Theory of Frames

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To: My wife, Hwee Hoong
A Special Gift
To our baby Samuel due in May 2002

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Chapter 1

Introduction

1.1 Background

The study of frames can be traced back to as early as Wallman's work in 1938, in which he initiated the study of topological properties from a lattice-theoretical point of view. C.Ehresmann and J.Benabou firstly regarded complete Heyting algebras as generalised topological spaces in their own right. Such lattices were called 'local lattices'. It was Dowker and Strauss who first used the term 'frame' in their systematic study of such structure. After that many people have made a significant contribution to the study of frames (or locales: the opposite categorical version of frames), such as Isbell, Banaschewski, Joyal, Johnstone, Simmons, etc.

On the other hand, inspired by frames, several types of generalised frames have been introduced and studied in relatively recent times. Three notable examples of generalised frames are σ-frames, κ-frames and preframes. The κ-frames, which were first systematically studied by Madden recently, generalise both frames and σ-frames. While preframes, which have been carefully studied by Johnstone and Vickers, belong to a different type of generalisation.

The emergence of Z-frames to unify the various concepts of frames, preframes, κ-frames and σ-frames became a natural development. Erne and Bandelt introduced Z-continuous posets which unifies various "continuous" structures and discussed most of their basic properties. In 1992, D.Zhao launched a similar programme in attempt to make a uniform approach to various frame-like structure by introducing Z-frames. The approach turns out to be very convenient and effective for further categorical treatment.

Category theory is an economical tool that provides a common framework for many branches of mathematics, especially in topology and algebra. In the process of my study of generalised frames, categorical concepts are employed extensively.

My three-years course of study has been constantly motivated by many important papers and publications. The first one, *Nuclearity* by K.A. Rowe ([19]), is an important paper. The concept of nuclearity aims to characterise finite-dimensionality in symmetric monoidal closed categories. Rowe made a systematic study of nuclearity via many different examples.

The second one is on *Nuclearity in the category of complete semilattices* by D.A. Higgs and K.A. Rowe ([11]). This paper demonstrated that the nuclear objects of the category of complete lattices are precisely the completely distributive lattices (CDL for short). In lattice theory, completely distributive lattices have always attracted special attention. Thus, the CDLs became one of the most important classes of lattices and have been extensively studied by many authors. This fact, together with many other examples in [19], lead to the following question: Are nuclear objects projective? The first part of my project indicates a positive answer with some minimal assumptions.

The book *A compendium of continuous lattices* ([10]), written by six expert lattice-theorists (G.Gierz et al), is an excellent guidebook for me in learning the ropes of continuous lattice theory. Difficult book it is indeed, but it gives a concise and in-depth treatment of continuity in lattice theory. It gives me a very sound foundation that prepares me to understand D.Zhao's approach to generalised frame theory via Z-theory.

The doctoral dissertation *Generalisation of Continuous Lattices and Frames* by D.Zhao ([21]) gives a detailed and clear introduction of Z-frames. It opened up a completely new and exciting area of research for me because the concepts and mathematical concepts that arise from Z-frame are very rich.

One natural question is whether the concept of nuclearity may be defined for the category of Z-frames. The very first step is, of course, to understand how tensor products may be set up in order that we have an autonomous categorical structure.

So the third paper *Tensor products and bimorphisms* by B.Banaschewski and E.Nelson ([1]) provides very handy information about conditions which will guarantee the existence of tensor multiplication in a concrete category.

Despite the promises that the Z-theory seemed to offer, there is one main obstacle that hinders a natural autonomous structure on **ZFrm**: It is not even clear how the internal hom may be established, let alone the tensor product. However, in the categories of complete join-semilattice, frames and preframes, various constructions have been made to show that they are autonomous categories (see [11], [16] and [17]). This branches off to two alternatives. One of them is to simplify the problem and focus on a less intricate category, namely the category of the Z-complete posets and the morphisms that preserve Z-sups. Although the internal hom exists, we still cannot enjoy the luxury of having a tensor product.

Another approach, which may be more difficult, is to generalise P.Johnstone's work ([16]). It seems that we can take advantage of the monadic nature of the category of Z-frames. Much work, involving Universal Algebra and Proof Theory, remains to be done in this direction.

The sixth is the paper *On projective z-frames* by D.Zhao ([23]) which characterises the **E**-projective objects in the category **zFrm** in adjunction to the category of semilattices. This leads to the study of **E**-projective objects in the category of frames in adjunction to the category of Z-frames. While working furiously at this problem, I ventured into the topic of generalised Scott-topology.

1.2 Research roadmap

My research work is rooted on two problems: (a) Looking for the suitable subset system Z for which the category **ZFrm** is autonomous and (b) characterising the **E**-projective frames (in adjunction to **PreFrm**).

Much effort was put in to understand nuclearity and Z-theory. Infusing these two concepts is an uphill task. Due to my lack of knowledge in Universal Algebra, I was not able to achieve the first target. However, in the midst of studying nuclearity, some original results have been obtained to help understand nuclearity better.

While I was attempting (b), I accidentally made some breakthrough in solving the open problem: Characterisation of the Scott-closed set lattices of non-continuous posets.

Instead of creating a meaningless amalgamation of all the materials, I choose to present the results under the umbrella of Z-theory and with the characterization of Scott-closed set lattices as our destination.

Chapter 2

Literature survey

In this chapter, we shall make a quick but concise survey of all the literature that led to the main result (presented in Chapter 5). For convenience, many definitions will not be explained but these can be found in Chapter 3 and 4 of this thesis. All the theorems and results mentioned in this section are due to the respective authors. They will be stated without proof.

2.1 Projectivity and injectivity in the category Sup

We start by recalling the definition of a projective (and dually, injective) object in a category.

Definition 2.1.0.1 *(**Projectivity**) An object P in category* **A** *is called a projective object if given any epimorphism $\sigma : B \to C$ and any morphism $f : P \to C$ (with B and C objects in* **A***), there is a morphism $f' : P \to C$ such that $\sigma \circ f' = f$. The dual notion is injectivity.*

Definition 2.1.0.2 *(**Residuated mappings**) A function $f : P \to Q$ between posets P and Q is said to be residuated if and only if for every $a \in Q$, there exists $b \in P$ such that $f^{-1}(\downarrow a) = \downarrow b$, where $\downarrow a = \{x \in Q : x \leq a\}$.*

Let $f : P \to Q$ be an order preserving map between complete lattices P and Q. It turns out that f is a residuated map if and only if f is a map preserving existing supremum.

Remark
Let P be a discrete poset with $|P| > 1$ and Q a poset with only one point. Then the unique map from P to Q preserves existing sups but is not a residuated mapping.

In [8], the category of interest is **Sup**, which consists of all complete lattices as objects and residuated mappings as morphisms.
The main result of [8] is:

Theorem 2.1.0.1 *(Crown) L is projective (respectively injective) in* **Sup** *if and only if L is completely distributive.*

This result characterises the projectives (and injectives) of the category **Sup**.

2.2 Nuclearity

The starting point of nuclearity begins in the problem of finding a categorical characterization of finite-dimensional objects in $\mathbf{Vec_K}$, the category of vector spaces over a field K (that is, the finite-dimensional vector spaces).
Let V denote a finite-dimensional vector space over K and let $f : V \to V$ denote any linear operator. Thus, $f \in Hom(V, V)$ in the language of category. For any ordered basis $\mathbf{a} = (a_1, a_2, ..., a_n)$ of V, let

$$f[\mathbf{a}] = \Sigma A_{ij} a_i \otimes a_j^* \in V \otimes_K V^*.$$

where $\otimes_K$ denotes the usual tensor product of K-modules, V^* is the dual space of V, i.e. $V^* = [V, K] = \{f : V \to K\}$, $A = (A_{ij})$ is the matrix representing f with respect to the basis $\mathbf{a}$, and $(a_1^*, a_2^*, ..., a_n^*)$ is the dual basis of $\mathbf{a}$. Now we shall verify that the element of $V \otimes_K V^*$ determined by f and $\mathbf{a}$ above is independent of the choice of the basis $\mathbf{a}$. To do this, we first show that the map $\varphi_{[\mathbf{a}]} : f \mapsto f[\mathbf{a}]$ is inverse to the "natural" linear map $\phi_V : V \otimes_K V^* \to Hom_K(V, V)$ given by $\phi_V(a \otimes \alpha)(x) = \alpha(x)(a)$. In fact, for any $f : V \to V$ and $j_0 \leq n$,

$$\begin{aligned} \phi_V(f[\mathbf{a}])(a_{j_0}) &= \phi_V(\Sigma A_{ij} a_i \otimes a_j^*)(a_{j_0}) \\ &= \Sigma a_j^*(a_{j_0}) A_{ij} a_i \\ &= \Sigma A_{ij_0} a_i \\ &= f(a_{j_0}) \end{aligned}$$

(since $a_j^*(a_{j_0}) = 1$ for $j = j_0$ and $a_j^*(a_{j_0}) = 0$ for $j \neq j_0$.)
Since $\{a_j\}$ is a basis, $\phi_V(f[\mathbf{a}])(x) = f(x)$ hold for all $x \in V$.
So, we have just shown that $\phi_V \circ \varphi_{[\mathbf{a}]} = id_{Hom(V,V)}$. It follows immediately that ϕ_V is surjective. It was claimed in [19] that ϕ_V is injective. But we supply the proof here for completeness sake.
We take any element (finite sum) $\Sigma \lambda_{ij} a_i \otimes a_j^*$ in $ker(\phi_V)$. Then, for any $x \in V$,

$$\begin{aligned} \phi_V(\Sigma ij \lambda_{ij} a_i \otimes a_j^*)(x) &= 0 \\ \Rightarrow \Sigma_{ij} \lambda_{ij} a_j^*(x)(a_i) &= 0 \\ \Rightarrow \Sigma_i(\Sigma_j \lambda_{ij} a_j^*(x)) a_i &= 0 \\ \Rightarrow \Sigma_j \lambda_{ij} a_j^*(x) &= 0 \quad \forall i \\ \Rightarrow (\Sigma_j \lambda_{ij} a_j^*)(x) &= 0 \quad \forall i \\ \Rightarrow \Sigma_j \lambda_{ij} a_j^* &= 0 \quad \forall i \\ \Rightarrow \lambda_{ij} &= 0 \quad \forall i, j \end{aligned}$$

Thus $Ker(\phi_V) = \{0\}$, and so ϕ_V is injective.
So, ϕ_V is an isomorphism for every finite-dimensional V.
Suppose we defined the map $\varphi_{[\mathbf{b}]}$ using the basis $\mathbf{b} = (b_1, b_2, ..., b_n)$, we would also have:

$$\phi_V \circ \varphi_{[\mathbf{b}]} = id_{Hom(V,V)}.$$

But the inverse of an isomorphism is unique, it follows that $f[\mathbf{a}] = f[\mathbf{b}]$. Hence it is verified that the element of $V \otimes_K V^*$ determined by f and $\mathbf{a}$ above is independent of the choice of $\mathbf{a}$.
Conversely, assume that ϕ_V is an isomorphism. Suppose that V is not finite-dimensional.

We observe that if f is in $\text{Im}(\phi_V)$, then there is a finite linear combination of elements in the tensor product $V \otimes_K V^*$ of the form $\Sigma A_{ij} a_i \otimes a_j^*$ for some bases $\{a_i\}$ for V and $\{a_j^*\}$ for V^* respectively such that $\phi_V(\Sigma A_{ij} a_i \otimes a_j^*) = f$. So, it follows that the rank of f is finite. But then the identity mapping id_V ,which has infinite rank, is not in the $\text{Im}(\phi_V)$. So ϕ_V is not onto. It follows that V is finite-dimensional if ϕ_V is an isomorphism.
It appears that similar morphisms as ϕ could be defined in the more general context of symmetric monoidal categories (also known as autonomous categories) and this provides us a way to define finite dimensional objects in a more general setting. This definition can be found in Chapter 3.
In the setting of such a category, Rowe proposed the following definition in [19]:

Definition 2.2.0.3 *(Nuclearity) If A is any object of the autonomous category* $\mathbf{A}$, *there is a morphism $\phi_A : A \otimes A^* \to [A, A]$ whose internal adjoint is given by the morphism*

$$(A \otimes A^*) \otimes A \to A \otimes K \cong A.$$

A is nuclear if ϕ_A is an isomorphism.

Later in Chapter 3, we shall state a refinement of this definition due to a later paper [11]. In [19], Rowe gave the characterisation of nuclear objects in many different categories. These are reproduced in Chapter 3. In many of these examples discussed, the nuclear objects are incidentally projective. This leads us to post the following question:

Are nuclear objects necessarily projective?

A partial answer to this question will be given in Chapter 3.

2.3 Nuclearity in the category Sup

In [11], Higgs and Rowe characterised the nuclear object in the category **Sup** of complete lattices and mappings which preserve arbitrary joins.

Theorem 2.3.0.2 *(Higgs & Rowe) An object of* **Sup** *is nuclear if and only if it is completely distributive.*

From Theorem(2.1.0.1), we know that projective objects in **Sup** are precisely completely distributive lattices. This result gives us yet another indication that a nuclear object may indeed be projective.
We now state our conjecture formally:
Conjecture: If A is a nuclear object in a given autonomous category $\mathbf{A}$, then it is projective.
Some implications:

1. If this conjecture were true, it would amount to one half of Theorem(2.3.0.2), i.e. If L is nuclear in **Sup**, then by the hypothesis, L will be projective in **Sup**. And by Theorem(2.1.0.1), we can deduce that L is a completely distributive lattice.

2. Given an autonomous category, we may use this property to look for possible candidate objects which are nuclear.

It turns out that this conjecture is true with some natural assumptions. Thus, the question is answered somehow completely by Theorem(3.3.0.7) in Chapter 3 which is the main result presented in the first part of this thesis.

2.4 Nuclearity and other categorical properties

It is discovered in Chapter 3 that the nuclear objects in an autonomous category (endowed a projective base object) are necessarily projective. So a natural question is: What are the other properties of a base object that can be passed on to the other nuclear objects? The second main result in Chapter 3 sheds some light to this problem:

Proposition 2.4.0.1 *Let* $\mathbf{K}$ *be an autonomous category equipped with a base object* I*, where* I *is the limit (dually colimit) of a diagram* F *in* $\mathbf{K}^{\mathbf{J}}$ *for some small indexing category* $\mathbf{J}$.

(i) If A *is reflexive (i.e.* $A \cong A^{**}$*), then* A *is the limit (dually colimit) of the diagram* $[A^*, F(-)]$ *in* $\mathbf{K}^{\mathbf{J}}$.

(ii) If A *is nuclear, then* A *is the limit (dually colimit) of the diagram* $F(-) \otimes A$ *in* $\mathbf{K}^{\mathbf{J}}$.

2.5 Z-frames and Smooth Lattices

The emergence of Z-frames arises out of an attempt to generalise many different types of "frames" such as frames, preframes, σ-frames and κ-frames. The key method employed here is to define a subset system for a given semilattice S.

Definition 2.5.0.4 *(System of sets on* Slat*)*
By definition, a system Z *of sets on* **Slat** *is a function which assigns to each semilattice* S *a collection of lower subsets of* S *such that the following conditions are satisfied:*

(S1) $\downarrow x \in Z(S)$ *for each* $x \in S$;

(S2) If $f : S \to T$ *is a semilattice morphism, then* $\downarrow f(D) \in Z(T)$ *for each* $D \in Z(S)$;

(S3) Each $Z(S)$ *is a semilattice with respect to the inclusion order of sets;*

(S4) For each $\mathcal{D} \in Z(Z(S))$, $\cup\mathcal{D} \in Z(S)$.

Definition 2.5.0.5 *(Z-frame)*
Given a set system Z *on* **Slat***, a semilattice* A *is called a* Z*-frame if it satisfies the following two conditions:*

(1) A *is* Z*-complete;*

(2) $x \wedge \vee D = \vee\{x \wedge d : d \in D\}$ *holds for each* $x \in A$ *and any* $D \in Z(A)$.

Likewise, one may consider a subset system Z of sets on **Poset** which satisfies similar properties as (S1) - (S4) (see [5]).
The set of axioms afforded by the theory of Z-frame (Z-theory for short) is a very rich one. In particular, it makes convenient further development of the theory of frame in a categorical setting. Concepts such as adjunction between certain categories, Z-continuity, stability and **E**-projectivity are also studied systematically by D.Zhao in [21] and [23]. The main result that was originally from [23] is also studied and reproduced (with the kind permission from Dr. Zhao) in this report:

Theorem 2.5.0.3 ***(D.Zhao)***
A Z-frame A is **E***-projective if and only if it is stably Z-continuous.*

It is also known that an adjunction exists between the category of Z-frames and the category of frames. A natural question is: How do we characterise the relative **E**-projective frames? In turn, this leads to an unresolved problem: How does one characterise $\sigma^{op}(P)$ - the lattice of Scott-closed subsets of a dcpo A?

An intense study was carried out and the theory of smooth lattices emerged. This concept runs parallel to that of the continuous lattices and establishes a strong relationship with the existing literature of continuous lattices. This will form the last and in fact the major result of my thesis.

Chapter 3

Nuclearity

3.1 Introduction

We start by recalling the definition of an autonomous category, as used by Rowe in [19].

Definition 3.1.0.6 ***(Autonomous category)*** *An autonomous category is a category* $\mathbf{A}$ *equipped with a 'base object'* I*, a 'tensor product'* $\otimes : \mathbf{A} \times \mathbf{A} \to \mathbf{A}$*, and an 'internal hom'* $[-,-] : \mathbf{A}^{op} \times \mathbf{A} \to \mathbf{A}$*. There are also given natural isomorphisms:*

$$i : I \otimes A \cong A,\ \tau : A \otimes B \cong B \otimes A \text{ and } \pi : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

satisfying the Mac Lane's coherence conditions ([18], XI. §1) which ensures that $(\mathbf{A}, \otimes, I)$ *is a symmetric monoidal category. Finally, it is required that for each object* B *of* $\mathbf{A}$ *the functor* $- \otimes B$ *is left adjoint to the functor* $[B, -]$*, from which it follows that there are natural isomorphisms* $\Omega : [A \otimes B, C] \cong [A, [B, C]]$ *which internalises this adjointness.*

Remark
In particular, we wish to highlight one important axiom that holds in an autonomous category $\mathbf{A}$ (see [6]): Given any three objects A, B and C in $\mathbf{A}$, there is a natural isomorphism

$$\omega_{A,B,C} : \mathrm{Hom}_{\mathbf{A}}(A \otimes B, C) \cong \mathrm{Hom}_{\mathbf{A}}(A, [B, C]).$$

Let $\mathbf{A}$ be an autonomous category. If A is any object of $\mathbf{A}$, we write $A^* = [A, I]$; that is, we denote the functor $[-, I] : \mathbf{A}^{op} \to \mathbf{A}$ by $(-)^*$. The following morphisms may be defined for any objects A, B and C of $\mathbf{A}$:

(a) $e_{A,B} : [A, B] \otimes A \to B$ ('internal evaluation'), which is obtained by adjointness from the identity morphism $[A, B] \xrightarrow{1_{[A,B]}} [A, B]$;

(b) $\eta_A : A \to A^{**}$, which is obtained by adjointness from the composition

$$A \otimes A^* \xrightarrow{\tau_{A,A^*}} A^* \otimes A \xrightarrow{e_{A,I}} I;$$

(c) $\Gamma_{A,B,C} : [B, C] \otimes [A, B] \to [A, C]$ ('internal composition') which is obtained by adjointness from the composition:

$$([B,C]\otimes[A,B])\otimes A \xrightarrow{\pi} [B,C]\otimes([A,B]\otimes A) \xrightarrow{1_{[B,C]}\otimes e_{A,B}} [B,C]\otimes B \xrightarrow{e_{B,C}} C;$$

(d) $\phi_{A,B} : B\otimes A^* \to [A,B]$;

(e) $\theta_{A,B} : B\otimes A^* \to [B,A]^*$.

The morphism $\phi_{A,B}$ in (d) is obtained by adjointness from the composition

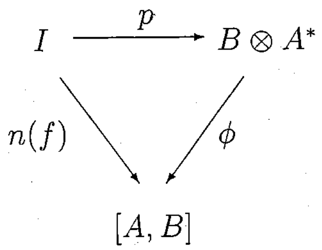
$$(B\otimes A^*)\otimes A \overset{\pi}{\cong} B\otimes(A^*\otimes A) \xrightarrow{1_B\otimes e_{A,I}} B\otimes I \overset{\tau_{B,I}}{\cong} I\otimes B \overset{i_B}{\cong} B,$$

while the morphism $\theta_{A,B}$ in (e) is obtained by adjointness from the composition

$$(B\otimes A^*)\otimes[B,A] \cong A^*\otimes([B,A]\otimes B) \xrightarrow{1_{A^*}\otimes e_{B,A}} A^*\otimes A \xrightarrow{e_{A,I}} I.$$

We say that A is reflexive if $\eta : A \to A^{**}$ is an isomorphism, a term which is borrowed from functional analysis.
For any morphism $f : A \to B$ in $\mathbf{A}$, we define the name of f to be the morphism $n(f) : I \to [A,B]$ obtained by adjointness from the composition $I\otimes A \overset{i_A}{\cong} A \xrightarrow{f} B$.

Definition 3.1.0.7 *(**Nuclearity**) A morphism $f : A \to B$ is nuclear if its name factors through the morphism ϕ; that is, if there exists a morphism $p : I \to B\otimes A^*$ such that the diagram*



commutes. An object A is nuclear if the identity morphism on A is nuclear.

We will refer to a morphism p satisfying the condition of Definition(3.1.0.7) as a *pseudoname* of f. We note that a pseudoname of f, when it exists, is not necessarily uniquely determined by f.
We shall list some useful theorems regarding nuclearity found in [19] and [11].

Proposition 3.1.0.2 *(Rowe [19])*
If A is nuclear, A is reflexive.

Proposition 3.1.0.3 *(Higgs & Rowe [11]) If $f : A \to B$ is nuclear, so are the following:*

(a) $f^* : B^* \to A^*$;

(b) $g\circ f : A \to C$ for any morphism $g : B \to C$;

(c) $f \circ h : D \to B$ *for any morphism* $h : D \to A$.

Proposition 3.1.0.4 *(Higgs & Rowe [11])*
If $f : A \to C$ *and* $g : B \to D$ *are nuclear, then so is* $f \otimes g : A \otimes B \to C \otimes D$.

Theorem 3.1.0.4 *(Rowe [19])*
Let $\mathbf{A}$ *be an autonomous category. Then the full subcategory of* $\mathbf{A}$ *determined by the nuclear objects of* $\mathbf{A}$ *is again an autonomous category.*

Corollary 3.1.0.1 *(Rowe [19]) If* A *is nuclear then so is* A^*.

Theorem 3.1.0.5 *(Higgs & Rowe [11])*
For any object A *of* $\mathbf{A}$*, the following are equivalent:*

(a) A *is nuclear.*

(b) $\phi : A \otimes A^* \to [A, A]$ *is an isomorphism.*

(c) $\phi : B \otimes A^* \to [A, B]$ *is an ismorphism for every object* B.

(d) $\phi : A \otimes B^* \to [B, A]$ *is an isomorphism for every object* B.

Corollary 3.1.0.2 *For any object* A *in* $\mathbf{A}$*, the following are equivalent:*

(a) A *is nuclear.*

(b) $\phi : - \otimes A^* \to [A, -]$ *is a natural isomorphism.*

Remark
To prove that an object A is nuclear, it is often convenient to use Corollary(3.1.0.2).

3.2 Some motivating examples

In [19] Rowe identified nuclear objects in several different categories. We shall show that in all these categories nuclear objects are projective.

1. The category $\mathbf{Vec_K}$ of vector spaces over a field K
   This is the same example cited in Chapter 2. Note that a vector space is nuclear if and only if it is finite-dimensonal. Note that a finite-dimensional vector space is projective. To see this, let V be a finite-dimensional vector space and $\mathbf{a}$ be an ordered basis $\{a_1, ..., a_n\}$. An epimorphism in $\mathbf{Vec_K}$ is precisely a surjective linear transformation. Take an epimorphism $\sigma : V_1 \to V_2$. Let $f : V \to V_2$ be a linear transformation. Since V is finite-dimensional, $\text{Im}(f)$ is finite-dimensional. More accurately, $\text{Im}(f)$ is spanned by the spanning set $\{f(a_1), ...f(a_n)\}$. Since σ is surjective, for each $1 \leq i \leq n$, there is a $b_i \in V_1$ such that $\sigma(b_i) = f(a_i)$. Then define $g : V \to V_1$, $g(\Sigma\alpha_i a_i) = \Sigma\alpha_i g(a_i) = \Sigma\alpha_i b_i$. It is easy to see that $\sigma \circ g = f$. hence V is projective. It is also noted that the "base object" K is 1-dimensional and hence projective.

2. The category $_R\mathbf{M}$ of R-modules where R is a commutative ring with unit
Since R is commutative, we need not distinguish between the left and right module. So tensor product of two left R-modules is well-defined via the usual universal mapping properties. Note that the base object in this category is R. Let A be nuclear. Then the canonical homomorphism $\phi_A : A \otimes Hom_R(A, R) \to Hom_R(A, A)$, given by

$$\phi_A(a \otimes \alpha)(x) = (\alpha(x))(a) \text{ where } a, x \in A, \alpha \in Hom_R(A, R),$$

has an inverse. Let $\phi_A^{-1}(1_A) = \Sigma_{k=1}^m a_k \otimes f_k$. Since for any $a \in A$, we have

$$a = \phi(\phi^{-1}(1_A))(a) = \Sigma_{k=1}^m f_k(a) a_k.$$

We immediately conclude that A is generated by $\{a_1, ..., a_m\}$.
By Proposition(3.1), p. 132 of [7], we deduce that A is projective.
In short, we have shown that if A is nuclear, then it is finitely generated and projective.
Conversely, suppose A is finitely generated, say by $\{a_1, a_2, ..., a_n\}$, and projective. Since A is finitely generated, define $f : R^n \to A$ by $f(e_i) = a_i$ where $e_i = (0, 0, ..., 1, ..., 0)$ (where it is a "1" at the i^{th} position). Then we have a split exact sequence

$$R^n \xrightarrow{f} A \longrightarrow 0.$$

Let $g : A \to R^n$ be the splitting map, i.e. $fg = 1_A$ and let $k = 1, ..., n$, $\pi_k : R^n \to R$ denote the "k^{th} projection" of g. Then define the map ϕ_A^{-1} as:

$$\phi_A^{-1}(h) = \Sigma h(a_k) \otimes (\pi_k g).$$

It is easy to see that:

$$\begin{aligned}(\phi_A(\phi_A^{-1}(h)))(a) &= \phi_A(\Sigma h(a_k) \otimes (\pi_k g))(a) \\ &= h\Sigma((\pi_k g)(a))(a_k) \\ &= h(a)\end{aligned}$$

and

$$\begin{aligned}(\phi_A^{-1}(\phi_A(\Sigma a_k \otimes f_k)(-)) &= (\phi_A^{-1}(\Sigma f_k(-)a_k)) \\ &= \Sigma((\Sigma f_k(a_j)a_k) \otimes (\pi_j g)) \\ &= (\Sigma a_j \otimes f_j)(-)\end{aligned}$$

It follows that A is nuclear. Therefore A is nuclear if and only if A is finitely generated and projective.

