle DEF$ contains a vertex of $\triangle ABC$. In this article we study the equilateral outscribed triangles of an arbitrary triangle and determine the area of the largest such triangles. We prove that the largest outscribed equilateral triangle of $\triangle ABC$ can be constructed by ruler and compass and its area equals $\frac{a^2+b^2+c^2}{2\sqrt{3}} + 2S_{\triangle ABC}$ where $S_{\triangle ABC}$ denotes the area of $\triangle ABC$.

Given two triangles $\triangle ABC$ and $\triangle DEF$, if each side of $\triangle DEF$ contains a vertex of $\triangle ABC$, then we call $\triangle DEF$ an outscribed triangle of $\triangle ABC$. Given $\triangle ABC$, let $\Phi_{\triangle ABC}$ be the set of all outscibed equilateral triangles of $\triangle ABC$. Clearly $\Phi_{\triangle ABC}$ is non-empty. In the following we will determine the area of the largest member of $\Phi_{\triangle ABC}$ and show that this largest member can be constructed by ruler and compass from $\triangle ABC$. The corresponding problem on quadrilaterals has been considered in [**1**].

1. Area of the largest outscribed equilateral triangle

Given a triangle $\triangle ABC$, let a and b denote the lengths of the sides BC and AC, respectively, and θ denote the angle $\angle ACB$. Let $\triangle DEF$ be any member in $\Phi_{\triangle ABC}$ as shown in Figure 1 and put $t = \angle DCB$.

Key words and phrases. Outscribed triangle, equilateral triangle.

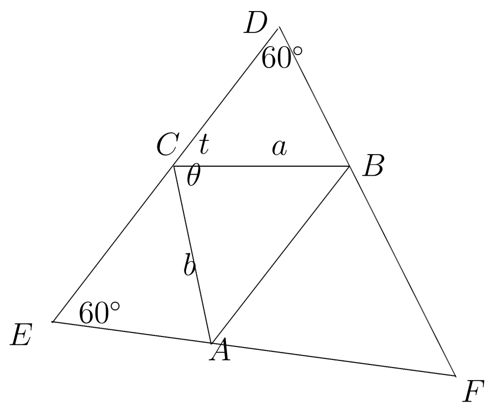

Figure 1

From the above diagram, we can see that the following conditions on t hold:

$$\begin{cases} 0 \leq t < 2\pi/3; \\ \pi/3 < t + \theta \leq \pi. \end{cases} \tag{1}$$

Also note that for any t satisfying the above conditions, there exists a member $\triangle DEF$ of $\Phi_{\triangle ABC}$ such that $t = \angle DCB$.

Using the Sine Rule, we can deduce that

$$DE = \frac{a}{\sin\frac{\pi}{3}}\sin(\frac{2\pi}{3} - t) + \frac{b}{\sin\frac{\pi}{3}}\sin(t + \theta - \frac{\pi}{3}), \tag{2}$$

or, equivalently,

$$\sin\frac{\pi}{3}DE = a\sin(\frac{2\pi}{3} - t) + b\sin(t + \theta - \frac{\pi}{3}). \tag{3}$$

Let $f(t)$ denote the right-hand side of the above expression. Then $f(t)$ is a function in variable t. We now determine the maximum value of $f(t)$.

The following is a key fact for proving the main result.

LEMMA 1.1. *Let a, b and α be constants with $0 < b \leq a$ and $0 < \alpha \leq \pi$. If $\cos\alpha \leq b/a$, then the function*

$$g(x) = a\sin x + b\sin(\alpha - x) \qquad (x \in [0, \alpha])$$

achieves its maximum value

$$(a^2 + b^2 - 2ab\cos\alpha)^{1/2},$$

at the unique root x_0 of the equation

$$a\cos x - b\cos(\alpha - x) = 0 \qquad (x \in [0, \alpha]).$$

In addition, if $\alpha = \pi$, then $x_0 = \pi/2$; otherwise, $x_0 \in (0, \pi/2) \cap [\alpha/2, \alpha]$, and $x_0 = \alpha$ if and only if $\cos\alpha = b/a$.

Proof. If $\alpha = \pi$, then $g(x) = (a + b)\sin x$ whose maximum value is $a + b$ when $x = \pi/2$. Now we assume that $0 < \alpha < \pi$.