3. The category $\mathbf{S}_*$ of pointed sets
The category $\mathbf{S}_*$ in which an object X is a pair consisting of a set $|X|$ and a "base point" $p(X) \in |X|$, while a morphism $f : X \to Y$ is a function $f : |X| \to |Y|$ such that $f(p(X)) = p(Y)$ (also known as "base-point" preserving maps). The internal hom in this category is the set of morphisms of which the "base point" is the morphism which takes every point in the domain to the specified point in the codomain.
More precisely, let the base object be $|K| = 2 = \{0, 1\}$, $p(K) = 0$. So, for any objects X and Y in $\mathbf{S}_*$, the internal hom is the set

$$[X,Y] = \{f : |X| \to |Y|, f(p(X)) = p(Y)\}$$

with "base point" the morphism $f_0 : |X| \to |Y|, f(X) = \{p(Y)\}$.
The tensor product of two objects is the usual "smash product" obtained from their ordinary cartesian product, their wedge which is the set $\{|X| \times \{p(Y)\}\} \cup \{\{p(X)\} \times |Y|\}$. In other words, the "smash product" is the quotient of the cartesian product over the wedge - so that geometrically speaking, the wedge effectively collapses to a point. More precisely, for any two objects A and B in this category, let $\pi : A \times B \to A \otimes B$ denote the canonical map such that

$$\pi(a,b) = [(a,b)]$$

where $[(a,b)]$ denotes the equivalence class, containing the point (a,b), which is generated by the following equivalence relation $\sim$ defined on $A \times B$:

$$(a,b) \sim (c,d) \Leftrightarrow \text{either } (a,b) = (c,d) \text{ or } (a,b) \text{ and } (c,d) \text{ both belong to the wedge.}$$

The isomorphism $\sigma : K \otimes A \to A$ is given by $\sigma(\pi(0,a)) = p(A)$, $\sigma(\pi(1,a)) = a$. The canonical morphism $\phi_A : A \otimes A^* \to [A,A]$ is given by

$$(\phi_A(\pi(a,f)))(x) = \begin{cases} a & \text{if } f(x) = 1; \\ p(A) & \text{if } f(x) = 0. \end{cases}$$

If ϕ_A has an inverse ϕ_A^{-1}, set $\phi_A^{-1}(1_A) = \pi(a_0, f)$. Then for all $x \in |A|$, we have

$$x = (\phi_A(\pi(a_0,f)))(x) = \begin{cases} a_0 & \text{if } f(x) = 1; \\ p(A) & \text{if } f(x) = 0. \end{cases}$$

Rowe highlighted that the nuclear objects are those underlying sets have cardinality 1 or 2. To see this, consider the case when A is nuclear. Then $\phi_A^{-1}(1_A)$ is given as above, and if $a \in |A|$, then $a = a_0$ if $f(a) = 1$; while if $f(a) = 1$, then $a = p(A)$. Hence, card($|A|$)$\leq$ 2. On the other hand, if card($|A|$)$\leq$ 2, it is easy to see that A is nuclear. Let A be an object with card($|A|$)$\leq$2. Given a morphism $f : A \to C$ and an epimorphism σ between the objects B and C. If card($|A|$)=1, then it is obviously projective. Otherwise, when $A = \{0,1\}$ (say) with $p(A) = 0$, define $c = f(1)$. By the surjectivity of σ, there is a point b in B such that $\sigma(b) = c$. Then let $g : A \to B$ be the mapping

$$g(x) = \begin{cases} b & \text{if } x = 1; \\ p(B) & \text{if } x = 0. \end{cases}$$

One readily sees that $\sigma \circ g = f$. Thus A is projective.

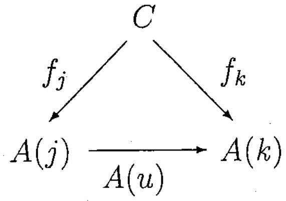
4. Category of real or complex Banach spaces, **Ban**
   Let **Ban** denote the category of real or complex Banach spaces in which the morphisms are just bounded linear operators. In [11], Rowe mentioned that an object in **Ban** is nuclear if and only if it is nuclear as a vector space, i.e. a finite dimensional vector space. Again, each nuclear object is projective.

5. Category of complete lattices, **Sup**
   The objects of **Sup** (also denoted in [11] as **CJSL**: Complete Join Semilattices) are all complete lattices and the morphisms are all the supremum-preserving functions between them. Such morphisms are exactly the residuated maps defined by Crown in [8]. According to [11], in the category of **Sup** the base object I is the two-element chain $2 = \{0, 1\}$ with $0 < 1$. The internal hom $[A, B]$ of A and B is the set of all morphisms from A to B with supremum defined pointwise. We shall write $A^* = [A, 2]$. The tensor product of A and B will be defined by $A \otimes B = [A, B^*]^*$. The various morphisms and natural isomorphisms equipped by **Sup** to be an autonomous category were meticulously defined in [11]. The main result is that of Theorem 4.10 which states that an object is nuclear if and only if it is completely distributive. In [8], Crown showed that a projective object in **Sup** is precisely a completely distributive lattice. It is also well known that the "base object" I, the two-element chain $2 = \{0, 1\}$ is projective. Thus in this case every nuclear object is also projective.

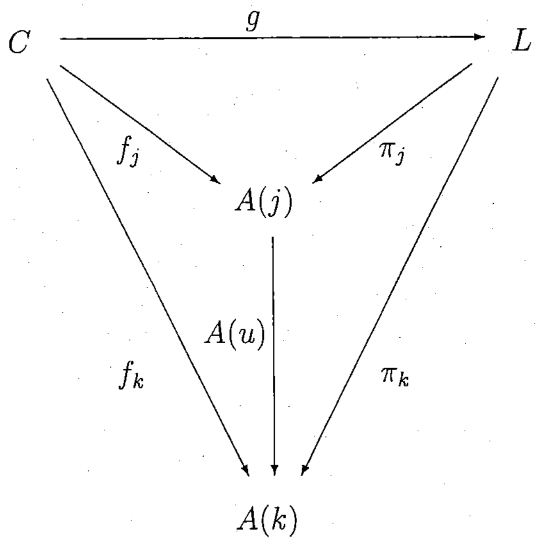
3.3 Projectivity of nuclear objects

We gather sufficient evidence to believe that a nuclear object in an autonomous category (which is equipped with a projective "base object") is necessarily projective. In this section, we are going to establish this main result. Hereafter, the category $\mathbf{A}$ is an autonomous category.
Before we establish the next lemma, it is timely to recall some categorical notions.

Definition 3.3.0.8 ***(Limit and Colimit)*** *Let $\mathbf{C}$ be a fixed category. For a small category $\mathbf{J}$ (the "indexing" category), we consider the category $\mathbf{C}^{\mathbf{J}}$ of functors from $\mathbf{J}$ to $\mathbf{C}$. An object of $\mathbf{C}^{\mathbf{J}}$ is also called a diagram in $\mathbf{C}$ of type $\mathbf{J}$. For each object C, there is a constant diagram $\triangle_{\mathbf{J}}(C) : \mathbf{J} \to \mathbf{C}$, $j \mapsto C$. Thus, $\triangle_{\mathbf{J}} : \mathbf{C} \to \mathbf{C}^{\mathbf{J}}$ defines the diagonal functor $\triangle_{\mathbf{J}}$.*
A natural tranformation π from the constant diagram $\triangle_{\mathbf{J}}(C)$ to some other diagram A of $\mathbf{C}^{\mathbf{J}}$, then consists of morphisms $f_j : C \to A_j$, one for each object j in $\mathbf{J}$, such that for any $u : j \to k$ in $\mathbf{J}$, the triangle



commutes.
Such a natural transformation is called a cone $f : C \to A$ on the diagram A with vertex C. A cone $\pi : L \to A$ with vertex L is universal to A when to every cone $f : C \to A$, there is a unique map $g : C \to L$ in $\mathbf{C}$ with $\pi_j \circ g = f_j$ for each j of $\mathbf{J}$, as in the commutative diagram:



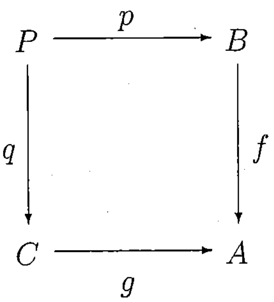
This universal cone $\pi : L \to A$ (or less accurately, its vertex) $L = lim_{\leftarrow \mathbf{J}} A$ is called the limit of the diagram A. The dual notion is colimit.

Examples of Limits
The following examples involve other common categorical concepts such as cartesian product, epimorphism, monomorphism, pullback and pushouts - the definitions of which may be obtained from [18].

1. Take $\mathbf{J} = \mathbf{2} = \{0, 1\}$ to be the category with two objects 0 and 1 and only 2 identity morphisms: $1_0 : 0 \to 0$ and $1_1 : 1 \to 1$. Let $\mathbf{C}$ be a category with finite limits. Let A and B be two objects in $\mathbf{C}$. Then the cartesian product $A \times B$ may be viewed as a limit of the diagram $F \in \mathbf{C}^{\mathbf{J}}$, where $F(0) = A, F(1) = B$ and $f_0 = \pi_0, f_1 = \pi_1$. Let $f : X \to A$ and $g : X \to B$ be any two morphisms. Since $A \times B$ is the product of A and B, there exists a unique morphism $h : X \to A \times B$ such that $\pi_0 h = f$ and $\pi_1 h = g$. So, $A \times B = \lim_{\leftarrow \mathbf{J}} F$.

2. Let $\mathbf{C}$ be a category with X as a terminal object and $\mathbf{J}$ be the empty category.
For any object C in $\mathbf{C}$, there is a unique morphism $f : C \to X$.
So we may view the terminal object $X = \lim_{\leftarrow \mathbf{J}} *$, where $*$ is the empty functor.

3. Suppose we have the following pullback:



Thatis, $fp = gq$ and for any $h : Q \to B, k : Q \to C$ such that $fh = gk$, there is a unique morphism $s : Q \to P$ such that $h = ps$ and $k = qs$.

Define $\mathbf{J}$ as the category consisting of the following 3 objects and two other non-identity arrows:

$$0 \xrightarrow{m} 1 \xleftarrow{n} 2$$

Let $F : \mathbf{J} \to \mathbf{C}$ be given by $F(0) = C, F(1) = A, F(2) = B$. Then, it is not hard to see that $P = \lim_{\leftarrow \mathbf{J}} F$. Dually, a pushout (see [18]) is a colimit.

4. For two parallel morphisms $f : A \to B$ and $g : A \to B$ in a category $\mathbf{C}$, the equalizer of f and g is defined as a morphism $e : E \to A$ such that $fe = ge$ and which is universal with this property; that is, given any other morphism $u : X \to A$ in $\mathbf{C}$ such that $fu = gu$, there is a unique $v : X \to E$ such that $ev = u$.
   Let $\mathbf{J}$ be the category containing 2 objects 0 and 1 with 2 non-identity arrows:

$$0 \underset{m}{\overset{n}{\rightrightarrows}} 1$$

   Define the diagram $F \in \mathbf{C}^{\mathbf{J}}$ such that $F(0) = A, F(1) = B$ and $F(n) = f, F(m) = g$. Then it is easy to see that $E = \lim_{\leftarrow \mathbf{J}} F$.

Definition 3.3.0.9 ***(Preservation of limits)***
Suppose $G : \mathbf{C} \to \mathbf{D}$ is a functor. If $\mathbf{J}$ is a small index category, G induces a functor $G^{\mathbf{J}} : \mathbf{C}^{\mathbf{J}} \to \mathbf{D}^{\mathbf{J}}$ of diagrams in the obvious way. If limits of type $\mathbf{J}$ exists in $\mathbf{C}$ and $\mathbf{D}$, one obtains a square of categories and functors

$$\begin{array}{ccc} \mathbf{C}^{\mathbf{J}} & \xrightarrow{\ lim_{\leftarrow \mathbf{J}}\ } & \mathbf{C} \\ {\scriptstyle G^{\mathbf{J}}}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbf{D}^{\mathbf{J}} & \xrightarrow[\ lim_{\leftarrow \mathbf{J}}\ ]{} & \mathbf{D} \end{array}$$

The universal property of limits implies that there is a canonical natural transformation

$$\alpha_{\mathbf{J}} : G \circ\, lim_{\leftarrow \mathbf{J}} \to lim_{\leftarrow \mathbf{J}} \circ G^{\mathbf{J}}.$$

One says that G preserves limits of type $\mathbf{J}$ if $\alpha_{\mathbf{J}}$ is a natural isomorphism.

Theorem 3.3.0.6 *([18]) G preserves limits of any type if G has a left adjoint. Dually, one defines preservation of colimits. The corresponding result is that left adjoints preserve colimits.*

Lemma 3.3.0.1 *Let A be an object in an autonomous category $\mathbf{A}$.*
The covariant functor $- \otimes A^*$ preserves epimorphisms in the sense that: Given any epimorphism $B \xrightarrow{\sigma} C$, the morphism $B \otimes A^* \xrightarrow{\sigma \otimes 1_{A^*}} C \otimes A^*$ is also an epimorphism.

Proof
Note that the functor $-\otimes A^*$ is left adjoint to the functor $[A^*, -]$. Firstly, it is well-known that an epimorphism may be represented as a pushout ([18]). Secondly, by Proposition(3.3.0.6), left adjoint functors preserve colimits. Since pushouts are special types of colimits, the left adjoint functor $- \otimes A^*$ preserves epimorphisms. □

Lemma 3.3.0.2 *Let A be a nuclear object in $\mathbf{A}$.*
Then the covariant functor $[A, -]$ preserves epimorphisms in the sense that:
Given any epimorphism $B \xrightarrow{\sigma} C$, then the morphism $[A, B] \xrightarrow{[A,\sigma]} [A, C]$ is also an epimorphism.

Proof
Since A is nuclear, by Corollary 3.1.0.2, we have the following commutative square:

$$\begin{array}{ccc} B \otimes A^* & \xrightarrow{\sigma \otimes 1_{A^*}} & C \otimes A^* \\ \phi_{A,B} \downarrow \cong & & \cong \downarrow \phi_{A,C} \\ [A, B] & \xrightarrow[{[A,\sigma]}]{} & [A, C] \end{array}$$

Given two parallel morphisms $[A, C] \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$ so that $\alpha \circ [A, \sigma] = \beta \circ [A, \sigma]$.

So, we have the following equal parallel morphisms:

$$B \otimes A^* \xrightarrow{\phi_{A,B}} [A, B] \xrightarrow{[A,\sigma]} [A, C] \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$$

By the above commutative square, we may replace the above arrows by their equivalent:

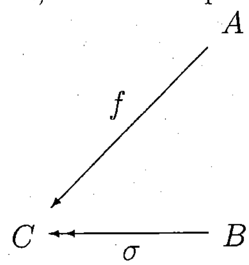
$$B \otimes A^* \xrightarrow{\sigma \otimes 1_{A^*}} C \otimes A^* \xrightarrow{\phi_{A,C}} [A, C] \underset{\beta}{\overset{\alpha}{\rightrightarrows}} Y$$

Since $\sigma \otimes 1_{A^*}$ is epic by Lemma 3.3.0.1, it follows that $\alpha \circ \phi_{A,C} = \beta \circ \phi_{A,C}$.
Since $\phi_{A,C}$ is an isomorphism, we have $\alpha = \beta$.
Therefore, $[A, -]$ preserves epimorphisms.□

Remark
If A is not nuclear, then it is not necessary that $[A, -]$ preserves epimorphism. For instance, take $A = C = \mathbf{M_3} := \{0, a, b, c, 1\}$ where $0 < a, b, c < 1$. Let $B = \{0, a, b, c, a \vee b, a \vee c, b \vee c, 1\}$ where $a \vee b, a \vee c$ and $b \vee c$ are three distinct elements which are the sup of $\{a, b\}$, $\{a, c\}$ and $\{b, c\}$ respectively. Choose $f : B \to C$ as $f(0) = 0$, $f(a) = a$, $f(b) = b$ and $f(c) = c$, $f(x) = 1$ for all other $x \in B$. Then f preserves sups and is epic. But there is no $g : A \to B$ satisfying $id_A = f \circ g$. In fact, if such a g exists, then $g(a) = a$, $g(b) = b$ and $g(c) = c$. But we cannot define $g(1)$. So $[A, f] : [A, B] \to [A, C]$ is not an epimorphism.

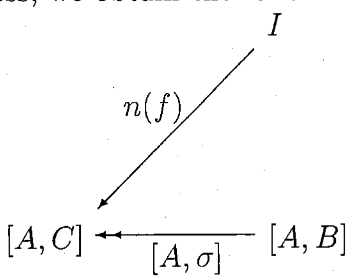
Theorem 3.3.0.7 *Let A be a nuclear object in* $\mathbf{A}$ *which is equipped with a projective base object I. Then A is projective in* $\mathbf{A}$.

Proof
Suppose we have the following objects B, C and morphisms in $\mathbf{A}$:

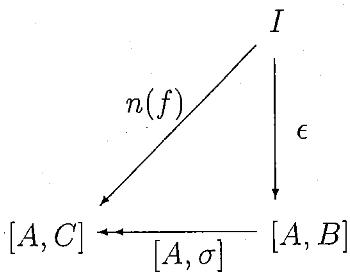


where σ is epic.

From Lemma 3.3.0.2 and adjointness, we obtain the following morphisms:



Since I is projective, there is a morphism $I \xrightarrow{\epsilon} [A, B]$ such that the following diagram commutes:



Via the naturality of the isomorphism Ω, we obtain the following commutative diagram:

$$\begin{array}{ccc} [I\otimes A,B] & \xrightarrow{[I\otimes A,\sigma]} & [I\otimes A,C] \\ {\scriptstyle \Omega_{I,A,B}}\Big\downarrow & & \Big\downarrow{\scriptstyle \Omega_{I,A,C}} \\ [I,[A,B]] & \xrightarrow[{[I,[A,\sigma]]}]{} & [I,[A,C]] \end{array}$$

Applying the set-valued functor $\mathrm{Hom}_{\mathbf{A}}(I,-)$ to the above commutative square gives rise to the inner square of the commutative diagram below. Since the Ω arrows are isomorphisms, $\mathrm{Hom}_{\mathbf{A}}(I,\Omega)$ are also isomorphisms. Also by one of the axiom of an autonomous category, there is a natural isomorphism $\omega_{I,X,Y} : \mathrm{Hom}_{\mathbf{A}}(I,[X,Y]) \cong \mathrm{Hom}_{\mathbf{A}}(I\otimes X,Y)$ for any two objects X and Y in $\mathbf{A}$. Note that $\mathrm{Hom}_{\mathbf{A}}(i_X,1_Y) : \mathrm{Hom}_{\mathbf{A}}(I\otimes X,Y) \cong \mathrm{Hom}_{\mathbf{A}}(X,Y)$. So we may define $\zeta_{X,Y} = \mathrm{Hom}_{\mathbf{A}}(i_X,1_Y)\circ\omega_{I,X,Y}$. It is routine, though tedious, to check that the following diagram commutes:

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathbf{A}}(I\otimes A,B) & & \xrightarrow{\mathrm{Hom}_{\mathbf{A}}(I\otimes A,\sigma)} & & \mathrm{Hom}_{\mathbf{A}}(I\otimes A,C) \\ & \nwarrow{\scriptstyle \zeta_{I\otimes A,B}} & & {\scriptstyle \zeta_{I\otimes A,C}}\nearrow & \\ & \mathrm{Hom}_{\mathbf{A}}(I,[I\otimes A,B]) & \xrightarrow{\mathrm{Hom}_{\mathbf{A}}(I,[I\otimes A,\sigma])} & \mathrm{Hom}_{\mathbf{A}}(I,[I\otimes A,C]) & \\ {\scriptstyle \omega_{I,A,B}}\Big\downarrow & {\scriptstyle \mathrm{Hom}_{\mathbf{A}}(I,\Omega_{I,A,B})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{A}}(I,\Omega_{I,A,C})} & \Big\downarrow{\scriptstyle \omega_{I,A,C}} \\ & \mathrm{Hom}_{\mathbf{A}}(I,[I,[A,B]]) & \xrightarrow[\mathrm{Hom}_{\mathbf{A}}(I,[I,[A,\sigma]])]{} & \mathrm{Hom}_{\mathbf{A}}(I,[I,[A,C]]) & \\ & \swarrow{\scriptstyle \zeta_{I,[A,B]}} & & {\scriptstyle \zeta_{I,[A,C]}}\searrow & \\ \mathrm{Hom}_{\mathbf{A}}(I,[A,B]) & & \xrightarrow[\mathrm{Hom}_{\mathbf{A}}(I,[A,\sigma])]{} & & \mathrm{Hom}_{\mathbf{A}}(I,[A,C]) \end{array}$$

Corresponding to the morphism $I \xrightarrow{\epsilon} [A,B]$ in $\mathrm{Hom}_{\mathbf{A}}(I,[A,B])$, there is a unique morphism $I\otimes A \xrightarrow{\epsilon'} B$ in $\mathrm{Hom}_{\mathbf{A}}(I\otimes A,B)$ such that $\omega_{I,A,B}(\epsilon') = \epsilon$. But $\mathrm{Hom}_{\mathbf{A}}(I,[A,\sigma])\circ\omega_{I,A,B}(\epsilon') = \mathrm{Hom}_{\mathbf{A}}(I,[A,\sigma])(\epsilon) = [A,\sigma]\circ\epsilon = n(f)$. By the commutativity of the above diagram, we also have: $\omega_{I,A,C}\circ \mathrm{Hom}_{\mathbf{A}}(I\otimes A,\sigma)(\epsilon') = \omega_{I,A,C}(\sigma\circ\epsilon')$. Thus, $\omega_{I,A,C}(\sigma\circ\epsilon') = n(f)$ which implies that $\sigma\circ\epsilon' = f\circ i_A$. Since i_A is an isomorphism, it follows that i_A^{-1} exists. Hence $\sigma\circ(\epsilon'\circ i_A^{-1}) = f$. It follows that A is projective in $\mathbf{A}$. $\square$

3.4 Limits transcend via nuclearity

From the previous section, we have gathered that a nuclear object in an autonomous category is projective, provided that the base object is projective. In other words, the projectivity of the nuclear base object has (so-called) transcended to all other nuclear objects in the category. Let $S(\mathcal{P}) = \{A \in \mathbf{C}_0 | A \text{ has the property } \mathcal{P}\}$. Given that the base object $I \in S(\mathcal{P})$. The question is:

Is it necessarily that any nuclear object B belongs to $S(\mathcal{P})$?

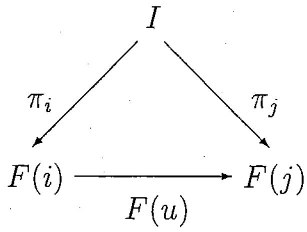
At first glance, the scope of the question may appear too wide since we have not restricted the nature of property $\mathcal{P}$. One natural way is to consider those properties which may be expressed in terms of limits (or colimits). It turns out that it is true in the sense of Theorem(3.4.0.8).

Theorem 3.4.0.8 *Let* $\mathbf{K}$ *be an autonomous category equipped with a base object* I*, where* I *is the limit (dually colimit) of a diagram* F *in* $\mathbf{K}^{\mathbf{J}}$ *for some small indexing category* $\mathbf{J}$.

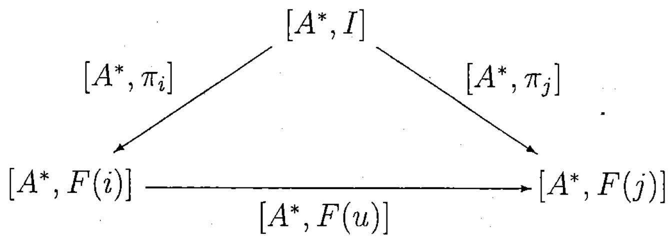
(i) If A *is reflexive, then* A *is the limit (dually colimit) of the diagram* $[A^*, F(-)]$ *in* $\mathbf{K}^{\mathbf{J}}$.

(ii) If A *is nuclear, then* A *is the limit (dually colimit) of the diagram* $F(-) \otimes A$ *in* $\mathbf{K}^{\mathbf{J}}$.

Proof
(a) Since I is the limit of F in $\mathbf{K}^{\mathbf{J}}$, there is a universal cone $\pi : I \longrightarrow F$. More precisely, there are natural transformations π_i and π_j such that the diagram below commutes:



where $u : i \to j$ is an arbitrary morphism in category $\mathbf{J}$. By applying the functor $[A^*, -]$ to the above diagram, we obtain:



By definition, $[A^*, I] = A^{**}$. Since A is reflexive, the arrow $\eta_A : A \to A^{**}$ (as defined in Definition(3.1.0.6)) is a natural isomorphism. We claim that $[A^*, \pi] \circ \eta_A : A \to [A^*, F]$ is the required universal cone.
In order to check this, we take any object C in $\mathbf{K}$ with natural transformation f_i and f_j such that the following triangle commutes for any $u : i \to j$ in $\mathbf{J}$:

$$
\begin{array}{ccc}
 & C & \\
{}^{f_i}\swarrow & & \searrow^{f_j} \\
[A^*, F(i)] & \xrightarrow[{[A^*, F(u)]}]{} & [A^*, F(j)]
\end{array}
$$

Via adjointness, there exist unique morphisms $\overline{f}_k$ $(k = i, j)$ in $\mathrm{Hom}_{\mathbf{K}}(C \otimes A^*, F(k))$ that corresponds to the morphisms f_k $(k = i, j)$ in $\mathrm{Hom}_{\mathbf{K}}(C, [A^*, F(k)])$. Moreover, by the naturality of ω, the following square commutes:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{K}}(C \otimes A^*, F(i)) & \xrightarrow{\mathrm{Hom}_{\mathbf{K}}(C \otimes A^*, F(u))} & \mathrm{Hom}_{\mathbf{K}}(C \otimes A^*, F(j)) \\
{\scriptstyle \omega_{C,A^*,F(i)}}\downarrow & & \downarrow{\scriptstyle \omega_{C,A^*,F(j)}} \\
\mathrm{Hom}_{\mathbf{K}}(C, [A^*, F(i)]) & \xrightarrow[\mathrm{Hom}_{\mathbf{K}}(C, [A^*, F(u)])]{} & \mathrm{Hom}_{\mathbf{K}}(C, [A^*, F(j)])
\end{array}
$$

Thus, the following triangle commutes:

$$
\begin{array}{ccc}
 & C \otimes A^* & \\
{}^{\overline{f}_i}\swarrow & & \searrow^{\overline{f}_j} \\
F(i) & \xrightarrow[F(u)]{} & F(j)
\end{array}
$$

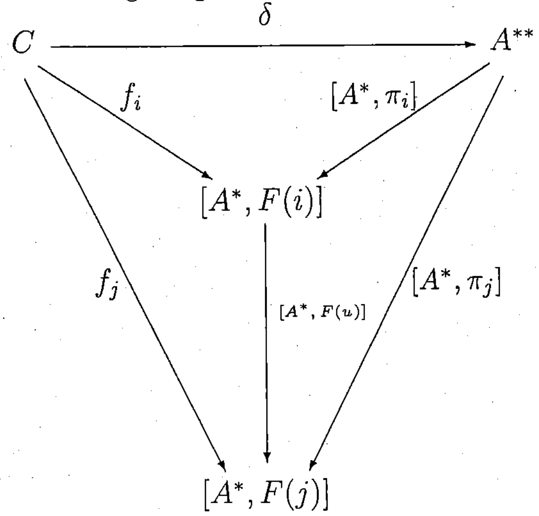
Since I is the limit of the diagram F in $\mathbf{C}^{\mathbf{J}}$, there is a unique morphism $\overline{\delta} : C \otimes A^* \to I$ such that $\pi_k \circ \overline{\delta} = \overline{f}_k$ for $k = i, j$, i.e. the following diagram commutes:

$$
\begin{array}{ccccc}
C \otimes A^* & & \xrightarrow{\overline{\delta}} & & I \\
 & \searrow^{\overline{f}_i} & & \swarrow_{\pi_i} & \\
 & & F(i) & & \\
 & {}_{\overline{f}_j}\searrow & \downarrow & \swarrow_{\pi_j} & \\
 & & F(j) & &
\end{array}
$$

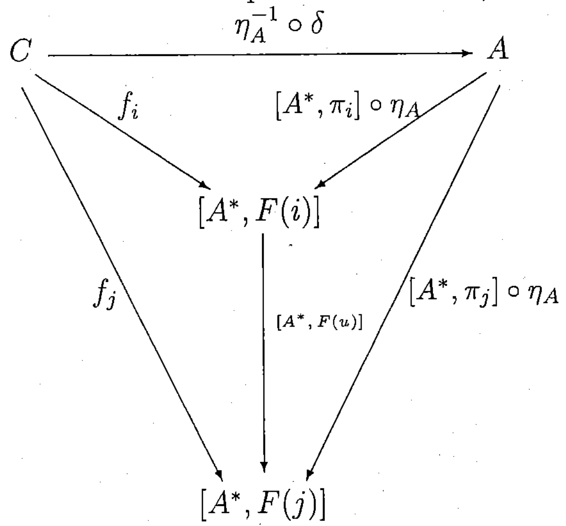
For $k = i, j$, the following commutative diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{K}}(C \otimes A^*, I) & \xrightarrow{\mathrm{Hom}_{\mathbf{K}}(C \otimes A^*, \pi_k)} & \mathrm{Hom}_{\mathbf{K}}(C \otimes A^*, F(k)) \\
{\scriptstyle \omega_{C,A^*,I}}\downarrow & & \downarrow{\scriptstyle \omega_{C,A^*,F(k)}} \\
\mathrm{Hom}_{\mathbf{K}}(C, A^{**}) & \xrightarrow[\mathrm{Hom}_{\mathbf{K}}(C, [A^*, \pi_k])]{} & \mathrm{Hom}_{\mathbf{K}}(C, [A^*, F(k)])
\end{array}
$$

From the above commutative diagram, corresponding to $\overline{\delta}$, there is a unique morphism $\delta : C \to A^{**}$ such that the following diagram commutes:

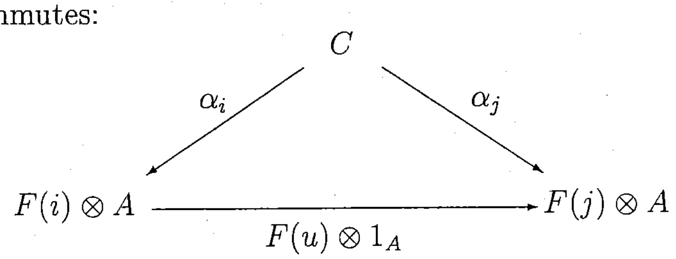


Since A is reflexive, η_A is an natural isomorphism. Therefore, the following diagram commutes:

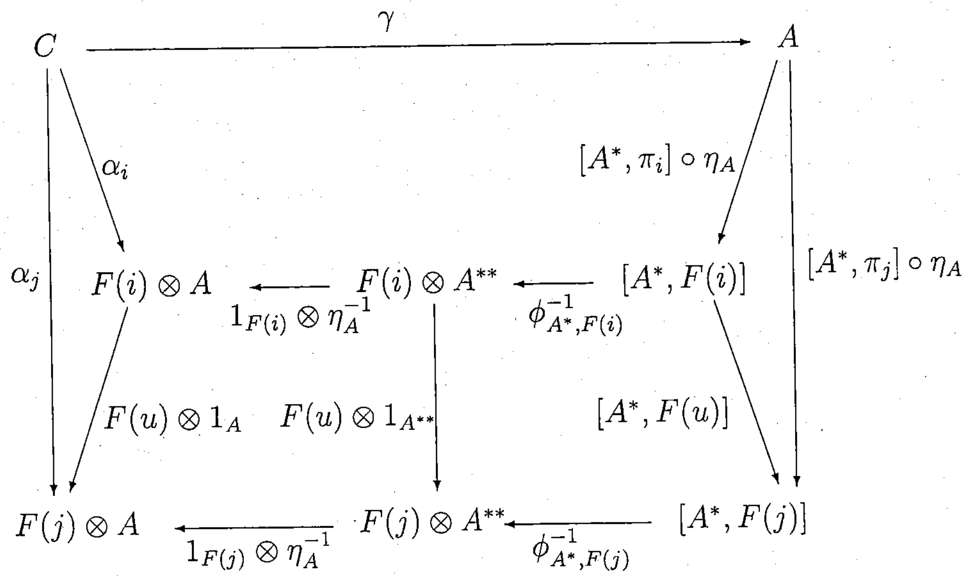


It follows that A is the limit of the diagram $[A^*, F(-)]$ in $\mathbf{C}^{\mathbf{J}}$. The dual result holds similarly.□

(b) Let C be any object from $\mathbf{K}$ equipped with natural transformation α_i and α_j such that the following triangle commutes:



If A is nuclear, then by Corollary(3.1.0.1), A^* is nuclear. Thus, by Corollary(3.1.0.2), $\phi_{A^*,F(i)}$ and $\phi_{A^*,F(j)}$ are natural isomorphisms. As a result, C is equipped with the natural transformations formed by the compositions $\phi_{A^*,F(i)} \circ (1_{F(i)} \otimes \eta_A) \circ \alpha_i$ and $\phi_{A^*,F(j)} \circ (1_{F(j)} \otimes \eta_A) \circ \alpha_j$. From (a), there exists a morphism $\gamma : C \to A$ so that the following diagram commutes:



The dual version of the above result may be established likewise. □

Examples

1. Take $\mathbf{J} = \mathbf{2} = \{0, 1\}$ to be the category with two objects 0 and 1 and only two identity morphisms 1_0 and 1_1. Let $\mathbf{C}$ be an autonomous category with finite products. Suppose that the base object I may be expressed as $X \times Y$ for some objects $X, Y \in \mathbf{C}$. Recall that a product may be regarded as a limit of a certain functor from $\mathbf{J}$ to $\mathbf{C}$. More precisely, define the functor $F : \mathbf{J} \to \mathbf{C}$ such that $F(0) = X$ and $F(1) = Y$. Thus, $I = X \times Y = \lim_{\leftarrow \mathbf{J}} F$.
   Suppose A is reflexive. By Proposition(3.4.0.8(a)), A is the limit of the diagram $[A^*, F(-)]$. It follows that $A \cong [A^*, X] \times [A^*, Y]$. Suppose further that A is nuclear, then by Proposition (3.4.0.8(b)), A is the limit of the diagram $- \otimes A$. So, $A \cong (X \otimes A) \times (Y \otimes A)$. This means that if I can be decomposed (via caretsian product) into two objects, then so can any nuclear object A.

2. Let $\mathbf{C}$ be an autonomous category equipped with a terminal (dually, initial) base object. Let $\mathbf{J}$ denote the empty category. One may view I as $\lim_{\leftarrow \mathbf{2}} *$ where $*$ is the empty functor. If A is reflexive, then A is also terminal (dually, initial).

3. Similar conclusions may be drawn when the situation involves other types of limits, such as pullbacks (dually, pushouts) and (co)equalisers.

Remark
Due to a later analysis, the proof of Theorem(3.4.0.8) may be shortened by invoking Theorem(3.3.0.6) ([18], p.118 (Theorem 5.1)). The simplified proof thus runs:
Proof
(i) Let A be reflexive. Then,

$$\begin{aligned} A &\cong A^{**} && (\because A \text{ is reflexive.}) \\ &= [A^*, I] && (\because [X, I] = X^*) \\ &= [A^*, \lim_{\leftarrow \mathbf{J}} F] && (\because I = \lim_{\leftarrow \mathbf{J}} F) \\ &= \lim_{\leftarrow \mathbf{J}} [A^*, F] && (\because [A^*, -] \text{ is a right adjoint functor which preserves limits.}) \end{aligned}$$

(ii) Let A be nuclear. It follows that A is reflexive, i.e. $A \cong A^{**}$. It follows that

$$\begin{aligned} A &\cong \lim_{\leftarrow \mathbf{J}} [A^*, F] && (\because A \text{ is reflexive and (i) holds.}) \\ &= \lim_{\leftarrow \mathbf{J}} (F(-) \otimes A^{**}) && (\because A \text{ is nuclear.}) \\ &= \lim_{\leftarrow \mathbf{J}} (F(-) \otimes A) && (\because A \text{ is reflexive.}) \end{aligned}$$

□

3.5 The category PreFrm

Following [3] and [16], we define the category **PreFrm** as follows: We call a poset A a preframe if it has finite meets and directed joins, with binary meets distributing over directed joins; and we call a function between preframes a preframe morphism if it preserves finite meets and directed joins.
Remarks

(i) If A is a preframe, then for any finite subset X of A, $\wedge X$ exists. In particular, A has a top element.

(ii) Let $f : A \to B$ be a preframe morphism. For any finite subset $X \subseteq A$, it follows from induction that $f(\wedge X) = \wedge f(X)$. In particular, $f(1_A) = 1_B$.

It is natural to ask if the category **PreFrm** of preframes is autonomous. It turns out that it is. The first step to justify this is to show that the internal hom exists.
Define $[A, B]$ to be the set of all preframe morphisms from a preframe A to another preframe B.

Proposition 3.5.0.5 *Let A and B be preframes. Define $\leq$ on $[A, B]$ pointwise, i.e. $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in A$. Then $([A, B], \leq)$ is a preframe.*

Proof
Since the map $\top : A \to B, : a \mapsto 1_B$ for each $a \in A$ is a preframe morphism, it is clear that $[A, B]$ is non-empty.
Next we prove that $[A, B]$ is a meet semilattice. Given any f and g in $[A, B]$, define $h : A \to B$ by $h(a) = f(a) \wedge g(a)$ for each $a \in A$.

(i) First we check that h preserves finite meets.

$$\begin{aligned} h(a \wedge b) &= f(a \wedge b) \wedge g(a \wedge b) \\ &= f(a) \wedge f(b) \wedge g(a) \wedge g(b) \\ &= f(a) \wedge g(a) \wedge f(b) \wedge g(b) \\ &= h(a) \wedge h(b). \end{aligned}$$

(ii) Next we show that h preserves directed joins. Take any directed set $D = \{d_i : i \in I\}$ of A. Then

$$\begin{aligned} h(\vee D) &= f(\vee D) \wedge g(\vee D) \\ &= \vee f(D) \wedge \vee g(D) \end{aligned}$$

Since g is monotone and D is directed, so $g(D)$ is a directed subset of B.
By treating $\vee f(D)$ as an element of B, we apply the distributivity over directed sets , so we have:

$$\begin{aligned} h(\vee D) &= \vee f(D) \wedge \vee\{g(d_j) : j \in I\} \\ &= \vee_{j \in I}(\vee f(D) \wedge g(d_j)) \\ &= \vee_{j \in I} \vee_{i \in I} \{f(d_i) \wedge g(d_j)\} \\ &= \vee\{f(d_i) \wedge g(d_j) : i, j \in I\} \end{aligned}$$

It is clear that $\vee\{f(d_i) \wedge g(d_i) : i \in I\} \leq \vee\{f(d_i) \wedge g(d_j) : i, j \in I\}$. So it suffices to show the reverse inequality. For fixed $i, j \in I$, there is $k \in I$ such that $d_i \leq d_k$ and $d_j \leq d_k$. Consequently,

$$f(d_i) \wedge g(d_j) \leq f(d_k) \wedge g(d_k) \leq \vee\{f(d_i) \wedge g(d_i) : i \in I\}$$

From this result, it follows that

$$\begin{aligned} h(\vee D) &= \vee\{f(d_i) \wedge g(d_j) : i, j \in I\} \\ &= \vee\{f(d_i) \wedge g(d_i) : i \in I\} \\ &= \vee\{h(d_i) : i \in I\} \\ &= \vee h(D). \end{aligned}$$

So, h preserves directed joins.
One can also show (in a similar way) that directed joins exists in $[A, B]$. More precisely, given a directed subset $\mathcal{D} = \{f_i : i \in I\}$ of $[A, B]$, the supremum denoted by $\vee_{i \in i} f_i$ exists in $[A, B]$ and is defined by $(\vee_{i \in I} f_i)(x) = \vee_{i \in I} f_i(x), \forall x \in A$.
Furthermore, one can show by direct calculation that binary meets distribute over directed joins in $[A, B]$.□

In [16], P.Johnstone and Vickers proved that given a pair of preframes A and B, it is possible to construct the tensor product $A \otimes B$ in a manner parallel to the formation of tensor product for two frames. In fact, it has been shown that **PreFrm** is an autonomous category. The question is:

"How does one characterize nuclearity in **PreFrm**?"

To date, it is still not known what exactly the nuclear preframes are. Having a better understanding about the theory of frames would definitely be our first step to solving this problem. This leads us to the next chapter, Z-theory, where the central theme is to unify the studies of several types of frames (such as frame, preframe, σ-frame, κ-frame, etc).

Chapter 4

Z-Frames

The purpose of this chapter is to draw a quick overview of Z-frame: including the fundamental definitions and some important results. The materials on Z-frames included in this chapter originally appeared in several different resources ([21],[22],[23]). Here we shall try to put them together in a way so that we can have a clearer and more complete picture of this new theory. On one hand, it helps us understand preframes better and also in a more general setting. On the other hand, we can use some of these results to formulate new problems - preparing the stage for the final chapter. As such, we shall omit some of the routine proofs.

4.1 Z-continuous posets

4.1.1 System of sets on Poset

Let **Poset** be the category of all posets and morphisms order-preserving functions between them.

Definition 4.1.1.1 ***(Subset system)***
Let $\mathcal{E}$ be a subcategory (not necessarily full) of ***Poset****, by a system of sets Z on $\mathcal{E}$ we mean a function that assigns to each object A a collection $Z(A)$ of down-closed subsets of A, i.e. $\forall D \in Z(A)$, $D \subseteq A$ and $D = \downarrow D$ $(:= \{x \in A : \exists y \in D, x \leq y\})$, such that the following conditions are satisfied:*

(S1) For each $x \in A$, $\downarrow x \in Z(A)$.

(S2) If $f : A \to B$ is any morphism in $\mathcal{E}$, then $\downarrow f(D) \in Z(B)$ for each $D \in Z(A)$;

(S3) $Z(A)$ is an object of $\mathcal{E}$ with respect to the inclusion relation for each object A of $\mathcal{E}$;

(S4) For each $\mathcal{D} \in Z(Z(A))$, $\cup\mathcal{D} \in Z(A)$.

Remarks

(1) The elements of $Z(A)$ are called Z-ideals.
A subset C of A is called a Z-set if $\downarrow C \in Z(A)$.

From condition (S2), it is seen that the image of Z-set under a morphism of ε is a Z-set.

(2) If the empty set $\phi \in Z(A_0)$ for some poset A_0, then $\phi \in Z(A)$ for every poset A.

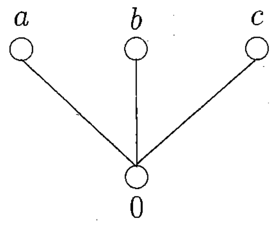
(3) The condition (S4) is independent of the others.
For example, if we define $Z(A) = \{\downarrow D : D \subseteq A \text{ and } |D| \leq 2\}$, then Z satisfies conditions (S1), (S2), (S3) but not (S4).

Proof

(a) For each $x \in A$, $|x| = 1 \leq 2$. Moreover, $\downarrow x \in Z(A)$ by definition of Z.

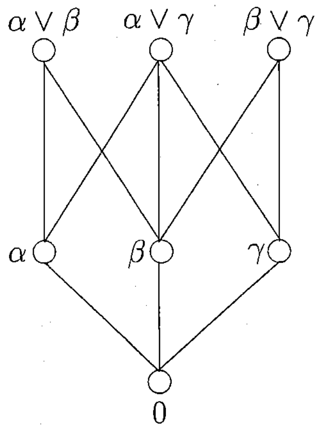
(b) If $f : A \to B$ is an ε-morphism, then for each $D \in Z(A)$, $\downarrow f(D)$ is obviously in $Z(B)$ since $f(D) \subseteq B$ and since $|D| \leq 2$, $|f(D)| \leq 2$.

(c) $Z(A) = \{\downarrow D : D \subseteq A \text{ and } |D| \leq 2\}$. With respect to $\subseteq$, $(Z(A), \subseteq)$ is clearly a poset. The axioms (S1), (S2) and (S3) are satisfied. But (S4) is not satisfied.
A typical element $\mathcal{D}$ of $Z(Z(A))$ will be of the form $\downarrow X$ for some $X \in Z(A)$ and $|X| \leq 2$. It seems that A may be made complicated enough so that there is a collection $\mathcal{D}$, an element of $Z(Z(A))$ with $\cup\mathcal{D} \notin Z(A)$. We shall consider a concrete example.
Concrete counterexample Consider A the poset as:



$$
\begin{aligned}
Z(A) &= \{\downarrow D : D \subseteq A \text{ and } |D| \leq 2\} \\
&= \{\downarrow 0, \downarrow a, \downarrow b, \downarrow c, \downarrow \{a,b\}, \downarrow \{a,c\}, \downarrow \{b,c\}\} \\
&= \{\{0\}, \{0,a\}, \{0,b\}, \{0,c\}, \{0,a,b\}, \{0,a,c\}, \{0,b,c\}\} \\
&:= \{0, \alpha, \beta, \gamma, \alpha \vee \beta, \alpha \vee \gamma, \beta \vee \gamma\}.
\end{aligned}
$$

The Hasse diagram for $Z(A)$ is:



One element of $Z(Z(A))$ will be $\downarrow\{\alpha, \beta \vee \gamma\} = \{0, \alpha, \beta, \gamma, \beta \vee \gamma\}$.
But $\cup\{0, \alpha, \beta, \gamma, \beta \vee \gamma\} = \{0, a, b, c\} \notin Z(A)$.
Thus, we have obtained a specific $\mathcal{D}$ of $Z(Z(A))$ with $\cup\mathcal{D} \notin Z(A)$.
Although in the following we only consider systems of sets on **Poset**, many of the results are still true for any system of sets on a subcategory of **Poset**.

Definition 4.1.1.2 ***(Z-complete poset)***
A poset A is Z-complete if $\bigvee D$ exists for each $D \in Z(A)$.

Remark
A poset A is a Z-complete if and only if every Z-set of A has a supremum in A.

Proposition 4.1.1.1 *For each poset A,*

(i) the map $\downarrow: A \to Z(A)$ which sends $x \in A$ to $\downarrow x$ is an order-preserving map.

(ii) A is Z-complete if and only if the map $\downarrow$ has a left adjoint. Moreover, the left adjoint, if it exists, must be the map $\vee : Z(A) \to A$.

The following are some examples of systems of sets on **Poset**.

(1) For each poset A, let $Z(A) = \{\downarrow x : x \in A\}$, then Z is a system of sets on **Poset**.

(2) Define $Z(A) = \downarrow A := \{C \subseteq A, C = \downarrow C\} \cup \{\phi\}$. Then Z is a system of sets on **Poset**. A poset A is Z-complete if and only if A is a complete lattice. If we do not allow the empty in $Z(A)$, then A is Z-complete if and only if A is join-complete.

(3) Let $Z(A) = Id(A)$ for each poset A, $Id(A)$ is the collection of all ideals of A, i.e. those lower sets of A which are directed. Then A is Z-complete if and only if it is up-complete.

In what follows, we assume that we are given a system of sets on **Poset**. Let A be any poset, then for each Z-ideal of $Z(Z(A))$, $\cup\mathcal{D} \in Z(A)$, thus $\vee\mathcal{D}$ exists in $Z(A)$ and is equal to $\cup\mathcal{D}$ is Z-complete. Hence $Z(A)$ is Z-complete.
Define the category **ZP** of Z-complete posets, the objects of the category are all Z-complete posets and the morphisms are those order-preserving maps $f : A \to B$ such that $f(\vee D) = \vee f(D)$ holds for each $D \in Z(A)$, i.e. f preserves the joins of Z-ideals.
It is easily seen that $f : A \to B$ preserves the joins of Z-ideals if and only if it preserves the joins of Z-sets.
For each system Z of sets on **Poset**, Z actually defines a functor from **Poset** to **ZP** and it is just left adjoint of the forgetful functor. Note that if $f : A \to B$ is a morphism in **Poset**, then $Z(f) : Z(A) \to Z(B)$ in **ZP** is defined by $Z(f)(D) = \downarrow f(D)$ for each $D \in Z(A)$. So by (S2), Z is a well-defined functor.

Theorem 4.1.1.1 *The assignment $A \mapsto Z(A)$ defines a functor from **Poset** to **ZP**, this functor is left-adjoint to the forgetful functor U : **ZP** $\to$ **Poset**.*

4.1.2 Z-continuous posets

Definition 4.1.2.1 ***(Z-continuous poset)***
A system Z of sets on ***Poset****, a poset A is called Z-continuous if and only if it satisfies the following two conditions:*

(1) A is Z-complete.

(2) For each $a \in A$, there is a smallest D in $Z(A)$ with $\vee D = a$.

Let A be a Z-complete poset, define the Z-below relation $\ll_z$ in A as $x \ll_z y$ for $x, y \in A$ if and only if for each $D \in Z(A)$, $y \leq \vee D$ implies $x \in D$.
A Z-complete poset A is Z-continuous if and only if for each $a \in A$,

$$\{x \in A : x \ll_z a\} \in Z(A) \text{ and } a = \vee\{x \in A : x \ll_z a\}.$$

We denote the set $\{x \in A : x \ll_z a\}$ by $\omega_z(a)$ (or simply by $\omega(a)$).

Remarks

(1) If A is Z-continuous, then $\vee\omega_z(a) = a$. Note that $\omega(a)$ is a Z-ideal by definition. So indeed we have a Z-ideal whose sup is a.
Suppose that $E \in Z(A)$ with $\vee E = a$. Since $\vee E = a$, $x \ll_z a$ implies $x \in E$. Since E is arbitrary, it follows that $\omega(a)$ is the intersection of all Z-ideals whose sups are equal to a, i.e. the smallest Z-ideal whose sup is a.

(2) In a Z-complete poset A, one has the following statements for $u, x, y, z \in A$:

(i) $x \ll_z y$ implies $x \leq y$.

(ii) $u \leq x \ll_z y \leq z$ implies $u \ll_z z$.

(iii) $0 \ll_z x$ if 0 exists.

(3) The main reason why $\ll_z$ fails to be an auxiliary relation is that finite joins of elements need not exists. Also, there is a lack of sufficient information about the subset system Z.