Observe that

$$g'(x) = a\cos x - b\cos(\alpha - x)$$

and

$$g''(x) = -a\sin x - b\sin(\alpha - x).$$

We first show that $g'(x_1) \neq g'(x_2)$ for all x_1, x_2 in $[0, \alpha]$ with $x_1 \neq x_2$.

Suppose that $g'(x_1) = g'(x_2)$ for some x_1, x_2 in $[0, \alpha]$ with $x_1 \neq x_2$. Then

$$\begin{aligned} a\cos x_1 - b\cos(\alpha - x_1) &= a\cos x_2 - b\cos(\alpha - x_2); \\ a\cos x_1 - a\cos x_2 &= b\cos(\alpha - x_1) - b\cos(\alpha - x_2); \\ -2a\sin((x_1+x_2)/2)\sin((x_1-x_2)/2)) &= -2b\sin((2\alpha-x_1-x_2)/2)\sin((x_2-x_1)/2); \\ \sin((x_1 - x_2)/2))(a\sin((x_1 + x_2)/2) &+ b\sin((2\alpha - x_1 - x_2)/2)) = 0. \end{aligned}$$

It is clear that $-\pi/2 < (x_1 - x_2)/2 < \pi/2$, and

$$a\sin((x_1 + x_2)/2) + b\sin((2\alpha - x_1 - x_2)/2)) > 0,$$

hence $x_1 - x_2 = 0$, a contradiction.

Thus $g'(x) = 0$ has at most one solution in $[0, \alpha]$. We shall then show that $g'(x)$ has a unique solution in $[0, \alpha]$. Note that

$$g'(x) = a\sin(\pi/2 - x) - b\sin(\pi/2 + x - \alpha).$$

Let $A'B'C'$ be the triangle shown in Figure 2, where $B'C' = a$, $A'C' = b$ and $\angle A'C'B' = \alpha$.

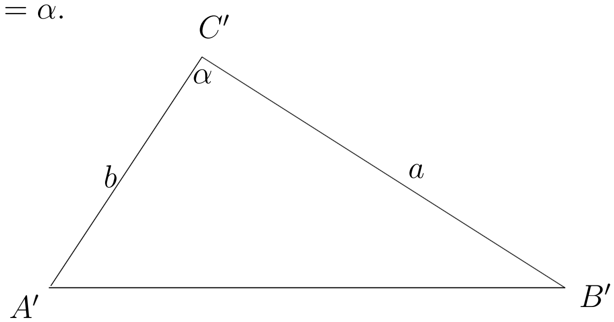

Figure 2

As $\cos\alpha \leq b/a$ and $b \leq a$, we have $\angle A' \leq \pi/2$ and $\angle B' \leq \angle A' \leq \pi/2$, which implies $0 < \angle B' < \pi/2$.

Let $x_0 = \pi/2 - \angle B'$. Then $0 < x_0 < \pi/2$. As $\angle A' \leq \pi/2$, we have

$$\alpha - x_0 = \alpha - (\pi/2 - \angle B') = \alpha + \angle B' - \pi/2 = \pi - \angle A' - \pi/2 \geq 0,$$

Thus $x_0 \leq \alpha$, and $x_0 = \alpha$ if and only if $\angle A' = \pi/2$, i.e., $\cos\alpha = b/a$. We also have $x_0 \geq \alpha/2$ because

$$2x_0 = \pi - 2\angle B' \geq \pi - \angle B' - \angle A' = \alpha.$$

It thus follows that $x_0 \in (0, \pi/2) \cap [\alpha/2, \alpha]$.

Finally we show that x_0 is a root of $g'(x) = 0$ and $g(x)$ has the maximum value at $x = x_0$. Note that $\angle B' = \pi/2 - x_0$ and $\angle A' = \pi - \alpha - (\pi/2 - x_0) = \pi/2 + x_0 - \alpha$. By the Sine Rule, we have

$$a \sin(\pi/2 - x_0) - b \sin(\pi/2 + x_0 - \alpha) = 0,$$

i.e., $g'(x_0) = 0$. So $x_0 = \pi/2 - \angle B'$ is the unique solution of $g'(x) = 0$.

Note that $g''(x_0) < 0$, so $g(x)$ achieves its maximum value at $x = x_0$. We now prove that $g(x_0) = A'B'$.

Let $C'D'$ be the height of $\triangle A'B'C'$ on the side $A'B'$. As both angles A' and B' are acute angles, D' is inside the side $A'B'$, as shown in Figure 3.