(4) If Z_1 and Z_2 are two systems of sets such that $Z_1(A) \subseteq Z_2(A)$ for each poset A, then we write $Z_1 \leq Z_2$. For example, let $Z_1(A) = \{\downarrow x : x \in A\}$, $Z_2(A) = Id(A)$. Clearly, any principally generated ideal is an ideal. So $Z_1 \leq Z_2$.
But even if $Z_1 \leq Z_2$,
(a) a Z_1-continuous poset need not be a Z_2-continuous poset. For instance, in the previous example we may restrict A to be a poset that is not up-complete.
(b) a Z_2-continuous poset need not be a Z_1-continuous poset. For example, set $Z_1(P) = \{\downarrow A : A \subseteq P \text{ is countable and directed}\}$ and $Z_2(P) = \{\downarrow A : A \subseteq P \text{ is directed}\}$. Then $Z_1 \leq Z_2$.
Let X be an uncountable set.
The power-set lattice of $X, \mathcal{P}(X)$, is continuous. Recall that $A \ll_{z_{1,2}} B$ if and only if A is a finite subset of B. Clearly, $\vee\omega_{z_2}(A) = A$ but $\omega_{z_1}(A)$ is uncountable.

Lemma 4.1.2.1 *If A is a Z-continuous poset, then the Z-below relation in A satisfies the interpolation property: For any $a, b \in A$, $a \ll_z b$ implies the existence of $x \in A$ such that $a \ll_z x \ll_z b$.*

Lemma 4.1.2.2 *If A is a Z-continuous poset, the following conditions are equivalent:*

(1) $x \ll_z y$.

(2) For each Z-set D of A, the relation $y \leq \vee D$ implies the existence of $d \in D$ with $x \ll_z d$.

Proof
By Remark(ii) which follows immediately after Definition(4.1.2.1), clearly (2) implies (1). Now suppose (1) and $y \leq \vee D$ for a Z-set $D \subseteq A$. By Lemma(4.1.2.1), there is an x^* such that $x \ll_z x^* \ll_z y$. Then we find a $d \in D$ with $x^* \leq d$ by Definition(4.1.2.1). Hence by the same remark(ii), we have $x \ll_z d$. □

4.2 Z-frames

4.2.1 System of sets on SLat

Definition 4.2.1.1 ***(Semilattice)***
In what follows, by a semilattice S, we mean a meet-semilattice which has a top element 1_S. Let **Slat** *be the category of all semilattices and mappings between the semilattices which preserve finite meets and the top of semilattices.*

Definition 4.2.1.2 ***(System of sets on*** **Slat***)*
By definition, a system Z of sets on **Slat** *is a function which assigns to each semilattice S a collection of lower subsets of S such that the following conditions are satisfied:*

(S1) $\downarrow x \in Z(S)$ *for each* $x \in S$;

(S2) If $f : S \to T$ is a semilattice morphism, then $\downarrow f(D) \in Z(T)$ for each $D \in Z(S)$;

(S3) Each $Z(S)$ is a semilattice with respect to the inclusion order of sets;

(S4) For each $\mathcal{D} \in Z(Z(S))$, $\cup\mathcal{D} \in Z(S)$.

Elements of $Z(A)$ are called the Z-ideals of A.
A subset E of A is called a Z-set if its lower part $\downarrow E$ is a Z-ideal.

Remarks

(1) Let S be a subsemilattice of A, i.e. a subset of A which is closed under finite meets and contains the top element of A, then the inclusion map $i : S \to A$ is a semilattice morphism, thus if $D \in Z(S)$, then

$$\downarrow D = \{x \in A : \exists s \in D, x \leq s\} \in Z(A).$$

(2) If $\mathcal{E}$ is a Z-set of $Z(A)$, then $\cup\mathcal{E}$ is a Z-ideal of A.
Proof
Recall that $(Z(A), \subseteq)$ is a semilattice with respect to the inclusion order of sets. So $\mathcal{E}$ is a Z-set of $Z(A)$ means that
$\downarrow \mathcal{E} \in Z(Z(A)) \Leftrightarrow \{C \subseteq D : D \in Z(A), D \in \mathcal{E}\} \in Z(Z(A))$. $\therefore \cup \downarrow \mathcal{E} = \cup\mathcal{E} \in Z(A)$ by (S4).

(3) If $f : A \to B$ is a semilattice morphism, then $f(E)$ is a Z-set of B for each Z-set E of A.
Proof
Let E be a Z-set of A. By definition, $\downarrow E \in Z(A)$. Since f is a semilattice morphism, $\downarrow f(\downarrow E) \in Z(B)$. We want to show that $\downarrow f(E) \in Z(B)$. First of all, it is obvious that $\downarrow f(E) \subseteq \downarrow f(\downarrow E)$. Let $u \in \downarrow f(\downarrow E)$. Then there is a $w \in \downarrow E$ such that $u \leq f(w)$. But there is an $e \in E$ such that $w \leq e$. Since f is a semilattice morphism, it preserves order; whence $f(w) \leq f(e)$. Therefore, $u \leq f(e)$ implies $u \in \downarrow f(E)$. Consequently, $\downarrow f(\downarrow E) \subseteq \downarrow f(E)$. It follows that $\downarrow f(E) \in Z(B)$; i.e. $f(E)$ is a Z-set of B.

(4) From each system Z of sets on **Slat**, we obtain a system Z_* of sets on **Slat** defined by:

$$Z_*(A) = Z(A) \cup \{\phi\}.$$

Generally, $Z = Z_*$ if and only if there is a semilattice S such that $\phi \in Z(S)$. In fact, if $Z = Z_*$, then for each semilattice S, $Z(S) = Z_*(S) = Z(S) \cup \{\phi\}$. This implies that $\phi \in Z(S)$. Conversely, if $\phi \in Z(S)$ for some semilattice S, then for each semilattice T, consider the semilattice morphism $f : S \to T$ given by $f(s) = 1_T, \forall s \in S$. Then as $\phi \in Z(S)$, so $f(\downarrow \phi) = \phi \in Z(T)$. Thus, $Z = Z_*$. Whether Z-complete semilattices have bottom elements will depend whether $Z = Z_*$.

Lemma 4.2.1.1 *If Z is a set-system on* **Slat**, *then the meet operation on the semilattice $(Z(A), \subseteq)$ is just the intersection of sets.*

Proof
Given $H, K \in Z(A)$. $H \wedge K \subseteq H$ and $H \wedge K \subseteq K$ imply that $H \wedge K \subseteq H \cap K$ obviously. If $x \in H \cap K$, then $\downarrow x \subseteq H$, $\downarrow x \subseteq K$. This means that $\downarrow x$ is a lower bound of $\{H, K\}$. Therefore, $\downarrow x \subseteq H \wedge K$. Thus, $x \in \downarrow x$ implies that $x \in H \wedge K$. We have $H \wedge K \subseteq H \cap K$ and hence $H \wedge K = H \cap K$. □
If follows (by induction) therefore that a finite number of Z-ideals of a semilattice is a Z-ideal, and clearly the condition (S3) can be replace by the condition:

(S3') $$C, D \in Z(A) \Rightarrow C \cap D \in Z(A).$$

Corollary 4.2.1.1 *Let Z be a system of sets on* **Slat** *and A a semilattice. Then for any $x \in A$ and any $D \in Z(A)$, the set*

$$x \wedge D = \{x \wedge d : d \in D\}$$

is a Z-set of A.

Proof
From (S1) and Lemma(4.2.1.1), we have $\downarrow x \cap D \in Z(A)$. But clearly $\downarrow x \cap D = \downarrow (x \wedge D)$. (To see this, $y \in \downarrow x \cap D$ implies $y \leq x$ and $y \in D$. Thus, $y = x \wedge y \in x \wedge D$. Conversely, $y \in \downarrow (x \wedge D)$ implies $y \leq x \wedge d$ for some $d \in D$, i.e. $y \leq x$ and $y \leq d$ for some $d \in D$. But D is a lower set so that $y \in \downarrow x \cap D$.) So $x \wedge D$ is a Z-set. □
Notation: We shall write $x \wedge D$ as xD for convenience.

Definition 4.2.1.3 ***(Z-complete)***
A semilattice A is called Z-complete if $\vee D$ exists for each Z-ideal D of A.

Remarks

(1) It is easily seen that A is Z-complete if and only if $\vee E$ exists in A for each Z-set E of A.
Proof
($\Rightarrow$) Let E be a Z-set of A. Then $\downarrow E \in Z(A)$. Since A is Z-complete, $\vee \downarrow E$ exists. But it is trivial that $\vee \downarrow E = \vee E$ so that $\vee E$ exists in A.
($\Leftarrow$) This direction is trivial. □

(2) By condition (S4), if $\mathcal{D}$ is a Z-ideal of $Z(A)$, then $\cup \mathcal{D} \in Z(A)$, thus $\vee \mathcal{D} = \cup \mathcal{D} \in Z(A)$. This shows that $Z(A)$ is a Z-complete semilattice. The top element of $Z(A)$ is obviously $\downarrow 1_A$.

Definition 4.2.1.4 ***(Maps preserving the joins of z-ideals)***
A semilattice morphism $f : A \to B$ is said to preserve joins of Z-ideals if $f(\vee D) = \vee f(D)$ for each Z-ideal D of A whenever the joins exist.

Remark
It is easy to check that a semilattice morphism $f : A \to B$ preserves joins of Z-ideals of A if and only if f preserves joins of Z-sets of A, that is, if $f(\vee E) = \vee f(E)$ holds for each Z-set E for which $\vee E$ exists.
Proof
($\Leftarrow$) is trivial since any Z-ideal is a Z-set.
($\Rightarrow$) $\vee \downarrow E = \vee E$ guarantees that $f(\vee E) = f(\vee \downarrow E) = \vee f(\downarrow E) = \vee f(E)$ for every Z-set E whenever $\vee E$ exists. □
Let **ZcS** denote the category of Z-complete semilattices and semilattice morphisms which preserve joins of Z-ideals.

Lemma 4.2.1.2 *Let A and B be two semilattices, and $f : A \to B$ be a map, then the map $Z(f) : Z(A) \to Z(B)$ defined by $Z(f)(D) = \downarrow f(D), \forall D \in Z(A)$ is a* **ZcS***-morphism if and only if f is a semilattice morphism.*

Proof
Note that $Z(f)(D) = \downarrow f(D)$ is well-defined by (S2). Suppose f is a semilattice morphism. If D_1 and D_2 are two elements of $Z(A)$, then

$Z(f)(D_1 \wedge D_2) = \downarrow f(D_1 \wedge D_2) \subseteq \downarrow f(D_i), i = 1, 2. \therefore \downarrow f(D_1 \wedge D_2) \leq \downarrow f(D_1) \wedge \downarrow f(D_2)$.
Whence $Z(f)(D_1 \wedge D_2) \leq Z(f)(D_1) \wedge Z(f)(D_2)$ obviously holds.
Let $x \in Z(f)(D_1) \wedge Z(f)(D_2) = Z(f)(D_1) \cap Z(f)(D_2)$. Then $x \in \downarrow f(D_1) \cap \downarrow f(D_2)$
$\Rightarrow x \leq f(d_1)$ and $x \leq f(d_2)$ fo some $d_1 \in D_1$ and $d_2 \in D_2$.
$\Rightarrow x \leq f(d_1) \wedge f(d_2)$
$\Rightarrow x \leq f(d_1 \wedge d_2)$ since f preserves binary meet.
$\Rightarrow x \in Z(f)(D_1 \wedge D_2) = Z(f)(D_1 \cap D_2)$ (by Lemma(4.2.1.1)).
Now this means $Z(f)(D_1 \wedge D_2) = Z(f)(D_1) \wedge Z(f)(D_2)$ and hence $Z(f)$ is a semilattice morphism.
If $\mathcal{D} \in Z(Z(A))$, then $\vee \mathcal{D} = \cup \mathcal{D}$. So
$Z(f)(\vee \mathcal{D}) = \downarrow f(\cup \mathcal{D}) = \cup\{\downarrow f(D) : D \in \mathcal{D}\} = \cup\{Z(f)(D) : D \in \mathcal{D}\}$. So $Z(f)$ preserves the joins of Z-ideals, i.e.$Z(f)$ is a **ZcS**-morphism.
For the converse direction, observe that for any element $x, y \in A$,

$$\downarrow f(x \wedge y) = \downarrow f(\downarrow x \cap \downarrow y).$$

To justify this, we prove the non-trivial inclusion. Let $w \in \downarrow f(\downarrow x \cap \downarrow y)$. This implies $w \leq f(u)$ for some $u \in \downarrow x \cap \downarrow y$. In particular, $u \in \downarrow x$ and $u \in \downarrow y$, i.e. $u \leq x$ and $u \leq y$ together imply $u \leq x \wedge y$. Therefore, $w \leq f(u)$ and thus $w \in \downarrow f(x \wedge y)$. This completes the proof of the observation. Thus,

$$\begin{aligned}
\downarrow f(x \wedge y) &= \downarrow f(\downarrow x \cap \downarrow y) \\
&= (\downarrow f)(\downarrow x \wedge \downarrow y)) \\
&= Z(f)(\downarrow x \wedge \downarrow y) \\
&= Z(f)(\downarrow x) \wedge Z(f)(\downarrow y) \\
&= \downarrow f(x) \cap \downarrow f(y) \\
&= \downarrow (f(x) \wedge f(y)).
\end{aligned}$$

This implies that $f(x \wedge y) = f(x) \wedge f(y)$. Thus, f is a semilattice morphism. □
Given two systems of sets, Z, Z' on **Slat**, define $Z \leq Z'$ if $Z(A) \subseteq Z(A')$ for each semilattice A. $\leq$ is a partial order on the collection of all systems of sets on **Slat**.
The biggest object for this order is the lower set system L where
$L(A) = \{D \subseteq A : \downarrow D = D\}$ - collection of all lower subsets of A, while the smallest one is the principal ideals system I where $I(A) = \{\downarrow x : x \in A\}$. We can further prove that any two system of sets have an infimum in this order.

Proposition 4.2.1.1 *If Z_1 and Z_2 are two systems of sets, define*

$$(Z_1 \wedge Z_2)(A) = Z_1(A) \wedge Z_2(A),$$

then $Z_1 \wedge Z_2$ is a system of sets, moreover, it is indeed the infimum of Z_1 and Z_2 with respect to the relation $\leq$ defined above.

Proof
$Z_1 \wedge Z_2$ obviously satisfies conditions (S1) and (S3). We now prove that it also satisfies (S2) and (S4).
Let $f : A \to B$ be any semilattice morphism. If $D \in (Z_1 \wedge Z_2)(A)$, then $D \in Z_1(A)$ and $D \in Z_2(A)$. So $\downarrow f(D) \in Z_1(B)$ and $\downarrow f(D) \in Z_2(B)$. It follows that
$\downarrow f(D) \in (Z_1 \wedge Z_2)(B)$. Thus $Z_1 \wedge Z_2$ satisfies condition (S2).

Suppose now that $\mathcal{D} \in (Z_1 \wedge Z_2)((Z_1 \wedge Z_2)(A))$, i.e. $\mathcal{D} \in Z_1((Z_1 \wedge Z_2)(A))$ and $\mathcal{D} \in Z_2((Z_1 \wedge Z_2)(A))$. Since $(Z_1 \wedge Z_2)(A) \subseteq Z_1(A)$ is obviously a subsemilattice of $Z_1(A)$, by Remark(2) that follows Definition(4.2.1.2), we see that $\downarrow \mathcal{D}$ belongs to $Z_1(Z_1(A))$, thus $\cup \downarrow \mathcal{D} = \cup\mathcal{D} \in Z(A)$. Similarly, we can show that $\cup\mathcal{D} \in Z_2(A)$. Thus, $\cup\mathcal{D} \in (Z_1 \wedge Z_2)(A)$. Hence $Z_1 \wedge Z_2$ satisfies condition (S4).
$Z_1 \wedge Z_2$ is obviously the infimum of Z_1 and Z_2 with respect to the partial order $\leq$. □
Thus the collection of all systems of sets on **Slat** constitutes a semilattice with a bottom as well as a top-element.
Given any semilattice A, consider the trace $\mathcal{S}(A)$ of systems of sets on A; the objects of $\mathcal{S}(A)$ are of the form $Z(A)$ where Z is a set system. With the inclusion order of sets, $\mathcal{S}(A)$ is a semilattice called the trace semilattice of A. This topic is worth further investigation.

4.2.2 Z-frames

In this section, we define the category **ZFrm** of all Z-frames for each system of sets on **Slat**, and investigate the basic properties of Z-frames. The category of Z-frames unifies various sorts of categories of frames, including the category **Frm** of all frames, σ**Frm** of σ-frames, **Prefrm** of preframes, as well as the categories **kFrm** of k-frames.

Definition 4.2.2.1 *(z-frame)*
Given a set system Z on **Slat**, *a semilattice A is called a Z-frame if it satisfies the following two conditions:*

(1) A is Z-complete;

(2) $x \wedge \vee D = \vee\{x \wedge d : d \in D\}$ holds for each $x \in A$ and any $D \in Z(A)$.

In other words, a Z-frame is a Z-complete semilattice such that the meet operation is distributive over joins of Z-ideals.
Remark
A Z-complete semilattice is a Z-frame if and only if its meet operation is distributive over the joins of Z-sets.
Proof
We shall only show that:
If a Z-complete semilattice is a Z-frame, then its meet operation is distributive over the joins of Z-sets.
Let E be a Z-set of a semilattice A and $x \in A$.

$$\begin{aligned} x \wedge \vee E &= x \wedge \vee \downarrow E \\ &= \vee\{x \wedge e : e \in \downarrow E\} \\ &= \vee\{x \wedge e : e \in E\} \end{aligned}$$

□

Denote by **ZFrm** the full subcategory of **ZcS** consisting of all Z-frames.
From the definition, it follows easily that if $Z_1 \leq Z_2$ are two systems of sets on **Slat**, then every Z_2-frame is also a Z_1-frame.
$\mathbf{Z_2Frm}$ is a subcategory (not necessarily full) of $\mathbf{Z_1Frm}$ if $Z_1 \leq Z_2$.

Examples

(1) Consider the lower set system L, where $L(A) = \{\downarrow D : D \subseteq A\} \cup \{\phi\}$. For any subset S of A, the lower set $\downarrow D \in L(A)$. Moreover, since A is L-complete, it follows that $\vee S = \vee \downarrow S$ exists in A. Distributivity of meet over joins follows from the fact that $\downarrow S$ is a lower set of A. Thus, it is clear that an L-frame is exactly the ordinary frame. The common examples of frames are the sets of all open sets of topological spaces.

(2) If $I(A) = \{\downarrow x : x \in A\}$ for every semilattice A, then I is a system of sets on **Slat**.
<u>Claim</u>: Every semilattice is an I-frame. (Note: By definition, every I-frame is a semilattice so that an I-frame is exactly a semilattice.)
Take any semilattice S and let $T \in I(S)$. Then $T = \downarrow x$ for some $x \in S$. Clearly, $\vee T = \vee \downarrow x = x \in S$. Pick any $s \in S$. $s \wedge \vee T = s \wedge x \leq x$ implies that $s \wedge x \in \downarrow x = T$. Thus, $s \wedge x = t$ for some $t \in T$. Trivially, $s \wedge \vee T \geq s \wedge t$ for all $t \in T$. Note that $(s \wedge x) \wedge s = s \wedge x$ so that $s \wedge \vee T = s \wedge (s \wedge x) = s \wedge t \in s \wedge T$. It follows that $s \wedge \vee T \leq \vee\{s \wedge t : t \in T\}$. Hence $s \wedge \vee T = s \wedge \downarrow x = \vee\{s \wedge t : t \in T\}$. □
Let $f : S \to T$ be a semilattice morphism between semilattices S and T.
<u>Claim</u>: f preserves joins of I-sets if and only if f preserves order.
Proof
($\Rightarrow$) Let $x \leq y$, i.e. $x \in \downarrow y$. Then $f(y) = f(\vee \downarrow y) = \vee f(\downarrow y)$. But $f(x) \in f(\downarrow y) \Rightarrow f(x) \leq \vee f(\downarrow y) = f(y)$.
($\Leftarrow$) Let $\downarrow x$ be an I-ideal for some $x \in S$. $f(\vee \downarrow x) = f(x)$. If $y \leq x$, then $f(y) \leq f(x)$. $\therefore f(\vee \downarrow x) \geq \vee f(\downarrow x)$. But in particular, $x \in \downarrow x$ so that $f(x) \in f(\downarrow x)$ which implies that $f(x) \leq \vee f(\downarrow x)$. Therefore, $f(\vee \downarrow x) = \vee f(\downarrow x)$, whence f preserves the joins of I-ideals. □
$\therefore$ The category of all I-frames is exactly **Slat**.

(3) Let $Id(A)$ denote the collection of all ideals of A, then Id defines a system of sets. Notice that $\phi \notin Id(A)$ A semilattice A is an Id-frame if and only if it is up-complete (i.e. each directed subset has a supremum) and the meet operation is distributive over the joins of up-directed sets. Such semilattices are called preframes. They are also known as meet-continuous semilattices. The category of all Id-frames is the same thing as the category of **Prefrm** of preframes studies by Johnstone and Vickers recently.

(4) Let κ be any regular cardinal.
Define a system $Z_\kappa(A) = \{\downarrow X : X \subseteq A, |X| < \kappa\}$. Then Z_κ satisfies conditions (S1)-(S4).
Proof
Let κ be a regular cardinal.
(S1) Let $x \in A$. Then $|x| = 1 < \kappa$. $\therefore \downarrow x \in Z_\kappa(A)$.
(S2) If $f : A \to B$ be a semilattice morphism, the $\downarrow f(D) - \{b \in B : b \leq f(d)$ for some $d \in D\}$ for any $D \in Z_\kappa(A)$. But f is at most bijective so that $|f(D)| \leq |D| < \kappa$. It follows that $f(D)$ is a subset of B with cardinality strictly less than κ, i.e. $\downarrow f(D) \in Z_\kappa(B)$.
(S3) It is clear that the intersection of two sets whose cardinality is less than κ - is still a set whose cardinality is less than κ. Thus, $(Z_\kappa(A), \subseteq)$ is a semilattice.

(S4) Let $\mathcal{D} \in Z_\kappa(Z_\kappa(A))$. Then,

$$\mathcal{D} = \{\downarrow X_i : X_i \subseteq A, |X_i| < \kappa, i \in I\} \text{ with } |I| < \kappa.$$

Clearly, $\cup\mathcal{D} \subseteq A$. Moreover, $\mathcal{D} =\downarrow (\cup\{X_i : i \in I\})$. But $|\cup \{X_i : i \in I\}| < \kappa$ since κ is regular, $|I| < \kappa$ and $|X_i| < \kappa$ for all $i \in I$.
Here $|\phi| = 0 < \kappa$. Thus, $\phi \in Z_\kappa(A)$ for each A. So each z_κ-complete semilattice has a bottom element. A semilattice A is a z_κ-frame if and only if it is a κ-frame, and the category of z_κ-frames is the same as the category of κ-frames.
In particular, taking $\kappa = \aleph_0$, then

$$\begin{aligned} Z_{\aleph_0}(A) &= \{\downarrow D : D \subseteq A, |D| < \aleph_0\} \\ &= \{\downarrow D : D \subseteq A, D \text{ is finite}\} \end{aligned}$$

A semilattice is an $\aleph_0$-frame if and only if it is a distributive lattice.
If $\kappa = \aleph_1$, then an $\aleph_1$-frame is exactly a σ-frame, i.e. a semilattice with countable joins and the meet operation is distributive over the joins of countable subsets. The category σ**Frm** of σ-frames has been studied by many authors.

(5) By Proposition(4.2.1.1), for any regular cardinal κ, we have a new system $Id \wedge Z_\kappa$ of sets on **Slat**. So we have many other categories of Z-frames.

What follows is a series of properties of Z-frames.

Proposition 4.2.2.1 *A Z-complete semilattice A is a Z-frame if and only if it satisfies one of the following three equivalent conditions:*

(1) $(\vee D) \wedge (\vee E) = \vee(D \cap E)$ *for any two Z-ideals D, E of A;*

(2) $(\vee E) \wedge (\vee H) = \vee\{x \wedge y : x \in E, y \in H\}$ *for any two Z-sets E, H of A;*

(3) The map $\sup : Z(A) \to A$ *is a semilattice morphism.*

Proof
We show first of all that the three statements (1), (2) and (3), are equivalent.
(1) $\Leftrightarrow$ (3):
Since the meet operation in $Z(A)$ is the intersection of sets, condition (1) is obviously equivalent to condition (3).
(2) $\Rightarrow$ (1):
If condition (2) holds, then for every pair of Z-ideals D and E of A, D and E are also Z-sets of A. So by (2),

$$(\vee D) \wedge (\vee E) = \vee\{x \wedge y : x \in D, y \in E\}.$$

It is clear that since D and E are lower sets, for any $x \in D$ and $y \in E$, $x \wedge y \leq x$ and $x \wedge y \leq y$ imply $x \wedge y \in D \cap E$. But any $w \in D \cap E$ may be written as $w \wedge w$, hence

$$\{x \wedge y : x \in D, y \in E\} = D \cap E.$$

Hence $(\vee D)\wedge(\vee E)=\vee(D\cap E)$ and condition (1) holds.
(1) $\Rightarrow$ (2):
Conversely, suppose condition (1) holds. Then for any pair of Z-sets E and H of A, $\downarrow E\in Z(A)$ and $\downarrow H\in Z(A)$. So by (1),

$$\vee(\downarrow E)\wedge\vee(\downarrow H)=\vee(\downarrow E\wedge\downarrow H)$$

holds. But $\vee(\downarrow E)\wedge\vee(\downarrow H)=(\vee E)\wedge(\vee H)$. Moreover,

$$\begin{array}{ll} & u\in\downarrow\{x\wedge y:x\in E,y\in H\}\\ \Leftrightarrow & u\leq x\wedge y \text{ for some } x\in E,y\in H\}\\ \Leftrightarrow & u\leq x \text{ and } u\leq y, x\in E,y\in H\\ \Leftrightarrow & u\in\downarrow x \text{ and } u\in\downarrow y, x\in E,y\in H\\ \Leftrightarrow & u\in\downarrow E \text{ and } u\in\downarrow H\\ \Leftrightarrow & u\in(\downarrow E)\cap(\downarrow H),\end{array}$$

i.e. $(\downarrow E)\cap(\downarrow H)=\downarrow\{x\wedge y:x\in E,y\in H\}$.
It follows immediately that

$$\vee((\downarrow E)\cap(\downarrow H))=\vee\downarrow\{x\wedge y:x\in E,y\in H\}=\vee\{x\wedge y:x\in E,y\in H\}.$$

Consequently, condition (2) holds.
Now we show that condition (1) is equivalent to the assertion that A is a Z-frame.
Suppose A is a Z-frame. Then for any two Z-ideals D,E of A, we have:

$$(\vee D)\wedge(\vee E)=\vee\{x\wedge(\vee E):x\in D\}.$$

For each $x\in D$, $x\wedge\vee E=\vee\{x\wedge y:y\in E\}\leq\vee(D\cap E)$, hence $(\vee D)\wedge(\vee E)\leq\vee(D\cap E)$.
The reverse inequality is obviously true since $D\cap E\subseteq D,\subseteq E$, and consequently, $\vee(D\cap E)\leq(\vee D)\wedge(\vee E)$. So, $(\vee D)\wedge(\vee E)=\vee(D\cap E)$.
Conversely, suppose condition (2) holds. Then for any $x\in A$ and any $D\in Z(A)$, $\downarrow x$ is a Z-ideal of A and by condition (2), we have:

$$\begin{aligned} x\wedge\vee D &=(\vee(\downarrow x))\wedge(\vee D)\\ &=\vee((\downarrow x)\cap D)\\ &=\vee\{x\wedge d:d\in D\}.\end{aligned}$$

So, A is a Z-frame. □

Lemma 4.2.2.1 *A semilattice A is a Z-frame if and only if the semilattice morphism $\downarrow: A\to Z(A)$ has a right inverse in* **Slat**.

Proof
If A is a Z-frame, then by Proposition(4.2.2.1), the map $\vee:Z(A)\to A$ is a semilattice morphism and clearly $\vee\circ\downarrow x=\vee(\downarrow x)=x$ for all $x\in A$, i.e. $\vee\circ\downarrow=id_A$. Hence $\vee$ is a left inverse of $\downarrow$ in **Slat**.
Conversely, if $h:Z(A)\to A$ is a semilattice morphism such that $h\circ\downarrow=id_A$. By Proposition(4.2.2.1), in order to show that A is a Z-frame, it is enough to show that h is $\vee$.
Let $X\in Z(A)$. For each $x\in A$, $\downarrow\leq X$ since X is a lower set. So $x=h(\downarrow x)\leq h(X)$ since h preserves finite meets and hence order. $\therefore$ $h(X)$ is an upper bound of X in A. If $a\in A$ is an upper bound of X, then $X\leq\downarrow a$. So $h(X)\leq h(\downarrow a)=a$. Hence $h(X)=\vee X$. □

Lemma 4.2.2.2 *Let S be a semilattice of a Z-frame A such that S is closed under the joins of Z-ideals, i.e. for each $D \in Z(S)$, $\vee S \in S$. Then, S is a Z-frame.*

Proof
Notice that S is closed under the joins of Z-ideals if and only if it is closed under the joins of Z-ideals. Since for each $D \in Z(S)$, $\vee D \in S$ exists, so S is Z-complete.
Since $i : S \hookrightarrow A$, the inclusion map, is a semilattice morphism, it follows from (S2) that $\downarrow i(D) = \downarrow D \in Z(A)$ for each $D \in Z(S)$. So, D is a Z-set of A.
If $s \in S$ and $D \in Z(S)$, then $s \wedge \vee D = \vee\{s \wedge x : x \in D\}$ holds in A. On the other hand, the set $\{s \wedge x : x \in D\}$ is a Z-set of S since by Corollary(4.2.1.1),
$\downarrow \{s \wedge x : x \in D\} = \downarrow s \cap D \in Z(S)$.
So, $s \wedge \vee D = \vee\{s \wedge x : x \in D\}$ holds in S. Thus, S is a Z-frame. □

4.2.3 Free Z-frame objects

A category **C** is said to be free over **Set** if the forgetful functor

$$\mathbf{C} \longrightarrow \mathbf{Set}$$

has a left adjoint.
In this section, we construct the free functor

$$\mathbf{Set} \longrightarrow \mathbf{ZFrm}$$

which is right adjoint to the forgetful functor

$$\mathbf{ZFrm} \longrightarrow \mathbf{Set}.$$

We use the factorisation

$$\mathbf{ZFrm} \longrightarrow \mathbf{Set} \longrightarrow \mathbf{SLat}$$

to obtain this free functor.
In the second part of the section we show that the forgetful functor **Frm** $\longrightarrow$ **ZFrm** has a left adjoint. Thus we get other ways to obtain the free functor

$$\mathbf{Set} \longrightarrow \mathbf{Frm}.$$

Lemma 4.2.3.1 *Given a system Z of sets on* **Slat**, *$Z(S)$ is a Z-frame for each semilattice S.*

Proof
We have seen that $Z(S)$ is a Z-complete semilattice by (S4), the meet and join operations are just the intersection and union of sets.
Let $E \in Z(S), \mathcal{D} \in Z(Z(S))$. Then we have:

$$E \wedge \vee\mathcal{D} = E \cap \cup\mathcal{D} = \cup\{E \cap D : D \in \mathcal{D}\} = \vee\{E \wedge D : D \in \mathcal{D}\}.$$

Hence $Z(S)$ is a Z-frame. □
The assignment $S \mapsto Z(S)$ actually defines a functor from **Slat** to **ZFrm**, which is precisely the free functor **Slat** $\longrightarrow$ **ZFrm**.
We shall make a small digression to the categorical concept of adjunction before we proceed with the next result.

Definition 4.2.3.1 ***(Adjoint functors)*** *Consider two functors*

$$F : \mathbf{C} \to \mathbf{D},\ G : \mathbf{D} \to \mathbf{C}.$$

between two categories $\mathbf{C}$ *and* $\mathbf{D}$*. One says that* G *is right adjoint to* F *(and* F *is left adjoint to* G*), denoted by* $F \dashv G$*, when for any two objects* C *from* $\mathbf{C}$ *and* D *from* $\mathbf{D}$ *there is a natural bijection* $\theta : Hom_{\mathbf{C}}(C, GD) \simeq Hom_{\mathbf{D}}(FC, D)$.

Remarks

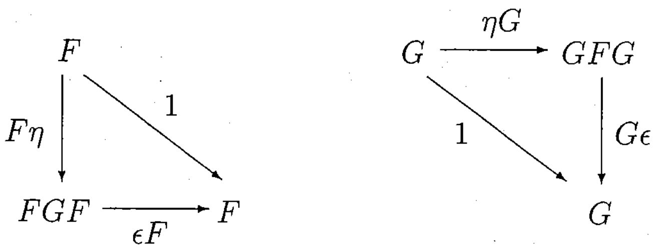
1. Let $F \dashv G$ and θ as above. For each $X \in C_0$, let $\eta_X : X \to GF(X)$ be the unique morphism such that $\theta(\eta_X) = id_{FX}$. This morphism η_X is called the unit of adjunction at X. Dually, there is the counit $\epsilon_Y : FG(Y) \to Y$ in $\mathbf{D}$.

2. $\eta_X : X \to GF(X)$ is universal among the arrows from X to an object of the form GA. On the other hand, if G is a functor such that a universal arrow $\eta_X : X \to GF(X)$ exists for each object X in $\mathbf{C}$, then G has a left adjoint. The dual result is also true.

3. If $F \dashv G$, then there exist two natural transformations

$$\eta : id_{\mathbf{C}} \to GF,\ \epsilon : FG \to id_{\mathbf{D}}$$

such that the following triangles commute:



Lemma 4.2.3.2 *Let* S *be a semilattice, consider the map*

$$\downarrow : S \to Z(S)$$

defined by $\downarrow a = \{x \in S : x \leq a\}$ *for all* $a \in S$*. Then,* $\downarrow$ *is a semilattice morphism. For any semilattice morphism* $r : S \to A$ *from* S *to a* Z*-frame* A*. there exists a unique* Z*-frame morphism* $\overline{r} : Z(S) \to A$ *such that* $r = \overline{r} \circ \downarrow$*.*

Proof
Take any $s_1, s_2 \in S$. Then, $\downarrow(s_1 \wedge s_2) = \{s \in S : s \leq s_1 \wedge s_2\} \subseteq \downarrow s_1 \cap \downarrow s_2$. If $s \in \downarrow s_1 \cap \downarrow s_2$, then $s \leq s_1$ and $s \leq s_2$. This implies that s is a lower bound of $\{s_1, s_2\}$ and hence is no match to the greatest lower bound $s_1 \wedge s_2$. Hence
$\downarrow(s_1 \wedge s_2) = (\downarrow s_1) \cap (\downarrow s_2) = (\downarrow s_1) \wedge_{Z(S)} (\downarrow s_2)$. Thus, $\downarrow: S \to Z(S)$ is a semilattice morphism.
Let $r : S \to A$ be any semilattice morphism and Z-frame, define

$$\overline{r} : Z(S) \to A$$

by $\overline{r} = \vee r(D)$, $\forall D \in Z(S)$.
As r is a semilattice morphism, by condition (S2), $\downarrow r(D) \in Z(A)$ for each $D \in Z(S)$, i.e. $r(D)$ is a Z-set of A. So $\vee r(D)$ exists in A, whence $\overline{r}$ is well-defined. We shall show that $\overline{r}$ is a Z-frame morphism.
Given any $\mathcal{D} \in Z(Z(S))$, we have

$$\begin{aligned}\overline{r}(\vee\mathcal{D}) &= \vee r(\cup\mathcal{D}) \\ &= \vee\{\vee r(D) : D \in \mathcal{D}\} \\ &= \vee\{\overline{r}(D) : D \in \mathcal{D}\}\end{aligned}$$

Hence $\overline{r}$ preserves joins of Z-sets.
Now given any two Z-ideals of S, $D, E \in Z(S)$, we have:

$$\overline{r}(D \wedge E) = \overline{r}(D \cap E) = \vee r(D \cap E)$$

Since r preserves meets and that D, E are both lower sets, it is clear that $r(D \cap E) = r(D) \cap r(E)$. Consequently,

$$\vee(r(D) \cap r(E)) = \vee(\downarrow r(D) \cap \downarrow r(E)) = (\vee \downarrow r(D)) \wedge (\vee \downarrow r(E))$$

by Lemma(4.2.2.1) and the fact that both $\downarrow r(D)$ and $\downarrow r(E)$ are Z-ideals of A. Thus, $\overline{r}(D \wedge E) = (\vee \downarrow r(D)) \wedge (\vee \downarrow r(E)) = (\vee r(R)) \wedge (\vee r(E)) = \overline{r}(D) \wedge \overline{r}(E)$. Therefore, $\overline{r}$ preserves meets. In addition, it is obvious that $\overline{r} \circ \downarrow(s) = \vee r(\downarrow s) = r(s)$ for all $s \in S$ so that $r = \overline{r} \circ \downarrow$.
Suppose $h : Z(S) \to A$ is any Z-frame morphism satisfying the equation $r = h \circ \downarrow$. For each $x \in S$, $\overline{r}(\downarrow x) = r(x) = h(\downarrow x)$. If $D \in Z(S)$, then the set $\mathcal{E} = \downarrow x : x \in D\}$ is a Z-set of $Z(S)$ because it is the image of D under the semilattice morphism $\downarrow: S \to Z(S)$. Moreover, $\vee\mathcal{E} = D$, so as both $\overline{r}$ and h preserves joins of Z-sets, we have:

$$\begin{aligned}h(D) &= h(\vee\mathcal{E}) \\ &= \vee h(\mathcal{E}) \\ &= \vee\{h(\downarrow x) : x \in D\} \\ &= \vee\{r(x) : x \in D\} \\ &= \overline{r}(D)\end{aligned}$$

This shows that $h = \overline{r}$. □

Lemma 4.2.3.3 *Let $f : S \to T$ be a semilattice morphism, then the unique Z-frame morphism $Z(f) : Z(S) \to Z(T)$ satisfying $Z(f) \circ \downarrow = \downarrow \circ f$ is the map which sends $D \in Z(S)$ to $\downarrow f(D)$.*

Proof
Define $Z(f) : Z(S) \to Z(T)$ by $Z(f)(D) = \downarrow f(D)$ for each $D \in Z(S)$. By (S2), $\downarrow f(D) \in Z(T)$ so that $Z(f)$ is a well-defined map. Let $D \in Z(S)$.

$$\begin{aligned} Z(f)(D \wedge E) &= Z(f)(D \cap E) \\ &= \downarrow f(D \cap E) \\ &= \downarrow f(D) \cap \downarrow f(E) \\ &= Z(f)(D) \cap Z(f)(E) \\ &= Z(f)(D) \wedge Z(f)(E). \end{aligned}$$

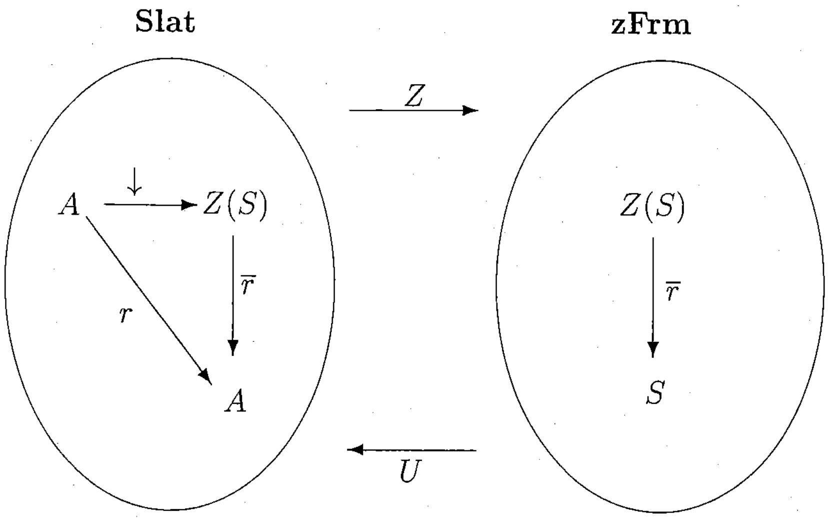
Let $\mathcal{D} \in Z(Z(S))$.

$$\begin{aligned} f(\vee \mathcal{D}) &= \downarrow f(\cup \mathcal{D}) \\ &= \downarrow f(\cup\{D \in Z(S) : D \in \mathcal{D}\}) \\ &= \cup \downarrow \{f(D) : D \in \mathcal{D}\} \\ &= \vee Z(f)(\mathcal{D}). \end{aligned}$$

On the other hand, suppose $\overline{r}$ also satisfies the required condition, i.e. $\overline{r} \circ \downarrow = \downarrow \circ f$. Let $k = \downarrow \circ f$. Clearly, k is a semilattice morphism. By Lemma(4.2.3.2), there is a unique Z-frame morphism $\overline{r} : Z(S) \to Z(T)$ such that $k = \downarrow \circ f = \overline{r} \circ \downarrow$. Therefore, $\overline{r} = \overline{f}$. So the lemma is proved. □
Combining the above two lemmas, we obtain the following result:

Theorem 4.2.3.1 *The functor Z :* **Slat** $\to$ **ZFrm** *which sends a semilattice S to the collection $Z(S)$ of all its Z-ideals, and sends a semilattice morphism f to $Z(f)$ is left adjoint to the forgetful functor* **ZFrm** $\to$ **Slat**.

Proof
The following diagram depicts the standard argument that shows $Z \dashv U$, where U is the forgetful functor **ZFrm** $\to$ **Slat**. □



Now we consider the free functor of U : **Slat** $\to$ **Set**. We now construct F : **Set** $\to$ **Slat**. Let X be a set. Consider the set $S(X) = \{A \in \mathcal{P}(X) : |A| < \infty\}$ and the superset relation

$\supseteq$ (which is the same as $\subseteq^{op}$). It is easy to see that $(S(X), \supseteq)$ is a semilattice where inf is exactly the set union. Now we shall show that $F \dashv U$.
Consider the singleton map $s : X \to F(X)$ which sends x to $\{x\}$. Clearly, this map is well defined as the singleton contains only one element (the fundamental finite subset of X).
Given a set map $r : X \to S$ to a semilattice S, define the map $\overline{r} : F(X) \to S$ by $\overline{r}(A) = r(A)$ for each $A \in F(X)$. We now show that $\overline{r}$ is a semilattice morphism. Given $A, B \in F(X)$.

$$\begin{aligned}\overline{r}(A \wedge B) &= \overline{r}(A \cup B)\\ &= \{r(x) : x \in A \text{ or } B\}\\ &= r(A) \cup r(B)\\ &= \overline{r}(A) \wedge \overline{r}(B).\end{aligned}$$

This complete the claim. Next we show that $\overline{r} \circ s = r$. Let $x \in X$ be given. Then $\overline{r} \circ s(x) = \overline{r}(\{x\}) = r(x)$ by definition.
Thus, $F \dashv U$ and this establishes the adjunction between the categories **Set** and **Slat**. □
By exploiting the factorisation

$$\mathbf{ZFrm} \longrightarrow \mathbf{Slat} \longrightarrow \mathbf{Set},$$

we obtained the desired free functor.
In the remaining part of this section, we construct the free functor **ZFrm** $\to$ **Frm**.

Definition 4.2.3.2 *(Z-closed)*
Given a Z-complete semilattice A, a subset $X \subseteq A$ is said to be Z-closed if it satisfies the conditions:

(1) $X = \downarrow X$.

(2) For any Z-set D of A, $D \subseteq X$ implies $\vee D \in X$.

The collection of all z-closed subsets of A is denoted by $C_z(A)$ (or just $C(A)$ if it causes no confusion).
Obviously, $C(A)$ is closed under intersection.
Proof
Let $\mathcal{C} = \{X_i : X_i \in C(A), i \in I\}$ be a collection of z-closed subsets of A. Assume without loss of generality that $\mathcal{C} \neq \phi$.

(1) $\downarrow \cap\mathcal{C} = \cap \downarrow \{X_i : i \in I\} = \cap\{X_i : i \in I\} = \cap\mathcal{C}$.

(2) For any Z-set D of A, $D \subseteq \cap\mathcal{C}$ implies $D \subseteq X_i, i \in I$. Then, $\vee D \in X_i$ for all $i \in I$. Because $X_i \in C(A)$, $\vee D \in \cap\mathcal{C}$.

In addition, $\downarrow A = A$ and of course $\sup D \in A$ for all Z-sets D of A since A is Z-complete. This means $A \in C(A)$. Of course, $C(A)$ is closed under arbitrary intersection. So, $(C(A), \subseteq)$ is a complete lattice. It is also obvious that for each $x \in A$, $\downarrow x \in C(A)$. Given any subsets X of A, the intersection of all z-closed sets which contain X is called the z-closure of X and is denoted by $c(X)$. Of course, $c(X) \in C(A)$ for any $X \subseteq A$.
For the purpose of discussion later, there is a need to digress to the following theorem regarding ordinal numbers.

Theorem 4.2.3.2 ***(Ordinal induction)***
Let S be a class of ordinal numbers such that

(1) $0 \in S$;

(2) $\alpha \in S$ *implies* $\alpha + 1 \in S$;

(3) If α *is a limit ordinal and* $\beta \in S$ *for all* $\beta \leq \alpha$, *then* $\alpha \in S$.

Then, $S = \mathcal{O}rd$, *which is the class of all ordinal numbers.*

For a given subset X of A, we define an ordinal sequence of subsets: $X^0 = \downarrow X$, $X^{n+1} = \downarrow \{\vee D : D \subset X^n, D \in Z(A)\}$, $X^m = \cup_{n<m} X^n$ if m is a limit ordinal.
<u>Claim</u>: $X^n \subseteq X^{n+1}$ for each ordinal n.
Proof
Let $x \in X^n$. Then $\downarrow x \in Z(A)$. Moreover, $\vee \downarrow x = x$ and $\downarrow \subseteq X^n$ by ordinal induction. Then in particular, $\downarrow x$ is a lower set contained in X^n so that $x \in \downarrow \{\vee D : D \subseteq X^n, D \in Z(A)\}$. Hence $X^n \subseteq X^{n+1}$. This justifies the claim.
There is an ordinal m such that $X^{m+1} = X^m$; otherwise we may have a strictly ascending chain $\{X^i : i \in \mathcal{O}rd\}$ of subsets of A. This contradicts A being a set. Note that if m is an ordinal such that $X^{m+1} = X^m$, then

(1) $X^{m+1} = \downarrow \{\vee D : D \subseteq X^m, D \in Z(A)\}$. So it is clear that X^m is a lower set.

(2) For any Z-set D of A, $D \subseteq X^m$ implies that $\vee D \in X^{m+1}$ by the manner in which X^{m+1} is constructed. But $X^{m+1} = X^m$. So $\vee D \in X^m$.

Hence X^m is a z-closed subset of A.
Since $X \subseteq X^m$, it follows easily that $c(X) \subseteq X^m$.
Now we shall prove by ordinal induction that

$$X^n \subseteq c(X) \text{ for each ordinal } n.$$

Firstly, for the case when $n = 0$. $X^0 = \downarrow X$. Clearly, $\downarrow X \subseteq \downarrow c(X) = c(X)$. So this case is trivially true.
Secondly assume that $X^n \subseteq c(X)$. Then $X^{n+1} = \downarrow \{\vee D : D \subseteq X^n, D \in Z(A)\}$.
Let $y \in X^{n+1}$. Then $y \leq \vee D$ for some $D \subseteq X^n, D \in Z(A)$. Since $X^n \subseteq c(X)$, $D \subseteq x(X)$. But $c(X)$ is z-closed, so $\vee D \in c(X)$. Therefore $y \in c(X)$ as $c(X)$ is a lower set.
$\therefore X^{n+1} \subseteq c(X)$.
Thirdly, if n is a limit ordinal, $X^n = \cup_{p<n} X^p$. But for all $p < n$, $X^p \subseteq c(X)$. It follows that $X^n = \cup_{p<n} X^p \subseteq c(X)$. By ordinal induction, $X^n \subseteq c(X)$ holds for all ordinals n.

□

This claim inevitably implies $X^m = c(X)$.
Hence $c(X) = X^m$ for some ordinal m.

Proposition 4.2.3.1 *For a semilattice* A, A *is a Z-frame if and only if* $C(A)$ *is a frame.*

Proof

Assume firstly that A is a Z-frame. Let $\{x_i : i \in I\} \subseteq C(A)$ be a set of z-closed sets of A and $X \in C(A)$, we wish to show that the equation:

$$X \wedge \vee\{X_i : i \in I\} = \vee\{X \wedge X_i\}$$

holds. Let $B = \cup\{X_i : i \in I\}$.
Then it is easily seen that $c(B)$ which is the smallest z-closed set containing all I_i's is equal to $\vee\{X_i : i \in I\}$ with sup taken in $C(A)$.
We will prove that $X \cap B^n \subseteq \vee\{X \cap X_i : i \in I\}$ holds for each n by induction.
Firstly, $X \cap B^0 = X \cap \downarrow \cup\{X_i : i \in I\}$. $y \in X \cap \downarrow \cup\{X_i : i \in I\}$ implies that $y \in X$ and $y \leq x_i$ for some $x_i \in X_i$. But $\downarrow X_i = X_i$ as X_i is z z-closed set. Therefore, $y \in X$ and $y \in X_i$. This implies that $y \in X \cap X_i$ for some $i \in I$. But $\wedge\{X \cap X_i : i \in I\}$ is the smallest z-closed set that contains $\{X \cap X_i : i \in I\}$. In particular, $y \in \vee\{X \cap X_i : i \in I\}$.
Secondly, suppose n is an ordinal such that $X \cap B^n \subseteq \vee\{X \cap X_i : i \in I\}$, then for each $x \in X \cap B^{n+1}$, by definition, $x \in X$ and $x \leq \vee D$ for some $D \in Z(A)$ and $D \subseteq B^n$. As A is a Z-frame, $x = x \wedge \vee D = \vee(x \wedge D)$ where $x \wedge D = \{x \wedge d : d \in D\}$ is a Z-set of A by Corollary(4.2.1.1). $x \wedge D$ is contained in $X \cap B^n$ since X and B^n are lower sets. By assumption, $X \cap B^n \subseteq \vee\{X \cap X_i : i \in I\}$, we see that $x \wedge D$ is a Z-set and is contained in the z-closed set $\vee\{X \cap X_i : i \in I\}$, thus
$x = \vee(x \wedge D) \in \vee\{X \cap X_i : i \in I\} = \vee\{X \wedge X_i : i \in I\}$.
Thirdly, if m is a limit ordinal and $X \cap B^n \subseteq \vee\{X \wedge X_i : i \in I\}$ for each $n < m$, then obviously $X \cap B^m \subseteq \vee\{X \wedge X_i : i \in I\}$. From the above discussion, it follows that $X \cap B^n \subseteq \vee\{X \wedge X_i : i \in I\}$ for all ordinals n.
$\therefore X \wedge \vee X_i = X \cap c(B) = X \wedge B^m$ for some ordinal m, and thus,
$X \wedge \vee\{X_i : i \in I\} = X \wedge B^m \subseteq \vee\{X \wedge X_i : i \in\}$. The reverse inequality is obviously true. So $C(A)$ is a frame.
Now suppose that $C(A)$ is a frame.
Consider the map $\downarrow: A \to C(A)$ which sends $x \in A$ to the z-closed set $\downarrow x$. ($\because$ For any Z-set D of A, $D \subseteq \downarrow x$ implies $\vee D \leq \vee \downarrow x = x$ which implies $\vee D \in \downarrow x$.)
Note also that $\downarrow$ is an order-embedding which preserves meets. ($\because \downarrow x \subseteq \downarrow y$ iff $x \leq y$. Moreover, $\downarrow(x \wedge y) = \downarrow x \cap \downarrow y$.) This makes $\downarrow$ an injective map. In addition, given any Z-set D of A, $\downarrow \vee D = \{y \leq \vee D : Y \in A\}$.

(1) $\downarrow(\downarrow \vee D)) = \downarrow \vee D$.

(2) For any Z-set E of A, $E \subseteq \downarrow \vee D$ implies that $\vee E \leq \vee \downarrow (\vee D) = \vee D$ so that $\vee E \in \downarrow \vee D$.