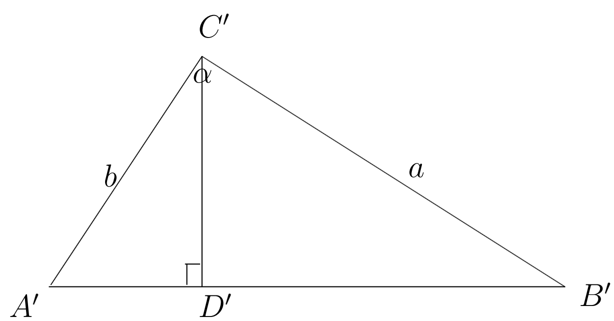

Figure 3

Note that

$$\angle B'C'D' = \pi/2 - \angle A' = \pi/2 - (\pi/2 - x_0) = x_0$$

and

$$\angle A'C'D' = \pi/2 - \angle B' = \pi/2 - (\pi/2 + x_0 - \alpha) = \alpha - x_0.$$

Thus

$$\begin{aligned} g(x_0) &= a \sin(x_0) + b \sin(\alpha - x_0) \\ &= B'D' + A'D' \\ &= A'B' \\ &= (a^2 + b^2 - 2ab \cos \alpha)^{1/2}. \end{aligned}$$

□

Recall that $f(t)$ is the following function:

$$f(t) = a \sin(2\pi/3 - t) + b \sin(t + \theta - \pi/3),$$

for $t \in [0, 2\pi/3) \cap (\pi/3 - \theta, \pi - \theta]$.

Proposition 1.2. *Assume that* $0 < b \le a$, $0 < \theta \le 2\pi/3$ *and* $\cos(\theta + \pi/3) \le b/a$. *Then, in the domain* $[0, 2\pi/3) \cap (\pi/3 - \theta, \pi - \theta]$, $f(t)$ *has the maximum value equal to*

$$(a^2 + b^2 - 2ab\cos(\theta + \pi/3))^{1/2}$$

at the only point t_0 *within the domain such that*

$$a\cos(2\pi/3 - t_0) - b\cos(t_0 + \theta - \pi/3) = 0.$$

Proof. Note that $(2\pi/3 - t) + (t + \theta - \pi/3) = \theta + \pi/3$. By Lemma 1.1, $f(t)$, when $2\pi/3 - t \in [0, \theta + \pi/3]$, has the maximum value equal to

$$(a^2 + b^2 - 2ab\cos(\theta + \pi/3))^{1/2},$$

at the only root t_0 of the following equation

$$a\cos(2\pi/3 - t) - b\cos(t + \theta - \pi/3) = 0.$$

Furthermore, either $2\pi/3 - t_0 = \pi/2$ when $\theta + \pi/3 = \pi$, or $2\pi/3 - t_0 \in (0, \pi/2) \cap [(\theta + \pi/3)/2, \theta + \pi/3]$, i.e., $t_0 \in (\pi/6, 2\pi/3) \cap (\pi/3 - \theta, \pi/2 - \theta/2)$. Thus t_0 is contained in the domain $[0, 2\pi/3) \cap (\pi/3 - \theta, \pi - \theta]$.

Note that if $t \in [0, 2\pi/3) \cap (\pi/3 - \theta, \pi - \theta]$, we have $2\pi/3 - t \in [0, \theta + \pi/3]$. Thus the result holds. □

Now we give the proof of the main result.

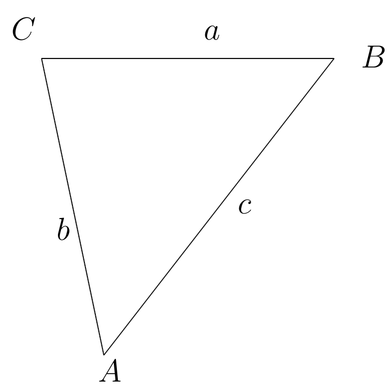

Figure 4

Proposition 1.3. *Let* $\triangle ABC$ *be any triangle with* a, b, c *as the lengths of its three sides, as shown in Figure 4. Then the largest equilateral triangle* $\triangle DEF$ *in* $\Phi_{\triangle ABC}$ *has the area equal to*

$$\frac{\sqrt{3}}{6}(a^2 + b^2 + c^2 + 4\sqrt{3}S_{\triangle ABC}),$$

where $S_{\triangle ABC}$ *denotes the area of* $\triangle ABC$.