Therefore, $\downarrow \vee D$ is a z-closed set that contains $\downarrow D$. Thus, $\downarrow(\vee D) = \vee(\downarrow D)$ where the second sup is taken in the context of $(C(A), \subseteq)$. So, $\downarrow$ preserves joins of Z-sets as well. It follows that $\downarrow$ identifies A with a subset $C(A)$ which is closed under meets and joins of Z-sets. Since $C(A)$ is a frame, any such a subset of it is a Z-frame by Proposition(4.2.2.1). Hence A is a Z-frame. □

Lemma 4.2.3.4 *Let A be a Z-frame. Then the map $\downarrow: A \to C(A)$ is a Z-frame morphism, and for any Z-frame morphism $h : A \to B$ to a frame B, there exists a unique frame morphism $\overline{h} : C(A) \to B$ such that $h = \overline{h} \circ \downarrow$.*

Proof

$\downarrow$ obviously preserves meets and the top element. Also $C(A)$ being a frame is obviously a Z-frame. If $D \in Z(A)$, then $\downarrow(\vee D) \geq \vee\{\downarrow d : d \in D\}$; on the other hand,
$D \subseteq \vee\{\downarrow d : d \in D\} \in C(A)$, so

$$\vee D \in \vee\{\downarrow d : d \in D\}.$$

Thus, $\downarrow(\vee D) \leq \vee\{\downarrow d : d \in D\}$. Hence $\downarrow(\vee D) = \vee\{\downarrow d : d \in D\}$. This shows that $\downarrow$ preserves joins of Z-sets, it is thus a Z-frame morphism.
Now suppose that $h : A \to B$ is a Z-frame morphism and B is a frame, define

$$\overline{h} : C(A) \to B,$$
$$\overline{h}(X) = \vee h(X) \text{ for any } X \in C(A).$$

We now show that $\overline{h}$ is a frame morphism.
Let X and Y be two z-closed sets of A, then obviously,
$\overline{h}(X \wedge Y) = \overline{h}(X \cap Y) \subseteq \overline{h}(X) \cap \overline{h}(Y)$.
On the other hand,

$$\begin{aligned}\overline{h}(X) \cap \overline{h}(Y) &= \vee h(X) \wedge \vee h(Y)\\ &= \vee\{h(x) \wedge h(y) : x \in X, y \in Y\}(\text{by Proposition}(4.2.2.1)\\ &= \vee\{h(x \wedge y) : x \in X, y \in Y\}\end{aligned}$$

Now, $\vee\{h(x \wedge y) : x \in X, y \in Y\} \leq \vee h(X \cap Y) = \overline{h}(X \wedge Y)$. Hence
$\overline{h}(X \wedge Y) = \overline{h}(X) \wedge \overline{h}(Y)$. So, $\overline{h}$ preserves meets.
Now let χ be a collection of z-closed sets of A, we need to show that

$$\overline{h}(\vee\chi) = \vee\{\overline{h}(X) : X \in \chi\}.$$

Put $E = \cup\chi$, we show that

$$\overline{h}(E^n) \leq \vee\{\overline{h}(X) : X \in \chi\}$$

for every ordinal n.
We do this by ordinal induction.
Firstly,

$$\begin{aligned}\overline{h}(E^0) &= \overline{h}(\downarrow E)\\ &= \overline{h}(\downarrow \cup\chi)\\ &= \overline{h}(\cup\chi)\\ &= \vee h(\cup\chi)\\ &\leq \vee\{\overline{h}(X) : X \in \chi\}\end{aligned}$$

Suppose n is an ordinal such that

$$\overline{h}(E^n) \leq \vee\overline{h}(X) : X \in \chi\}$$

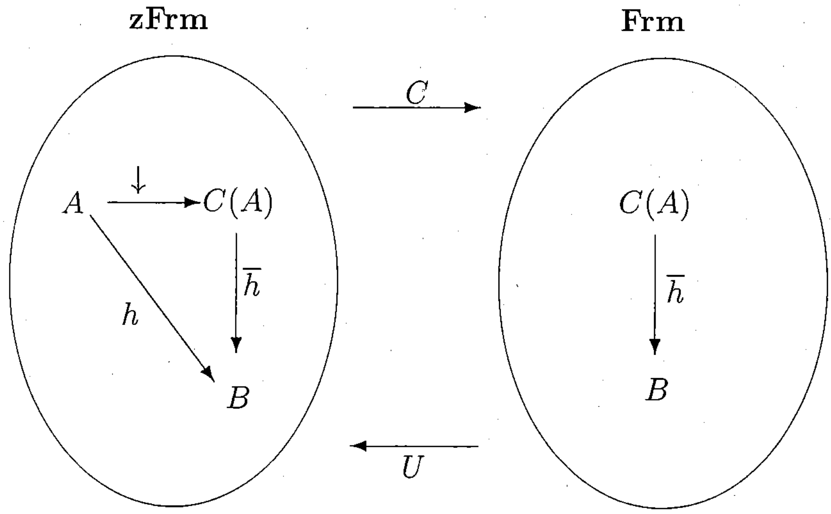
Then, for each $x \in E^{n+1}$, there is a $D \in Z(A)$, $D \subseteq E^n$ which implies that $x \leq \vee D$. Since $h(d) \leq \vee\{\overline{h}(X) : X \in \chi\}$ for all $d \in D$ and h preserves the joins of Z-sets, so

$$h(x) \leq h(\vee D) \leq \vee\{\overline{h}(X) : X \in \chi\}.$$

Hence $\overline{h}(E^{n+1}) \leq \vee\{\overline{h}(X) : X \in \chi\}$.
If m is a limit ordinal such that $\overline{h}(E^n) \leq \vee\{\overline{h}(X) : X \in \chi\}$, then obviously, $\overline{h}(E^m) \leq \vee\{\overline{h}(X) : X \in \chi\}$.
By ordinal induction, we have proven that
$\overline{h}(E^n) \leq vee\{\overline{h}(X) : X \in \chi\}$ for each ordinal n. Thus, $\overline{h}(\vee\chi) =\leq \vee\{\overline{h}(X) : X \in \chi\}$. It follows then that $\overline{h}$ preserves arbitrary joins since the other inequality is trivial. So, $\overline{h}$ is a frame morphism.
Let $a \in A$.
$(\overline{h}\circ \downarrow)(a) = \overline{h}(\downarrow a) = \vee h(\downarrow a) = h(a)$.
$\therefore h = \overline{h}\circ \downarrow$. This is obvious.
If $r : C(A) \to B$ is another frame morphism with $h = r\circ \downarrow$, then it follows that for any $X \in C(A)$, i.e. X is a z-closed set of A,
$r(X) = r(\downarrow X) = (r\circ \downarrow)(X) = h(X)$.
$\therefore$ it follows easily that $r = \overline{h}$. Thus, the lemma is proved. □
The following theorem is an immediate consequence of the preceding lemma.

Theorem 4.2.3.3 *The forgetful functor U : **Frm** $\to$ **ZFrm** has a left adjoint.*

Proof
The following picture and a standard argument suffice to prove:



□

4.2.4 Z-frames freely generated by sites

The main aim of this section is to construct the Z-frame freely generated by a z-site.

Definition 4.2.4.1 *(**z-site**) Given a system Z of sets on* **Slat**, *by a z-coverage (or briefly a coverage if the system is explicitly refered) C on a semilattice A, we mean a function C which assigns to each element a of A a collection $C(a)$ of subsets satisfying the conditions:*

(i) each $S \in C(a)$ *is a* Z*-set of* A *and* $S \subseteq \downarrow a$*;*

(ii) $S \in C(a)$ *and* $b \leq a$*, then* $\{x \wedge b : x \in S\} \in C(b)$*.*

A z-site is a pair (A, C) *with* A *a semilattice and* C *a z-coverage on* A*.*

Definition 4.2.4.2 ***(Transforms covers to joins)***
Let (A, C) *be a z-site, a map* $f : A \to B$ *to a* Z*-frame* B *is said to transform covers to joins if for any* $a \in A$ *and* $S \in C(a)$*,* $f(a) = \vee f(S)$*.*

Proposition 4.2.4.1 *Let* A *be a* Z*-complete semilattice and define:*

$$C_A(a) = \{D : \downarrow D \in Z(A) \text{ and } \vee D = a\}.$$

A is a Z*-frame if and only if* C_A *is a z-coverage on* A*.*

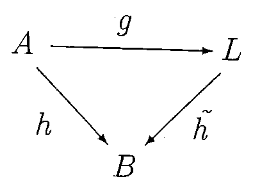
Proof
($\Rightarrow$) Since A is a Z-frame, it is trivially a semilattice. Let $a \in A$.

(i) Let $S \in C_A(a)$. So, $\downarrow S \in Z(A)$ means that S is a Z-set of A. Clearly, $\vee S = a$ implies $S \subseteq \downarrow \vee S = \downarrow S = \downarrow a$.

(ii) Let $S \in C_A(a)$ and $b \leq a$ in A. Then the map $(x \mapsto x \wedge b) : A \to A$ is a semilattice morphism since for any $x_1, x_2 \in A$, $(x_1 \wedge x_2) \wedge b = (x_1 \wedge b) \wedge (x_2 \wedge b)$. So, by (S2), $\downarrow \{x \wedge b : x \in S\} \in Z(A)$ ($\because \downarrow S \in Z(A)$ and $\downarrow f(\downarrow S) = \downarrow f(S) \in Z(A)$ for any semilattice morphism f).
Since S is a Z-set of A and A is a Z-frame, it follows that $\vee\{x \wedge b : x \in S\} = \vee(b \wedge S) = b \wedge \vee S$. But $\vee S = a$ so that $b \wedge \vee S = b \wedge a = b(\because b \leq a)$. Whence, $S \in C_A(b)$.

Therefore, C_A is a z-coverage on A.
($\Leftarrow$) Given that C_A is a z-coverage on A. Since A is a Z-complete semilattice, it suffices to show that for any $x \in A$ and any Z-set D of A, $x \wedge \vee D = \vee\{x \wedge d : d \in D\}$. Let $a = \vee D$. Clearly, $\downarrow D \in Z(A)$. Thus, $D \in C_A(a)$. Since $x \wedge a \leq a$ and $D \in C_A(a)$, it follows that $\{d \wedge (x \wedge a) : d \in D\} \in C_A(x \wedge a)$, i.e. $\vee\{x \wedge (a \wedge d) : d \in D\} = a \wedge x$. But $\vee D = a$ so that $a \wedge d = d$. Therefore, $\vee\{x \wedge d : d \in D\} = x \wedge \vee D$. This completes the proof. □

Definition 4.2.4.3 ***(Freely generated by site)***
A Z*-frame* L *is freely generated by a site* (A, C) *if there is a semilattice morphism* $g : A \to L$ *which transforms covers to joins and is universal in the sense that for any semilattice morphisms* $h : A \to B$ *from* A *to a* Z*-frame, if* h *transforms covers to joins, then there is a unique* Z*-frame morphism* $\tilde{h}: L \to B$ *with* $h = \tilde{h} \circ g$*.*



We are now to show that for each z-site (A, C), there is a Z-frame freely generated by (A, C).

Lemma 4.2.4.1 *Let $f : A \to B$ be a frame morphism from a frame A to a frame B, then for any sub-Z-frame B_1 of B, $f^{-1}(B_1)$ is a sub-Z-frame of A.*

Proof
Let $x, y \in f^{-1}(B_1)$. Then $f(x) \in B_1$, $f(y) \in B_1$. So $f(x \wedge y) = f(x) \wedge f(y) \in B_1$ because B_1 is a sub-Z-frame of B and thus is closed under finite meets. $\therefore x \wedge y \in f^{-1}(B_1)$. So $f^{-1}(B_1)$ is closed under finite meets.
Since $f(1_A) = 1_B \in B_1$, it follows $1_A \in f^{-1}(B_1)$.
Suppose now that $D \subseteq f^{-1}(B_1)$ is any Z-set. Then as $f(D) \subseteq B_1$ and is a Z-set of B by (S2), so $f(\vee D) = \vee f(D) \in B_1$. Hence $\vee D \in f^{-1}(B_1)$. Thus, $f^{-1}(B_1)$ is a sub-Z-frame. □

Lemma 4.2.4.2 *Let B be a subset of a frame A. For each ordinal n, define a subset B^n of A by induction:*

(i) $B^0 = B \cup \{1_A\}$,
$B^{n+1} = \{(\vee D_1) \wedge (\vee D_2) \wedge ... \wedge (\vee D_k) :\downarrow D_i \in Z(B^n), k = 0, 1, ...\}$.

(ii) $B^m = \cup_{n<m} B^n$ *if m is a limit ordinal.*

Then there is an ordinal m' such that $B^{m'+1} = B^{m'}$ and $B^{m'}$ is the smallest sub-Z-frame of A containing B.

Proof
Obviously, $B^n \subseteq B^{n+1}$ so there must be an m' with $B^{m'+1} = B^{m'}$.
Take any two elements of $B^{m'}$, they are of the form: $\wedge_{i=1}^{k_1} D_i$ and $\wedge_{j=1}^{k_2} D_j$. So, their meets is still in $B^{m'}$. Moreover, $1_A \in B^{m'}$ and any join of Z-sets in $B^{m'}$ is still in $B^{m'}$. Therefore, $B^{m'}$ is closed under finite meets, and joins of Z-sets, and contains the top element of A; thus it is a sub-Z-frame of A.
$B^{m'}$ is clearly the smallest sub-Z-frame of A containing B. □

Lemma 4.2.4.3 *Let $f, g : B \to A$ be two Z-frame morphisms. Then for a subset E of B, if B is the smallest sub-Z-frame containing E and $f|_E = g|_E$, then $f = g$.*

Proof
By Lemma(4.2.4.2), we only need to show that $f|_{E^n} = g|_{E^n}$ for each ordinal n. We prove this by ordinal induction.
Firstly, by assumption and that $f(1_B) = 1_A = g(1_B)$, we see that $f|_{E^0} = g|_{E^0}$; suppose $f|_{E^n} = g|_{E^n}$ holds for some ordinal n, then for each $x \in E^{n+1}$, there are some Z-sets $D_i \subseteq E^n, i = 1, 2, ..., k$, with $x = (\vee D_1) \wedge (\vee D_2) \wedge ... \wedge (\vee D_k)$, so

$$\begin{aligned} f(x) &= f((\vee D_1) \wedge ... \wedge (\vee D_k)) \\ &= f(\vee D_1) \wedge ... \wedge f(\vee D_k) \\ &= (\vee f(D_1)) \wedge ... \wedge (\vee f(D_k)) \\ &= (\vee g(D_1)) \wedge ... \wedge (\vee g(D_k)) \\ &= g(\vee D_1) \wedge ... \wedge g(\vee(D_k)) \\ &= g((\vee D_1) \wedge ... \wedge (\vee(D_k))) \\ &= g(x). \end{aligned}$$

So, $f|_{E^{n+1}} = g|_{E^{n+1}}$.
If m is a limit ordinal and assume that $f|_{E^n} = g|_{E^n}$ for each $n < m$, then it follows directly ($\because E^m = \cup_{n<m} E^n$) that $f|_{E^m} = g|_{E^m}$.
Hence we have proven that $f|_{E^n} = g|_{E^n}$ for each ordinal n. $\square$
Now we digress to a theorem due to P.T.Johnstone (see [15]). Let A be a semilattice. By a coverage on A, we mean a function C assigning to each $a \in A$ a set $C(a)$ of subsets of $\downarrow a$, called coverings of a, with the following 'meet-stability' property:

$$S \in C(a) \Rightarrow \{s \wedge b : s \in S\} \in C(b) \text{ for all } b \leq a.$$

By a site we mean a semilattice equipped with a coverage. We say a frame B is freely generated by a site (A, C) if there is a semilattice morphism $f : A \to B$ which transforms covers to joins in the sense that for every $a \in A$ and every $S \in C(a)$ we have

$$f(a) = \vee\{f(s) : s \in S\}$$

and which is universal among such maps, i.e. $f' : A \to B'$ satisfying the same conditions factors uniquely through f by a frame morphism $B \to B'$.
At this juncture, we note that the above-mentioned concepts coincide with the definitions of z-coverage and z-site, when one chooses

$$Z(A) = I(A) = \{\downarrow a : a \in A\}.$$

Definition 4.2.4.4 ***(C-ideal)***
Given a coverage C on a semilattice A, we define a subset I of A to be a C-ideal if it is a lower set and satisfies the following condition:

$$\textit{For each } a \in A,\ S \in C(a), S \subseteq I \Rightarrow a \in I.$$

We write C-$Id(A)$ for the set of all C-ideals of A, ordered by inclusion.

Theorem 4.2.4.1 ***(P.T.Johnstone)***
For any site (A, C), C-$Id(A)$ is a frame, and is freely generated by (A, C).

We shall omit the proof here but would like to mention that the map $i : A \to C\text{-}Id(A)$, defined by composition of the following arrows:

$$A \xrightarrow{\ \downarrow\ } \downarrow A \xrightarrow{\ j\ } C\text{-}Id(A)$$

which consists of the map:

$$\downarrow : A \to \downarrow A,\ \downarrow : a \mapsto \downarrow a.$$

where $\downarrow A$ is the collection of lower sets of A, ordered by inclusion and thus a frame, and also the map:

$$j : \downarrow A \to C\text{-}Id(A),\ j : S \mapsto \cap\{I \in C\text{-}Id(A) : I \supseteq S\}.$$

transforms covers to joins.
Denote $\breve{A}$ to be the smallest sub-Z-frame of $C\text{-}Id(A)$ containing $i(A)$, then we have the following result:

Theorem 4.2.4.2 ***(D.Zhao)***
For any z-site (A, C), there is a Z-frame freely generated by (A, C) and the Z-frame is just the $\breve{A}$ defined above.

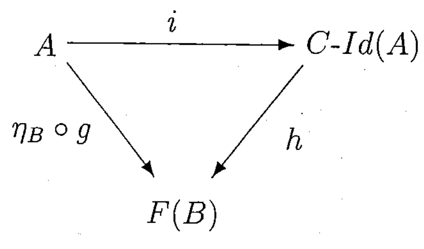
Proof
Denote by $j : A \to \breve{A}$ the restriction to the codomain of i. Given any $a \in A, S \in C(a)$m we have $i(a) = \vee i(S)$ as i is the universal semilattice morphism

$$i : A \to C\text{-}Id(A)$$

that transforms covers to joins.
By (S2), $\downarrow i(S) \in Z(C\text{-}Id(A))$ for each $S \in C(a)$ since $S \in Z(A)$. Now we see that $i(S)$ is a Z-set contained in $i(A)$. So, $\vee i(S) \in \breve{A}$ and hence $j(a) = i(a) = \vee i(S) = \vee j(S)$ holds in $\breve{A}$. thus, j transforms covers to joins and is a semilattice morphism.
Now suppose $g : A \to B$ is a semilattice morphism to a Z-frame B and that g transforms covers to joins, let $F(B) = C_z(B)$ (i.e. the reflection of B in **Frm**, i.e. the set of all z-closed subsets of B).
The unit morphism

$$\begin{aligned} \eta_B \ &: B \to F(B) \\ &: x \mapsto \downarrow x, \qquad \forall x \in B. \end{aligned}$$

is an order embedding.
η_B preserves the joins of Z-sets and so $\eta_B(B)$ is a sub-Z-frame of $F(B)$. Now for each $a \in A, S \in C(a), S$ is a Z-set by assumption, so $g(S)$ is a Z-set of B by (S2), and $g(a) = \vee_B g(S)$; hence $\eta_B(g(a)) = \eta_B(\vee_B g(S)) = \vee \eta_B(g(S))$. This shows that $\eta_B \circ g : A \to F(B)$ transforms covers to joins.
By the universality of i, there is a unique frame morphism $h : C\text{-}Id(A) \to F(B)$ such that $h \circ i = \eta_B \circ g$.



Now $h^{-1}(\eta_B(B))$ is a sub-Z-frame of $C\text{-}Id(A)$ and contains $i(A)$. This is because $\eta_B(B)$ is a sub-Z-frame of $F(B)$, by Lemma(4.2.4.1), $h^{-1}(\eta_B(B))$ is a sub-Z-frame of $C\text{-}Id(A)$. Moreover, $hi(A) = \eta_B \circ g(A) =\subseteq \eta_B(B)(\because g(A) \subseteq B), \therefore i(A) \subseteq h^{-1}(\eta_B(B))$. So, $\breve{A}$ is contained in $h^{-1}(\eta_B(B))$.
As η_B is an order embedding and preserves finite meets and joins of Z-sets, the map $h' : \breve{A} \to F(B)$, obtained by restricting h to domain $\breve{A}$, factors the embedding η_B through

a map $\breve{h} : \breve{A} \to B$. More precisely, let $x \in \breve{A}$. Since $\breve{A}$ is the smallest Z-frame containing $i(A)$ which is in turn contained in $h^{-1}(\eta_B(B))$, $h'(x) = h(x) \in \eta_B(B)$. Now define $\breve{h} : \breve{A} \to B$ by $\breve{h}\,(x) = \eta_B^{-1}(h'(x))$. Then it is trivial that $h' = \eta_B \circ \breve{h}$.
Moreover, h preserves finite meets and joins of Z-sets. Hence we have a Z-frame morphism $\breve{h} : \breve{A} \to B$ with $g = \breve{h} \circ j$.
Suppose there is another Z-frame morphism $xi : \breve{A} \to B$ with $\xi \circ j = g$. Then, $\xi|_{i(A)}(i(a)) = g(a) = \breve{h}\,|_{i(A)}(i(a))$. Invoking Lemma(4.2.4.3), $\xi = \breve{h}$. It follows that $\breve{h}$ is the unique Z-frame morphism with $g = \breve{h} \circ j$. And the theorem is thus proved. □
The following three statements are equivalent:

(1) A Z-complete semilattice A is a Z-frame.

(2) The function C_A which assigns to each $a \in A$ the set $C_A(a)$ of all Z-sets with sup a defines a z-coverage on A.

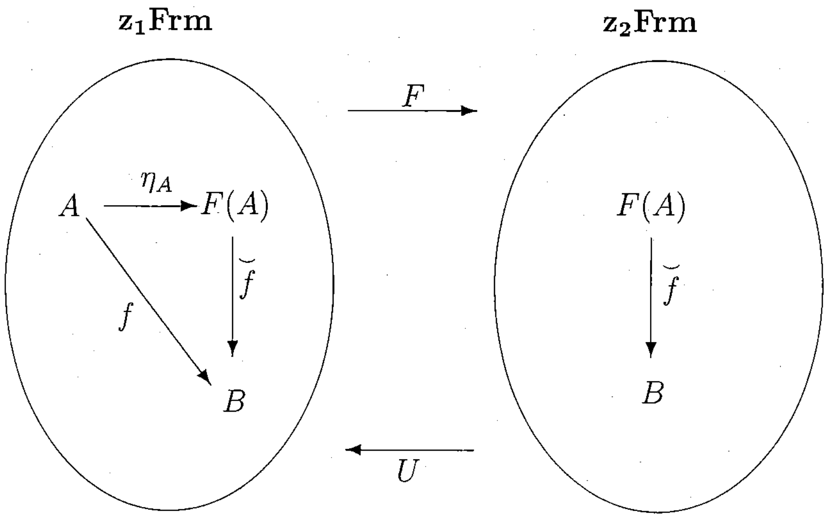
(3) (A, C_A) is a z-site.

Now given two systems of sets Z_1 and Z_2 with $Z_1 \leq Z_2$. If A is a z_1-frame, then we have a z_1-cite. (A, C_A) is also a z_2-site by definition since $Z_1 \leq Z_2$. By Theorem(4.2.4.2), there is a z_2-frame freely generated by (A, C_A). Suppose

$$\eta_B : A \to \breve{A}$$

is the universal map which transforms covers to joins. By its definition η_A is a semilattice morphism and given any $D \in Z_1(A)$ with $\vee D = a$, $\eta_A(\vee D) = \eta_A(a) = \vee \eta_A(D)$ since η_A transforms covers to joins. So, η_A also preserves the join of Z-sets. Thus, η_A is a z_1-frame morphism.
On the other hand, by its universality, for any z_1-frame morphism $f : A \to B$ to a z_2-frame B, there is a unique z_2-frame morphism $\breve{f} : \breve{A} \to B$ such that $f = \breve{f} \circ \eta_A$. This argument proves the following main result:

Theorem 4.2.4.3 *If $Z_1 \leq Z_2$ are two systems of sets on* **Slat**, *then there is a functor F :* $\mathbf{z_1Frm} \to \mathbf{z_2Frm}$ *which is left adjoint to the forgetful functor U :* $\mathbf{z_2Frm} \to \mathbf{z_1Frm}$.



4.3 E-projectivity of z-frames

Characterizing the projectives and injectives have been frequent topics of discussion in the mathematics. Sometimes, the concept of projective (dually injective) may be too restricted when applied to certain categories. For instance, the only projective frame is the **2**-chain. The quest is to look for a way to extend this concept to include more objects (of course, in a meaningful way). In this section, we shall make use of the adjunction derived in Theorem(4.2.4.3) to generalise projectivity (dually injectivity).

4.3.1 Definitions and preliminary results

Definition 4.3.1.1 *(Stable)*
A Z-continuous semilattice A is said to be stable if the Z-below relation $\ll_z$ is multiple in the sense that if $x \ll_z a$ and $x \ll_z b$, then $x \ll_z a \wedge b$ and $1_A \ll_z 1_A$, where 1_A is the top element of A.

Definition 4.3.1.2 *(z-compact)*
An element a of A is called z-compact if $a \ll_z a$.

Definition 4.3.1.3 *(z-algebraic semilattice)*
A z-algebraic semilattice A is a Z-complete semilattice such that for each $a \in A$,

$$\downarrow \{x \in A : x \ll_z x, x \leq a\} \in Z(A) \text{ and}$$
$$a = \vee \downarrow \{x \in A : x \ll_z x, x \leq a\}.$$

Definition 4.3.1.4 *(Coherent z-frame)*
A Z-frame is said to be coherent if it is Z-algebraic and is stable as Z-continuous semilattice.

It can be proven that a coherent Z-frame is exactly a Z-algebraic semilattice such that it is stable as Z-continuous semilattice, or equivalently, the Z-compact elements form a subsemilattice.

Proposition 4.3.1.1 *Every Z-algebraic semilattice is Z-continuous.*

Proof
Let A be a Z-algebraic semilattice and $a \in A$ be fixed. It suffices to establish the fact that

$$\downarrow \{x \in A : x \ll_z x, x \leq a\} = \omega_z(a) \text{ for any } a \in A.$$

Let $u \in A$ such that $u \leq x$ for some $x \in A$ and $x \ll_z x, x \leq a$. Clearly, $u \ll_z a$ by the transivity of $\ll_z$. So $u \in \omega(a)$ and "$\subseteq$ " is proven.
Now we show the reverse inclusion. Let $u \in \omega(a)$. Note that since A is a Z-algebraic, the set $\downarrow \{x \in A : x \ll_z x, x \leq a\} \in Z(A)$. By the definition of $\ll_z$, since $\vee \downarrow \{x \in A : x \ll_z x, x \leq a\} = a \geq a$ and $u \ll_z a$, we have $u \in \downarrow \{x \in A : x \ll_z x, x \leq a\}$. Thus the reverse inclusion is established.
That $\vee\omega(a) = a$ follows immediately. □

Lemma 4.3.1.1 *For each semilattice S, $Z(S)$ is a coherent Z-frame.*

Proof
For each $x \in S, \downarrow x \in Z(S)$.
Claim: $\downarrow x \ll_z \downarrow x$ in $(Z(S), \subseteq)$.
To prove this claim, let $\mathcal{D}$ be a Z-set of $Z(S)$ with $\vee\mathcal{D} \geq \downarrow x$. Then, we may write $\mathcal{D} = \{D_i : i \in I\}$. (Note that by (S4), $\cup\mathcal{D} = \vee\mathcal{D} \in Z(S)$. Since $\downarrow x \subseteq \cup\mathcal{D}$, this implies that $x \in D_i$ for some $i \in I$. But since D_i is a lower set, $\downarrow x \subseteq D_i$ for some $i \in I$. In turn, this means that $\downarrow x \in \mathcal{D}$. Hence $\downarrow x \ll_z \downarrow x$. Whence, $\downarrow x$ is z-compact in $(Z(S), \subseteq)$.
Given any $D \in Z(S)$, $D = \cup\{\downarrow x : x \in D\}$. Observe that $\{\downarrow x : x \in D\}$ is the image of the Z-set D under the semilattice morphism

$$\eta_s = \downarrow : S \to Z(S)$$

and by (S2), it follows that $\{\downarrow x : x \in D\}$ is a Z-set of $Z(S)$. From this, it is easily seen that the set

$$\{B \in Z(S) : B \ll_z B, B \leq D\}$$

is a Z-set and

$$D = \vee\{B \in Z(S) : B \ll_z B, B \leq D\}.$$

Hence $Z(S)$ is algebraic. We now prove that $Z(S)$ is stable.
Suppose $E \ll_z D$ and $E \ll_z F$ hold in Z(S). Since $\{\downarrow x : x \in D\}$ is a Z-set of $Z(S)$ whose sup (which is the union) is equal to D, applying the definition of $\ll_z$, we conclude that $E \in \downarrow \{\downarrow x : x \in D\}$ in the context of $(Z(S), \subseteq)$. Hence $E \subseteq \downarrow x$ for some $x \in D$. Likewise, for the relation $E \ll_z F$, there is an element $y \in F$ with $E \subseteq \downarrow y$. So $E \subseteq \downarrow x \cap \downarrow y = \downarrow (x \wedge y)$. But $x \wedge y \in D \wedge F (= D \cap F)$. $\therefore E \ll_z D \wedge F$.
Let 1_S be the top element of S. Then $\downarrow 1_S = S$. Since $\downarrow x \ll_z \downarrow x$ for each $x \in S$, we have $\downarrow 1_S \ll_z \downarrow 1_S$, i.e. $S \ll_z S$. Note that S is the top element of $(Z(S), \subseteq)$. Hence $Z(S)$ is a coherent Z-frame. □

Definition 4.3.1.5 ***(Retract of an object)***
In a category **C**, *an object A is called the retract of the object B if there are morphisms $f : A \to B$ and $r : B \to A$ in* **C** *such that*

$$r \circ f = id_A.$$

Let **ZComSlat** denote the category of all Z-complete semilattices and Z-frame morphisms. Thus **ZFrm** is a full subcategory of **ZComSlat**.

Lemma 4.3.1.2 *In the category* **ZComSlat**, *the following notions are stable under retraction:*

(i) Being a Z-frame,

(ii) Z-continuity, and

(iii) stable Z-continuity.

Proof

(i) Let A be a Z-complete semilattice and B Z-frame. Let $f : A \to B$ be a Z-frame morphism between them. Suppose thee is a morphism $r : B \to A$ such that $r \circ f = 1_A$. For any $a \in A, D \in Z(A)$,

$$\begin{aligned} a \wedge \vee\{x : x \in D\} &= rf(a) \wedge \vee\{rf(x) : x \in D\} \\ &= rf(a) \wedge r(\vee\{f(x) : x \in D\}) \\ &= r(f(a) \wedge \vee\{f(x) : x \in D\}) \\ &= r(\vee\{f(a) \wedge f(x) : x \in D\}) \\ &= \vee\{rf(a \wedge x) : x \in D\} \\ &= \vee\{a \wedge x : x \in D\} \end{aligned}$$

(ii) Suppose that A is a Z-complete semilattice that is a retract of a Z-continuous Z-complete semilattice L. Then there are Z-frame morphisms $r : L \to A$ and $f : A \to L$ such that $r \circ f = 1_A$. Let $a \in A$. L is Z-continuous, so

$$f(a) = \vee\{x \in L : x \ll_z f(a)\}$$

and $\{x \in L : x \ll_z f(a)\} \in Z(L)$.
r is a Z-frame morphism, so

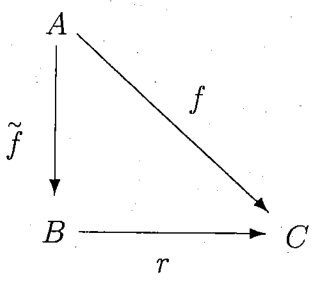
$$\begin{aligned} a &= rf(a) \\ &= r(\vee\{x \in L : x \ll_z f(a)\}) \\ &= \vee\{r(x) : x \in L, x \ll_z f(a)\}. \end{aligned}$$

Now, $\{r(x) : x \ll_z f(a)\}$ is a Z-set of which the join equals to a. We now only need to verify that for each $x \ll_z f(a)$, $r(x) \ll_z a$. Let D be an arbitrary Z-set of A with $\vee D \geq a$. Then $\{f(y) : y \in D\}$ is a Z-set of L and $\vee\{f(y) : y \in D\} = f(\vee D) \geq f(a)$. So there exists a $d \in D$ with $f(d) \geq x$ and hence $d = rf(d) \geq r(x)$. This shows that $r(x) \ll_z a$. It then follows immediately that $a = \vee\{y \in A : y \ll_z a\}$ and $\{y \in A : y \ll_z a\}$ is a Z-set of A. Thus, A is Z-continuous.

(iii) Suppose now that L is also stable. We show that A is stable as well. First, $1_L \ll_z 1_L = f(1_A)$. From the above discussion, $1_A = r(1_L) \ll_z 1_A$, i.e. 1_A is z-compact. If $y, a, b \in A$ such that $y \ll_z a, y \ll_z b$, then there are elements $x_1 \ll_z f(a)$ and $x \ll_z f(b)$ such that $y \leq r(x_1)$ and $y \leq r(x_2)$. But L is stable, so $x_1 \wedge x_2 \ll_z f(a) \wedge f(b) = f(a \wedge b)$. This then indicates that $r(x_1 \wedge x_2) \ll_z a \wedge b$. As $y \leq r(x_1) \wedge r(x_2) = r(x_1 \wedge x_2)$, it follows that $y \ll_z a \wedge b$. Hence A is stable. □

4.3.2 E-projective Z-frames

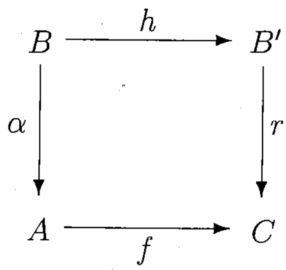
Definition 4.3.2.1 *(E-projective)*
Let **C** *be a category and* **E** *be a collection of morphism in* **C**. *An object A of* **C** *is called a* **E**-*projective object if for any morphism $r : B \to C$ in* **E** *and $f : A \to C$ in* **C**, *there exists a morphism $\tilde{f}$ in* **C** *such that $f = r \circ \tilde{f}$.*



Remark
If **E** is a collection of all epimorphisms , then **E**-projective objects are exactly the so-called projective objects of **C**.

Proposition 4.3.2.1 *It is well known that the retracts of an* **E***-projective object are* **E***-projective.*

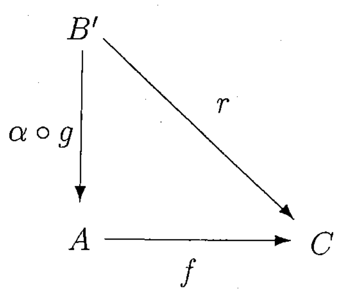
Proof
Let B' be a retract of B. So there are morphisms $g : B' \to B$ and $h : B \to B'$ so that $h \circ g = 1_{B'}$.
Suppose B is **E**-projective. Given a morphism $r : B' \to C$ to an object C and any **E**-morphism $f : A \to C$, we may form the composition $r \circ h : B \to C$ which is a **C**-morphism. Since B is **E**-projective, there exists a morphism $\alpha : B \to A$ such that the following diagram commutes:



i.e. $f \circ \alpha = r \circ h$.
Now apply g on the right,

$$f \circ (\alpha \circ g) = r$$

since $h \circ g = 1_{B'}$. It follows that $\alpha \circ g : B' \to A$ is the required morphism that completes the following commutative diagram:



□

If $G : \mathbf{C} \to \mathbf{D}$ and $F : \mathbf{D} \to \mathbf{C}$ are functors such that F is left adjoint to G, with counit (also known as back-adjunction) $\epsilon : F \circ G \to id_{\mathbf{C}}$, then it is natural to consider the $\mathbf{E}$-projective objects of $\mathbf{C}$ for the collection $\mathbf{E}$ of all $f : A \to B$ such that $G(f)$ has a section, that is, a right inverse. The basic result concerning these is the following:

Lemma 4.3.2.1 *For any $A \in \mathbf{C}$, the following are equivalent:*

(1) A is $\mathbf{E}$-projective.

(2) $\epsilon_A : F \circ GA \to A$ has a right inverse.

(3) A is a retract of some FX.

Proof

(1) $\Rightarrow$ (2):

We first show that the counit morphism ϵ_A is an $\mathbf{E}$-morphism. Since F is left adjoint to G, there is a natural bijection

$$\theta_{DC} : \mathrm{Hom}_{\mathbf{D}}(D, GC) \cong \mathrm{Hom}_{\mathbf{C}}(FD, C).$$

Let $A \in \mathbf{C}$ be fixed. Then $GFG(A) \in \mathbf{D}$.

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathbf{C}}(FG(A), FG(A)) & \xrightarrow{\mathrm{Hom}_{\mathbf{C}}(FG(A), \epsilon_A)} & \mathrm{Hom}_{\mathbf{C}}(FG(A), A) \\
\Big\downarrow{\scriptstyle \theta_{G(A),FG(A)}} & & \Big\downarrow{\scriptstyle \theta_{G(A),A}} \\
\mathrm{Hom}_{\mathbf{D}}(G(A), GFG(A)) & \xrightarrow[\mathrm{Hom}_{\mathbf{D}}(G(A), G(\epsilon_A))]{} & \mathrm{Hom}_{\mathbf{D}}(G(A), G(A))
\end{array}$$

Via adjointness, we obtain the unit morphism $\eta_{G(A)}$ in the hom-set $\mathrm{Hom}_{\mathbf{D}}(G(A), GFG(A))$. From the above commutative rectangle, one deduces that $G(\epsilon_A) \circ \eta_{G(A)} = 1_{G(A)}$ in $\mathbf{D}$. So ϵ_A is an $\mathbf{E}$-morphism. Since A is $\mathbf{E}$-projective, there is a morphism $f : A \to FG(A)$ so that the following diagram commutes:

$$\begin{array}{ccc}
A & & \\
\Big\downarrow{\scriptstyle f} & \searrow^{1_A} & \\
FG(A) & \xrightarrow[\epsilon_A]{} & A
\end{array}$$

Then f is the right inverse of the counit morphism ϵ_A.
(2) $\Rightarrow$ (3):
If $\epsilon_A : FG(A) \to A$ has a right inverse, say $r : A \to FG(A)$, i.e. $\epsilon_A \circ r = 1_A$. Then clearly, A is the retract of $F(GA)$, where X may be set as GA. So this direction is trivial.
(3) $\Rightarrow$ (1):
We shall show that FX is **E**-projective.
Given objects and an **E**-morphism between them $g : B \to C$ and $f : FX \to C$ in **C**. Let $k : G(C) \to G(B)$ be a section of $G(g)$, i.e. $G(g) \circ k = 1_{G(C)}$. Put
$h = k \circ G(f) \circ \eta_X : X \to G(B)$. Clearly, h is a morphism in **D**. By the adjunction between the categories, there exists a unique **C**-morphism $h' : FX \to B$ such that $G(h') \circ \eta_X = h$.
But

$$\begin{aligned} G(g) \circ G(h') \circ \eta_X &= G(g) \circ h \\ &= G(g) \circ k \circ G(f) \circ \eta_X \\ &= G(f) \circ \eta_X \end{aligned}$$

Thus, by adjunction again, we deduce that $g \circ h' = f$.
Thus, FX is **E**-projective. But by Proposition(4.3.2.1), the retracts of **E**-projectives are **E**-projective. So A is **E**-projective. □

Remark

For the forgetful functor U : **ZFrm** $\to$ **Slat** and its left adjoint Z : **Slat** $\to$ **ZFrm**, **E** is the collection of all the Z-frame morphisms which have a section in **Slat**, and the counit morphism is given by the map $\vee : Z(A) \to A$. For this, we now have the main result:

Theorem 4.3.2.1 *(D.Zhao [23])*
A Z-frame A is **E**-*projective if and only if it is stably Z-continuous.*

Proof

By Lemma(4.3.2.1), it is enough to establish that $\vee : Z(A) \to A$ has a right inverse in $zFrm$ if and only if A is stably Z-continuous.
($\Rightarrow$) From Lemma(4.3.1.1), for each semilattice A, $Z(A)$ is a coherent Z-frame, i.e. it is stably Z-continuous. $\vee : Z(A) \to A$ has a right inverse means A is a retract of $Z(A)$. But by Lemma(4.3.1.2), we are done.
($\Leftarrow$) We claim that

$$\omega : A \to Z(A), \omega(A) = \{x \in A : x \ll_z a\}, a \in A.$$

defines the desired right inverse of $\vee : Z(A) \to A$.
Since A is Z-continuous, $\omega(a) \in Z(A)$ and $\vee\omega(a) = a$ for each $a \in A$.
Take any $x, y \in A$.

$$\omega(x \wedge y) = \{w \in A : w \ll_z x \wedge y\} \subseteq \omega(x) \cap \omega(y).$$

But the reverse follows from $w \ll_z x$ and $w \ll_z y$, which together imply that $w \ll_z x \wedge y$ since A is stable. Therefore, ω is a meet-semilattice morphism.
Finally, for any $D \in Z(A)$, if $x \ll_z \vee D$, then $x \ll_z y \ll_z \vee D$ for some y by the interpolation property of the $\ll_z$ relation (see Proposition(4.1.2.1). Hence $x \in \omega(y)$ and $y \in D$, and therefore $x \in \cup\omega(D)$. It follows that $\omega(\vee D) \subseteq \cup\omega(D) = \vee\omega(D)$, the non-trivial part of the identity

$$\omega(\vee D) = \vee\omega(D).$$

So ω preserves the joins of Z-sets. Hence ω is a Z-frame homomorphism, and is the right inverse of $\vee$. □

4.3.3 Some applications

The above theorem easily applies to many special cases.

(1) For each semilattice S, take $Z(S) = \mathcal{D}(S)$, the set of all down sets of S. Then, A is a Z-frame if and only if it is a frame, i.e. iff A is a complete lattice such that for any $a \in A$ and any non-empty set $X \subseteq A$, the following equation holds:

$$a \wedge \vee X = \vee\{a \wedge x : x \in X\}$$

By Raney's characterisation, for this Z, A is Z-continuous if and only if it is a completely distributive lattice. A frame morphism $f : A \to B$ is a semilattice morphism that preserves joins of arbitrary sets. Hence every frame morphism $f : A \to B$ has a right adjoint $f_* : B \to A$, and f_* is a section of f if and only if f is surjective. **E** is the collection of all surjective frame morphisms, which in turn are exactly the regular epimorphisms in **Frm**. So **E**-projective frames are exactly the regular-projective frames. Theorem(4.3.2.1) then says that the regular-projective frames are exactly the stably completely distributive lattices. This is the main result obtained by Banaschewski and Niefield in [4] where completely distributive lattices are called supercontinuous lattices.

(2) For each semilattice S, let $Z(S) = Id(S)$, the collection of all ideals of S. Here $D \in Id(S)$ if and only if it is a down-set and up-directed. Then Z is a set system on **Slat**. In this case, a Z-frame A is a semilattice in which every up-directed set has a join and the equation

$$a \wedge \vee D = \vee\{a \wedge x : x \in D\}$$

holds for any $a \in A$ and $D \in Id(A)$. thus Z-frames are exactly the meet-continuous semilattices [10], or preframes that have been studied by Banaschewski and Vickers [16]. Now a Z-continuous semilattice is exactly a continuous semilattice in the sense of [10]. Hence Theorem(4.3.2.1) implies that a preframe is **E**-projective if and only if it is a stably continuous semilattice.

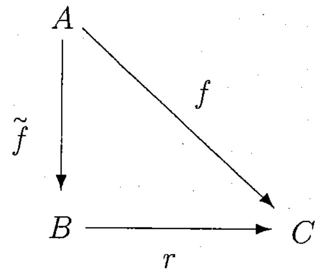
(3) For each semilattice S, let $Z(S) = \{\downarrow E : E \text{ is a finite subset of } S\}$. Z defines a subset system on **Slat**. A Z-complete semilattice is just a lattice. A Z-frame now is exactly a distributive lattice. A Z-continuous semilattice A is a lattice which satisfies the property that for each element $a \in A$, there is a finite set $D = \{d_i : i = 1, 2, ..., n\}$ with $\vee D = a$ and for each d_i, if $x \vee y \geq a$, then $x \geq d_i$ or $y \geq d_i$.

(4) For each semilattice S, define $Z(S) = \{\downarrow E : E \subseteq S \text{ is a countable set}\}$. then a Z-frame is exactly a σ-frame which has been studied by many authors especially Banaschewski. Theorem(4.3.2.1) here characterises the **E**-projective σ-frames.

(5) For the smallest set system $Z(S) = \mathcal{P}(S)$, Z-frames are exactly the semilattices, and $\ll_z = \leq$. So, in this case every Z-frame is **E**-projective.

4.4 Relative projectivity

By Theorem(4.2.4.3), there exists an adjunction between the categories $\mathbf{Z_1Frm}$ and $\mathbf{Z_2Frm}$, where $Z_1 \leq Z_2$. So, we may apply the concept of **E**-projective in this special adjunction. Let **E** be the class of $\mathbf{Z_2Frm}$ morphisms which have a retract in $\mathbf{Z_1Frm}$. We say that a Z_2-frame is relatively Z_1-projective if for any morphism $r : B \to C$ in **E** and $f : A \to C$ in $\mathbf{Z_2Frm}$, there exists a morphism $\tilde{f}$ in $\mathbf{Z_2Frm}$ such that $f = r o \tilde{f}$.



The question is:

How can one characterize the relatively Z_1-projective Z_2-frames?

The first difficulty is to deal with very complicated left-adjoint functor from $\mathbf{Z_1Frm}$ to $\mathbf{Z_2Frm}$. This functor is defined via sites and characterizing the relatively Z_1-projectives can really be an uphill task. So the natural direction is to simplify the problem by looking at the adjunction between the categories **PreFrm** (preframes) and **Frm** (frames). Here, we have the left adjoint functor $\mathbf{C} : \mathbf{PreFrm} \to \mathbf{Frm}$, $\mathbf{C}(P) = C(P)$ where $C(P)$ denotes the collection of all the Scott-closed subsets of P. Before one characterizes the relative projectives, one must first answer a very difficult question:

For a given preframe P, how can one characterize the frame $C(P)$?

Only after one characterizes $C(P)$ (as some properties) can one start to ask whether such properties are stable under retraction.
One can also continue to simplify the problem by asking:

For a given dcpo P, how can one characterize the complete lattice $C(P)$?

In an attempt to answer these open problems, new definitions and concepts need to be discovered. This leads us to the next chapter.

Chapter 5

On Scott-closed set lattices

5.1 Introduction

A subset A of a poset P is called Scott closed if (i) $A = \downarrow A$ where $\downarrow A := \{x \in P : x \leq a$ for some $a \in A\}$; and (ii) for any up-directed set $D \subseteq A$, if $\vee D$ exists then $\vee D \in A$. Let $\mathbf{C}(P)$ denote the set of all Scott closed subsets of P. Then $\mathbf{C}(P)$ is closed under arbitrary intersection, hence it is a complete lattice with respect to the inclusion relation $\subseteq$. Notice that $\mathbf{C}(P)$ is also closed under finite union.
A general question arising here is : Given a class of dcpos $\mathbf{P}$, can we find a inner characterization of those complete lattices which are isomorphic to $\mathbf{C}(P)$ for some poset P.
A complete lattice L is called a closed-set lattice if it is isomorphic to the lattice of all closed sets of some topological space.
An element r of a lattice L is called irreducible (or join-prime) if $r \leq a \vee b$ implies $r \leq a$ or $r \leq b$.
It is well known that a complete lattice L is a closed set lattice if and only if the $\vee$-prime elements of L are join dense in L(see [10]).
A poset P is called a dcpo (brief of direct complete poset) if every directed subset of P has a supremum in P. Note that the empty set is not directed, thus a dcpo need not have a bottom element. Let P be a dcpo, the way-below relation $\ll$ on P is defined by $a \ll b$ for $a, b \in P$ iff for any directed set D, $b \leq \vee D$ implies $a \leq d$ for some $d \in D$. A dcpo P is called a continuous poset if for any $a \in P$, $W(a) = \{x \in P : x \ll a\}$ is a directed set and $a = \vee W(a)$. A continuous poset which is also a complete lattice is called a continuous lattice. Note that for any directed set D of a dcpo P, $W(\vee D) = \cup\{W(d) : d \in D\}$ and $\cup\{W(d) : d \in D\}$ is also a directed set if each of the $W(d)$ is directed.
Hoffmann and Lawson actually prove that a complete lattice is isomorphic $\sigma(P)$ for a continuous P if and only if L is a completely distributive lattice([10]). A complete lattice L is completely distributive if for each $a \in L$, $a = \vee\{x \in L : x \lhd a\}$. Here $x \lhd a$ if for any subset $D \subseteq L$, $a \leq \vee D$ implies $x \leq d$ for some $d \in D$. Since L is completely distributive if and only if its dual L^{op} is completely distributive, it follows that Scott closed set lattices of continuous posets are exactly completely distributive lattices.
In [2], Banaschewski characterizes $\sigma(L)$, where L is a continuous lattice, as the multiplicatively supercontinuous lattice.

One natural question is: How do we characterize $\sigma^{op}(L)$?
If P is not continuous, there is still no property identified which is valid for all $\mathbf{C}(P)$, let alone characterizing them. The main difficulty lies in recovering the poset P from $\mathbf{C}(P)$. In case where P is continuous one can show that P is isomorphic to the poset of all join-prime elements.
In this chapter we introduce a new auxiliary relation which can be used to formulate a characterization of the lattice $\sigma^{op}(P)$ with P a dcpo. When P is a complete lattice we can recover P as the subset of all Scott compact elements of $\sigma^{op}(P)$. Next we shall explore some categorical aspects entailed by this new structure.

5.2 Smooth lattices

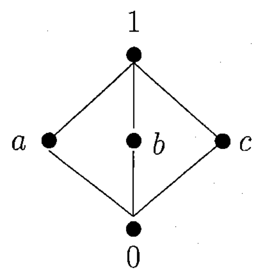
Given a complete lattice L, we are going to define a new auxiliary relation on it. This relation is crucial for us to characterize lattices $\sigma^{op}(P)$.

Definition 5.2.0.1 *Let L be a complete lattice and $x, y \in L$. The element x is Scott-below y, denoted by $x \prec_S y$, if for each Scott-closed set C of P, $\vee C \geq y$ implies $x \in C$.*

Examples

1. Consider the unit interval $I = [0, 1]$ with the usual order $\leq$. It is easy to see that $x \prec_S y$ if and only if $x \leq y$. In particular, $x \prec_S x$ for all $x \in [0, 1]$. However, the only compact element (i.e. $y \ll y$) of $[0, 1]$ is 0. Hence $x \prec_S y$ does not imply $x \ll y$.

2. Consider the lattice $A = \mathbf{M_3}$.



Since A is a finite lattice, for any two elements x and y in A, $x \ll y$ if and only if $x \leq y$. The set $C = \{0, b, c\}$ is Scott-closed whose supremum equal to 1. Although $\vee C \geq a$, a is not in C. So, $a \not\prec_S a$. So the $a \ll a$ does not imply $a \prec_S a$.

However, it is trivial to see that $x \lhd y$ implies $x \prec_S y$ (of course also $x \ll y$).

Proposition 5.2.0.1 *In a complete lattice L one has the following statements for all $u, v, x, y \in L$:*

(i) $x \prec_S y$ *implies* $x \leq y$.

(ii) $u \leq x \prec_S y \leq v$ *implies* $u \prec_S v$.