PROOF. Assume that $\angle A \geq \angle B \geq \angle C$. Then $a \geq b \geq c$ and $\angle C \leq \angle B < \pi/2$. We claim that $\cos(B+\pi/3) \leq c/a$ or $\cos(C+\pi/3) \leq b/a$.

Suppose, on the contrary, that $\cos(B+\pi/3) > c/a$ and $\cos(C+\pi/3) > b/a$. Then

$$\cos(B+\pi/3)+\cos(C+\pi/3) > c/a+b/a = (c+b)/a > 1.$$

However, we have $\cos(B+\pi/3) \leq \cos(C+\pi/3) < \cos(\pi/3) = 1/2$, a contradiction. Thus either $\cos(B+\pi/3) \leq c/a$ or $\cos(C+\pi/3) \leq b/a$.

Without loss of generality, assume that $\cos(C+\pi/3) \leq b/a$. Let $\theta = \angle C$. By (3) and Proposition 1.2, the maximal value of the sides of members in $\Phi_{\triangle ABC}$ equals

$$\frac{(a^2+b^2-2ab\cos(\theta+\pi/3))^{1/2}}{\sin(\pi/3)}.$$

Thus the area of the largest triangle in $\Phi_{\triangle ABC}$ is

$$\begin{aligned}
& \frac{1}{2}\sin(\pi/3)\left[\frac{(a^2+b^2-2ab\cos(\theta+\pi/3))^{1/2}}{\sin(\pi/3)}\right]^2 \\
= & \frac{\sqrt{3}}{4}(a^2+b^2-2ab\cos(\theta+\pi/3))\cdot 4/3 \\
= & \frac{\sqrt{3}}{3}(a^2+b^2-2ab(\cos\theta\cos\pi/3-\sin\theta\sin\pi/3)) \\
= & \frac{\sqrt{3}}{3}(a^2+b^2-ab\cos\theta+\sqrt{3}ab\sin\theta) \\
= & \frac{\sqrt{3}}{3}((a^2+b^2+c^2)/2+2\sqrt{3}S_{\triangle ABC}) \\
= & \frac{\sqrt{3}}{6}(a^2+b^2+c^2+4\sqrt{3}S_{\triangle ABC}).
\end{aligned}$$

□

2. Construction of the maximal outscribed equilateral triangle

Having determined the area of the largest outscribed equilateral triangle $\triangle DEF$ of $\triangle ABC$, it is then natural to wonder whether $\triangle DEF$ can be constructed from $\triangle ABC$ by Ruler and Compass. In this section, we demonstrate that this is possible.

We first construct $\triangle DEF$ shown in Figure 5 by the following steps and then prove it is the largest member of $\Phi_{\triangle ABC}$.

Step 1: Construct the equilateral triangles $A'BC$, $AB'C$ and ABC' outside $\triangle ABC$, as shown below.

Step 2: Draw segments AA', BB' and CC'.

Step 3: Draw the line passing A and perpendicular to AA'. Similarly, draw the line passing B and perpendicular to BB' and draw the line passing C and perpendicular to CC'.

Step 4: Let D, E and F be the three meeting points of the above three lines.

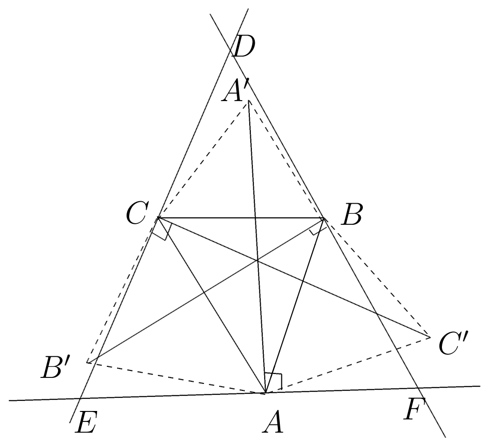

Figure 5

PROPOSITION 2.1. *The triangle DEF constructed above is an equilateral triangle with*

$$S_{\triangle DEF} = \frac{\sqrt{3}}{6}(a^2 + b^2 + c^2 + 4\sqrt{3}S_{\triangle ABC}).$$

Hence triangle $\triangle DEF$ is the largest member of $\Phi_{\triangle ABC}$.

PROOF. We first show that $\triangle DEF$ is equilateral. Observe that $\triangle A'BA \cong \triangle CBC'$, as $A'B = CB, BC' = BA$ and $\angle A'BA = \angle CBA + 60^o = \angle CBA + \angle C'BA = \angle CBC'$. Thus $\angle A'AB = \angle CC'B$, which implies that the points O, A, C', B are concyclic, where O is the meeting point of AA' and CC', as shown in Figure 6.

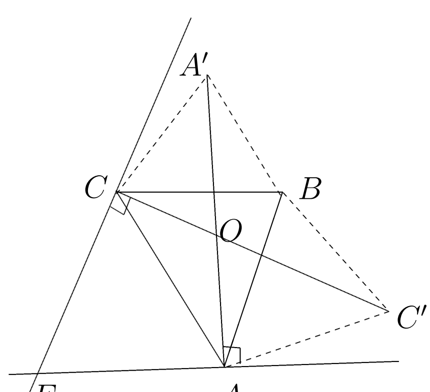

Figure 6

Since O, A, C', B are concyclic, we have $\angle AOC' = \angle ABC' = \pi/3$. Thus $\angle AOC = 2\pi/3$ and so $\angle DEF = \pi/3$, as $A'A \perp EA$ and $CC' \perp EC$. Similarly we can show that $\angle EDF = \angle DFE = \pi/3$, so $\triangle DEF$ is equilateral.

Now we determine the area of $\triangle DEF$. Let $t = \angle BCD$ and $\theta = \angle ACB$, as shown below. By (3), we have

$$\sin(\pi/3)DE = a\sin(\frac{2\pi}{3} - t) + b\sin(t + \theta - \frac{\pi}{3}),$$

where $a = BC$ and $b = AC$.

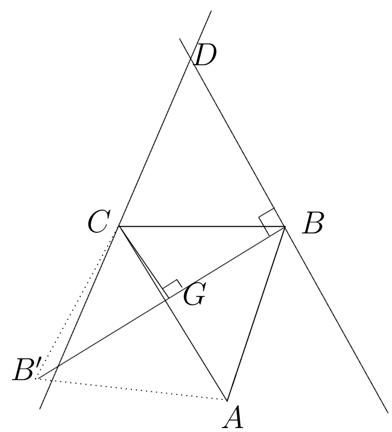

Figure 7

Let CG be the height of $\triangle BB'C$ on the base BB' as shown in Figure 7. As both DB and CG are perpendicular to line BB', we have $CG//DB$, implying that $\angle DCG = \pi - \angle BDC = \pi - \pi/3 = 2\pi/3$. Thus $\angle BCG = 2\pi/3 - \angle DCB = 2\pi/3 - t$. As

$$\angle B'CB = \angle B'CA + \angle ACB = \pi/3 + \theta,$$

we have

$$\angle B'CG = \pi/3 + \theta - \angle BCG = \pi/3 + \theta - (2\pi/3 - t) = t + \theta - \pi/3.$$

Thus

$$\begin{aligned} BB' &= BG + GB' = BC\sin\angle BCG + B'C\sin B'CG \\ &= a\sin(2\pi/3 - t) + b\sin(t + \theta - \pi/3), \end{aligned}$$

implying that

$$\sin(\pi/3)DE = BB'.$$

By the Cosine Rule, we have

$$BB'^2 = BC^2 + B'C^2 - 2BC \times B'C\cos\angle BCB' = a^2 + b^2 - 2ab\cos(\theta + \pi/3).$$

As $DE = BB'/\sin(\pi/3)$, so by a similar argument as the proof of Proposition 1.3, we see that the area of $\triangle DEF$ is

$$\frac{\sqrt{3}}{6}(a^2 + b^2 + c^2 + 4\sqrt{3}S_{\triangle ABC}).$$

From Proposition 1.3, it follows that $\triangle DEF$ is indeed the largest member in $\Phi_{\triangle ABC}$. □

References

[1] Zhao D.: Maximal outboxes of quadrilaterals, *Int. J. Math.Educ. Sci. Technol.*, 42(2011), 4:534-540.

(F. Dong) Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,Singapore 637616
E-mail address: `fengming.dong@nie.edu.sg`

(D. Zhao) Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,Singapore 637616
E-mail address: `dongsheng.zhao@nie.edu.sg`

(W.K. Ho) Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk,Singapore 637616
E-mail address: `wengkin.ho@nie.edu.sg`