(iii) $0 \prec_S x$ *for all* $x \in P$.

Proof

(i) Choose $C = \downarrow y$. $\vee C \geq y$ implies that $x \in C = \downarrow y$, i.e. $x \leq y$.

(ii) This can be proven readily using the fact that any Scott-closed set is a lower set and that $\leq$ is transitive.

(iii) 0, if it exists, is in every Scott-closed subset of A. □

Proposition 5.2.0.2 *Let $\{d_i : i \in I\}$ be an up-directed subset of A and $d_i \prec_S d$ for each $i \in I$, then*

$$\vee d_i \prec_S d.$$

Proof
Let $F \in \sigma^{op}(A)$ such that $\vee D \geq d$. Then for each $i \in I$, as $d_i \prec_S d$, so $d_i \in F$. Thus $\{d_i : i \in I\} \subseteq F$, and so $\vee\{d_i : i \in I\} \in F$ because F is Scott closed and $\{d_i : i \in I\}$ is up-directed. □
Notice that neither $\lhd$ nor $\ll$ has this property.

Corollary 5.2.0.1 *For any element a of a complete lattice L, the set*

$$\{x \in L : x \prec_S a\}$$

is a Scott-closed subset of L.

Proof
First note that $\{x \in L : x \prec_S a\}$ is down-closed. Next, take any directed subset $\{x_i : i \in I\}$ of $\{x \in L : x \prec_S a\}$. Then for each $i \in I$, $x_i \prec_S a$. By Proposition(5.2.0.2), $\vee x_i \prec_S a$ so that $\vee_{i \in I} x_i \in \{x \in L : x \prec_S a\}$. □

Using the relation $\prec_S$ we can define a new class of complete lattices.

Definition 5.2.0.2 *A complete lattice L is called smooth if for each $a \in P$*

$$a = \vee\{x \in L : x \prec_S a\}.$$

The following proposition can be proved in a similar way as for continuous lattices.

Proposition 5.2.0.3 *For a complete lattice L, the following statements are equivalent:*

(i) L is smooth.

(ii) For any collection $\{F_i : i \in I\}$ of Scott closed subsets F_i's of L, the following equation holds:

$$\bigwedge_{i \in I} \vee F_i = \bigvee \bigcap_{i \in I} F_i.$$

Proof

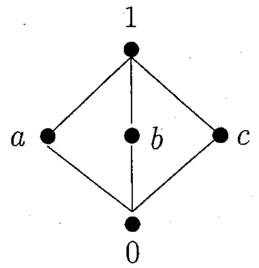
(i) $\Rightarrow$ (ii): For convenience let lhs denote the left-hand side of (iii) and rhs be the right-hand side. It is obvious that for every complete lattice $lhs \geq rhs$. Assuming that L is smooth, all we have to do to prove the reverse inequality is to show that whenever $t \prec_S lhs$, then $t \leq rhs$. Suppose then that $t \prec_S lhs$; we conclude that $t \prec_S \vee F_i$ for all $i \in I$. This implies that $t \in \cap_{i \in I} F_i$. And so $t \leq rhs$ must follow.

(ii) $\Rightarrow$ (i): Let a be a fixed element in L. Consider the collection of Scott-closed sets $\{F_i : i \in I\}$ where $\vee F_i \geq a$ for each $i \in I$. It is immediate that $\{x \in L : x \prec_S a\} = \cap_{i \in I} F_i$. But this implies that $\vee\{x \in L : x \prec_S a\} = \vee \cap_{i \in I} F_i = \wedge_{i \in I} \vee F_i \geq a$. □

Examples

1. Since for any two elements a and b, $a \lhd b$ implies $a \prec_S b$, so every completely distributive lattice is smooth.

2. The finite lattice $\mathbf{M}_3$ (see diagram below) is continuous but not smooth.



Note that the set $C = \{0, a, b\}$ is Scott-closed. Although $\vee C = 1 \geq c$, $c \notin C$. Thus, $c \not\prec_S c$ and it follows that $\mathbf{M}_3$ is not smooth.

In next section we shall give an example of smooth lattice which is not continuous. However by the following result every smooth lattice is distributive.

Proposition 5.2.0.4 *Let L be a smooth lattice. Then for any collection $\{D_i : i \in I\}$ of finite subsets D_i's of L the following equation hold:*

$$\bigwedge_{i \in I} \vee D_i = \bigvee_{f \in \Pi_{i \in I} D_i} \wedge_{i \in I} f(i).$$

Proof

It is enough to show the left side is less than or equal to the the right side.
Let $a = \bigwedge_{i \in I} \vee D_i$ and $x \prec_S a$. For each $i \in I$, the set $\downarrow D_i$ is a Scott-closed set and $x \prec_S a \leq \vee D_i$, so there is a $d_i \in D_i$ with $x \leq d_i$. Let $f \in \Pi_{i \in I} D_i$ with $f(i) = d_i$. Then $x \leq \wedge_{i \in I} f(i)$ which is less than or equal to the right side. Then $a = \vee\{x \in L : x \prec_S a\}$ is less than or equal to the right side. □
In particular, every smooth lattice is distributive.

Theorem 5.2.0.1 *Let L be a complete lattice. Then the following statements are equivalent:*

(i) L is smooth and continuous.

(ii) L is completely distributive.

Proof
It suffices to show that (i) implies (ii), i.e. L is completely distributive. Since L is continuous, for each $a \in L$, $a = \vee\{x \in L : x \ll a\}$. Now for each $x \ll a$, $x = \vee\{y \in L : y \prec_S x\}$. It follows that $a = \vee\{y \in L : \exists x, y \prec_S x \ll a\}$. Next let $y \prec_S x \ll a$. We show that $y \lhd a$ (long-way below in the sense of Raney). Let $D \subseteq L$ with $\vee D \geq a$. Construct the set $E = \{\vee A : A \text{ is a finite subset of } D\}$. Then E is an up-directed set and $\vee E = \vee D \geq a$. Since $x \ll a$ there is a finite subset $A \subseteq D$ such that $x \leq \vee A = \vee \downarrow A$, the last set is Scott-closed. Then $y \prec_S x$ means that $y \leq d$ for some $d \in A \subseteq D$. This implies that $y \lhd a$. And so L is completely distributive. □
Example
Consider the lattice of all open sets of the unit interval $[0,1]$, denoted by $\mathcal{O}[0,1]$. It is a well-known fact that $\mathcal{O}[0,1]$ is both continuous and distributive but not completely distributive, hence it can not be smooth.

5.3 Some properties of lattices $\mathbf{C}(P)$

In this section we establish the smoothness of lattices arising as $\mathbf{C}(P)$ for a dcpo P.

Proposition 5.3.0.5 *Let P be a dcpo and $\mathcal{C} \in \mathbf{C}(\mathbf{C}(P))$. Then $\vee\mathcal{C} = \cup\mathcal{C}$.*

Proof
Let $\mathcal{C} = \{C_i : i \in I\}$. Note that each of the C_i's is a Scott-closed subset of P. It suffices to show that $\cup\mathcal{C}$ is a Scott-closed subset of $\mathbf{C}(P)$, that is

$$\cup\{C_i : i \in I\} \in \mathbf{C}(P).$$

Let D be an up-directed subset of P contained in $\cup\{C_i : i \in C\}$. Then obviously $\{\downarrow d : d \in D\}$ is an up-directed subset of $\mathbf{C}(P)$. Moreover, $\{\downarrow d : d \in D\} \subseteq \mathcal{C}$ because the later is down-closed. Since $\mathcal{C}$ is a Scott-closed set of $\mathbf{C}(P)$, so the supremum of $\{\downarrow d : d \in D\}$ in $\mathbf{C}(P)$ is still in $\mathcal{C}$. But the supremum of $\{\downarrow d : d \in D\}$ in $\mathbf{C}(P)$ is $\downarrow e$, where $e = \vee D$. Thus $\downarrow e \in \mathcal{C}$ implies $\vee D = e \in \cup\{C_i : i \in I\}$. Hence $\cup\mathcal{C} \in \mathbf{C}(P)$. □
Remark
Thus in the language of [5], the function that assigns P to $\mathbf{C}(P)$ defines a union-complete subset system.

Definition 5.3.0.3 *We call an element x of L Scott-compact if $x \prec_S x$. We shall use $\Gamma(L)$ to denote the set of all Scott-compact elements of L.*

Recall that an element a of a complete lattice L is called *join*-prime if $a \leq x \vee y$ implies $a \leq x$ or $a \leq y$. An element x is call compact if $x \ll x$.

Proposition 5.3.0.6 *Let L be a complete lattice.*

(1) If $r \in L$ is Scott-compact, then r is join-prime.

(2) If L is a completely distributive lattice, then every join-prime element is Scott-compact.

(3) The set $\Gamma(L)$, if not empty, is a dcpo.

Proof

(1) Suppose r is Scott-compact and $r \leq x \vee y$. Let $D = \downarrow \{x, y\}$. Then D is Scott closed and $\vee D = x \vee y$. Hence $r \in D$, so $r \leq x$ or $r \leq y$. Thus r is join-prime.

(2) Let L be a completely distributive lattice and $r \in L$ be join-prime. By [20] the set $\beta(r) = \{x \in L : x \lhd r\}$ is an up-directed set and $\vee \beta(r) = r$. Since $x \lhd r$ implies $x \prec_S r$, from Proposition (5.2.0.2). it then follows that $\vee \beta(r) \prec_S r$, that is $r \prec_S r$. Hence r is Scott-compact.

(3) Follows from Proposition(5.2.0.2).

Examples

(1) Let $F([0,1])$ be the lattice of all closed subsets of the unit interval $[0,1]$. Then each singleton point set is join-prime, but they are not Scott-compact. For instance, let $A = \{1/2\}$ and $\mathcal{C} = \{\{x\} : x \neq 1/2\} \cup \{\emptyset\}$. Then obviously $\mathcal{C}$ is a Scott closed set of $F([0,1])$ and sup $\mathcal{C} = [0,1] \geq A$, but $A \notin \mathcal{C}$.

(2) In a finite lattice, every element is compact. Thus it is easy to find a compact element that is not Scott-compact.

(3) In [13] Isbell constructed a complete lattice L such that the Scott space ΣL is not sober, this implies that there exists an irreducible Scott closed set F such that $F \neq \downarrow x$ for any $x \in L$. From the following Corollary(5.4.0.10), Scott-compact elements of $\mathbf{C}(L)$ are exactly $\downarrow x, x \in L$. Hence the above F is not Scott-compact.

(4) Question: What are the Scott-compact open sets of $\mathbb{R}$?

Proposition 5.3.0.7 *Let P be a dcpo and X be a Scott-closed subset of P. Then for each $x \in X, \downarrow x \prec_S X$ in $\mathbf{C}(P)$.*

Proof
Let $x \in X$. Suppose $\mathcal{C}$ is a Scott-closed subset of $\mathbf{C}(P)$ such that $\vee \mathcal{C} \supseteq X$. Then, by Proposition(5.3.0.5), $X \subseteq \cup \mathcal{C}$. Hence there exists $C_i \in \mathcal{C}$ such that $x \in C_i$, hence $\downarrow x \in \mathcal{C}$. □

Corollary 5.3.0.2 *Let P be a dcpo. Then for each $x \in P$, $\downarrow x$ is a Scott-compact element of $\mathbf{C}(P)$.*

Definition 5.3.0.4 *A complete lattice P is said to be algebraically smooth if for each* $a \in P$,

$$a = \vee\{x \in P : x \prec x, x \leq a\}.$$

In other words, P is algebraically smooth if the set of all Scott-compact elements is join dense in P.

Obviously every algebraically smooth lattice is smooth.

Proposition 5.3.0.8 *For any dcpo P, the lattice* $\mathbf{C}(P)$ *is algebraically smooth.*

Proof
This follows from the fact the for each $x \in P$, $\downarrow x$ is Scott-compact, and $F = \vee\{\downarrow x : x \in F\}$ for every $F \in \mathbf{C}(P)$. □

Examples
Using above result we can find a smooth lattice that is not continuous. To do this, let us consider any non-continuous dpos, say P. Since P is not continuous, it follows that $\mathbf{C}(P)$ cannot be completely distributive. But $\mathbf{C}(P)$ is smooth so $\mathbf{C}(P)$ cannot be continuous. Otherwise $\mathbf{C}(P)$ is completely distributive so from [10] P would be a continuous poset. But $\mathbf{C}(P)$ is always smooth.

5.4 Characterization of $\mathbf{C}(P)$ with P an upper bounded complete dcpo

Recall that an upper bounded complete dcpo (abbreviated as ubc dcpo) is one of which every upper bounded subset has a supremum.
Although we are still unable to give a complete characterization of lattices $\sigma^{op}(P)$ for arbitrary dcpo, we can do this for ubc dcpo P.

Proposition 5.4.0.9 *Let P be an ubc dcpo. If* $X \in \mathbf{C}(P)$ *is Scott compact, then* $\mathcal{D} = \downarrow\{\downarrow x : x \in X\}$ *is a Scott-closed subset of* $\mathbf{C}(P)$.

Proof
To show this, take any up-directed subset of $\mathbf{C}(P)$, say $\mathcal{E} \subseteq \mathcal{D}$. Now for each Scott-closed set $E \in \mathcal{E}$, $E \subseteq\downarrow x$ for some $x \in X$ and since P is upper bounded complete, $\vee E$ exists. Clearly the set $\{\vee E : E \in \mathcal{E}\}$ is an up-directed subset of P. Let $e = \bigvee\{\vee E : E \in \mathcal{E}\}$. X is a lower set implies that $\vee E \in X$ for each $E \in \mathcal{E}$. So $e = \bigvee\{\vee E : E \in \mathcal{E}\} \in X$. Next we claim that $\downarrow e = sup\ \mathcal{E}$. For any $Y \in \mathcal{E}$, $e = \vee \downarrow e \geq \vee Y$. This means $Y \subseteq\downarrow \vee Y \subseteq\downarrow e$ for each $Y \in \mathcal{E}$. Since sup $\mathcal{E}$ is the smallest Scott-closed set of $\mathbf{C}(P)$ that contains $\mathcal{E}$, it follows that sup $\mathcal{E} \subseteq\downarrow e$. But $e \in X$. So sup $\mathcal{E} \in \mathcal{D}$. □

Proposition 5.4.0.10 *Let P be an ubc dcpo. If* $X \in \mathbf{C}(P)$ *is Scott-compact, then there is an element* $a \in P$ *such that* $X = \downarrow a$.

Proof
Suppose $X \in \mathbf{C}(P)$ is Scott-compact. By Proposition (5.4.0.9), $\mathcal{D} = \downarrow \{\downarrow x : x \in X\}$ is a Scott-closed subset of $C(P)$. It is clear that sup $\mathcal{D} =$ sup $\downarrow \{\downarrow x : x \in X\} = X$. Since $X \prec_S X$, $X \in \mathcal{D}$. So $X \subseteq \downarrow a$ for some $a \in X$. But this means $X = \downarrow a$. □
Remark Proposition(5.4.0.10) reveals that we can recover P from $\mathbf{C}(P)$ as the set of all Scott-compact elements if P is an ubc dcpo.

Corollary 5.4.0.3 *Let P be a ubc dcpo and $M = \mathbf{C}(P)$. Then the principal map $\eta : P \to \Gamma(M), : x \mapsto \downarrow x$ is an order-isomorphism.*

Corollary 5.4.0.4 *If P and Q are two ubc dcpos such that the lattices $\mathbf{C}(P)$ and $\mathbf{C}(Q)$ are isomorphic, then P and Q are isomorphic.*

Definition 5.4.0.5 *Let L be a complete lattice. We say that L is Scott-stable if the following condition holds:*

(i) If $x \prec_S d_i$ for each $d_i \in L (i \in I)$, then $x \prec_S \wedge_{i \in I} d_i$;

(ii) $1_L \prec_S 1_L$.

In the case when L satisfies only condition (i), we say that L is weakly Scott-stable.

Proposition 5.4.0.11 *The following statements are equivalent for an algebraically smooth lattice L:*

(i) L is Scott-stable.

(ii) $\Gamma(L)$ is a complete lattice.

Proof
(i) ⇒ (ii): By (ii), $1_L \in \Gamma(L)$. So we consider any non-empty subset $\{x_i : i \in I\}$ of $\Gamma(L)$. Denote $\wedge_L \{x_i : i \in I\}$ by x. Clearly, $x \prec_S x_i$ for each $i \in I$. By the Scott-stability of L, $x \prec_S x$.
(i) ⇐ (ii): Note that $\Gamma(L) = \downarrow 1_L \cap \Gamma(L)$. Since L is an algebraically smooth lattice, it follows that $\vee \Gamma(L) = 1_L \in \Gamma(L)$, i.e. $1_L \prec_S 1_L$.
Suppose that $x \prec_S d_i (i \in I)$ in L. For each $i \in I$, $\downarrow (\Gamma(L) \cap \downarrow d_i)$ is Scott-closed.
In fact, given any directed set $\{c_j : j \in J\} \subseteq \downarrow (\Gamma(L) \cap \downarrow d_i)$.
For each $j \in J$, let $\overline{c_j} = \wedge_{\Gamma(L)} \{x : c_j \leq x, x \in \Gamma(L) \cap \downarrow d_i\}$. Then $\overline{c_j} \prec_S a$.
Since $\{\overline{c_j} : j \in J\}$ is directed, so $\vee \{\overline{c_j}\} \in \downarrow (\Gamma(L) \cap \downarrow d_i)$ by Proposition(5.2.0.2).
So in turn $\vee \{c_j : j \in J\} \leq \vee \{\overline{c_j} : j \in J\}$ which implies that $\vee \{c_j : j \in J\} \in \downarrow (\Gamma(L) \cap \downarrow d_i)$.
Moreover for each $i \in I$, $\vee \downarrow (\Gamma(L) \cap \downarrow d_i) = d_i$ since L is algebraically smooth.
For each $i \in I$, $x \prec_S d_i$ implies that $x \leq e_i \prec_S d_i$ for some $e_i \in \Gamma(L) \cap \downarrow d_i$. Since $\Gamma(L)$ is complete, it follows that the $\wedge_{\Gamma(L)} \{e_i : i \in I\}$ exists. Denote this infimum by e. It follows that $x \leq e \prec_S e \leq \wedge_L \{d_i : i \in I\}$ and thus $x \prec_S \wedge_L \{d_i : i \in I\}$. □
Similar arguments result in the following proposition:

Proposition 5.4.0.12 *The following statements are equivalent for an algebraically smooth lattice L:*

(i) L is weakly Scott-stable.

(ii) $\Gamma(L)$ is an ubc dcpo.

Theorem 5.4.0.2

(i) *A complete lattice M is isomorphic to the lattice $\mathbf{C}(L)$ for a complete lattice L if and only if M is Scott stable and algebraically smooth.*

(ii) *A complete lattice M is isomorphic to the lattice $\mathbf{C}(P)$ for an ubc dcpo P if and only if M is weakly Scott stable and algebraically smooth.*

Proof

For convenience, we shall only prove (ii).
Let P be an ubc dcpo. That $\mathbf{C}(P)$ is algebraically smooth follows from Proposition (5.3.0.8). Also by Corollary(5.4.0.10), $\Gamma(\mathbf{C}(P))$ is isomorphic to P. Thus $\Gamma(\mathbf{C}(P))$ is an ubc dcpo. By Proposition(5.4.0.12), $\mathbf{C}(P)$ is weakly Scott stable.
Now assume that M is a stable and algebraically smooth lattice. Let $P = \Gamma(M)$. If $\{x_i : i \in i\} \subseteq P$, then as M is stable it follows that $\wedge_{i\in I} x_i \prec_S \wedge_{i\in I} x_i$. Thus $\wedge_{i\in I} x_i$ is the infimum of $\{x_i : i \in I\}$ in P. By Proposition(5.4.0.12), this implies that P is an ubc dcpo.
We now show that M is isomorphic to $\mathbf{C}(P)$. We claim that the map ζ defined by $\zeta : M \to \mathbf{C}(P), x \mapsto (\downarrow x) \cap P$ is an isomorphism of complete lattices. Clearly ζ is order-preserving. In order to show ζ is an isomorphism, it suffices to show that the map $\vee : \mathbf{C}(P) \longrightarrow M$ is an inverse of the map ζ. For each $x \in M, \vee\zeta(x) = \vee(\downarrow x \cap P) = x$ holds for each $x \in M$ since M is algebraically smooth. Now let $C \in \mathbf{C}(P)$ and $\vee_M C = a$, we claim that $(\downarrow a) \cap P = C$. Since $(\downarrow a) \cap P \supseteq C$ is trivial, we only need to show that $(\downarrow a) \cap P \subseteq C$. Let $x \in (\downarrow a) \cap P$, that is $x \prec_S x$ and $x \leq a$. This implies that $x \prec_S a = \vee_M C$. Let $Q = \downarrow C = \{y \in M : \exists c \in C, y \leq c\}$. Then $\vee_M C = \vee_M Q$. If we can show Q is a Scott closed set of M, then $x \prec_S a = \vee_M C = \vee_M Q$ will deduce $x \in Q$, thus $x \leq c$ for some $c \in C$, and therefore $x \in C$.
To show $\downarrow C$ is a Scott-closed set, let $D \subseteq \downarrow C$ be any directed subset. For each $d \in D$, let $\overline{d} = \inf\{y \in C : d \leq y\}$, where the infimum is taken in P. Then each $\overline{d} \in C$, and obviously the set $\overline{D} = \{\overline{d} : d \in D\}$ is a directed subset of P. Since C is a Scott-closed set of P, so $\vee_P \overline{D} \in C$. However it is clear that $\vee_M D \leq \vee_M \overline{D} \leq \vee_P \overline{D}$, hence $\vee_M D \in \downarrow C$. Since $\downarrow C$ is obviously a down-closed set, it is thus a Scott-closed set of M. The proof is complete. □

Question: Suppose L is a distributive continuous lattice. Can we show that every irreducible element is Scott-compact?

Remarks

(1) Recall that a topological space X is called a *sober* space if every irreducible closed set is the closure of a singleton set. For an arbitrary dcpo P, the Scott space ΣP need not be sober(see [14]). Actually even if P is a complete lattice ΣP may not be sober([13]).

We call a topological space X pre-sober, if for any Scott-compact closed set F there is a point $x \in X$ such that $F = cl\{x\}$, the closure of the set $\{x\}$. Then it follows that every sober space is pre-sober. For any complete lattice L, the Scott space ΣL is pre-sober by Proposition(5.4.0.10). Then the example constructed in [13] is pre-sober but not sober.

(2) We then ask: is the Scott space ΣP of every dcpo pre-sober? The answer is no. A counterexample is the one constructed by Johnstone in [14]. We now explain this. Let $X = \mathbf{N} \times (\mathbf{N} \cup \{\infty\})$, with partial order defined by

$$(m,n) \leq (m',n') \iff \text{either } m = m' \text{ and } n \leq n' \\ \text{or } n' = \infty \text{ and } n \leq m'.$$

Then $(X, \leq)$ is a dcpo and X is is not a principal ideal. Suppose that there is a collection $\mathcal{F} = \{F_i : i \in I\}$ of Scott closed subsets F_i of X such that $\mathcal{F}$ is a Scott closed subset of $\sigma^{op}(X)$, and sup$\mathcal{F} = X$. Hence $\cup\mathcal{F} = X$. For each $k \in \mathbf{N}$, let $E_k = \{(m,n) : n \leq k\}$, which is clearly a Scott closed subset of X. For each k there exists F_i such that $(k,\infty) \in F_i$, hence $E_k \subseteq\downarrow (k,\infty) \subseteq F_i$ imply that $E_k \in \mathcal{F}$. Clearly $\{E_k : k \in \mathbf{N}\}$ is an up-directed set. Since $\mathcal{F}$ is a Scott closed set of $\sigma^{op}(X)$, so $\sup\{E_k : k \in \mathbf{N}\} \in \mathcal{F}$. But trivially $\sup\{E_k\} = X$, so $X \in \mathcal{F}$. Hence X is Scott-compact.

5.5 A categorical equivalence

After proving Theorem(5.4.0.2), it is natural to conjecture that there exists an equivalence between a category of all upper bounded complete posets and Scott-stable and algebraically smooth lattices. In this section we first establish an adjunction between the category of all dcpo's and the category of all complete lattices using the function $\mathbf{C}(P)$ which assign to every dcpo P the complete lattice $\mathbf{C}(P)$ of all Scott-closed subsets of P. The restriction of this adjunction will give the equivalence as mentioned above.
First let **DCPO** be the formilar category of all dcpo's and mappings that preserve joins of up-directed subsets.
Let $f : M \longrightarrow N$ and $g : N \longrightarrow M$ be a pair of order-preserving mappings between posets M and N. Then f is said to be left adjoint to g if for any $x \in M$ and $y \in N$, $f(x) \leq y$ if and only if $x \leq g(y)$. In this case g is also called the right adjoint to f. It is well known that if M and N are both complete lattices then f is a left adjoint if and only if it preserves joins of arbitrary sets and g is a right adjoint if and only if it preserves meets of arbitrary sets, and f and its right adjoint g has the following relationship:

$$f(a) = \bigwedge\{y \in N : a \leq g(y)\}, \forall a \in M,$$

$$g(b) = \bigvee\{x \in M : f(x) \leq b\}, \forall b \in N.$$

In the literature of order theory, such a pair of mappings (f,g) is called a Galois connection. A mapping $h : M \longrightarrow N$ between complete lattices is said to preserve the relation $\prec_S$ if $x \prec_S y$ implies $h(x) \prec_S h(y)$. Now let $\mathbf{Sup}^*$ be the category of all complete lattices and mappings which are left adjoints that preserve the relation $\prec_S$.

Proposition 5.5.0.13 *Let (f,g) be a Galois connection between complete lattices L and M. If g preserves the sups of Scott-closed subsets, then f preserves the relation $\prec_S$.*

Proof
Let $x \prec_S y$ in M. Let C be a Scott-closed set of M such that $\vee C \geq f(y)$. It follows that $g(\vee C) \geq y$ since g is right adjoint to f. Thus $\vee g(C) \geq y$. But it is easily seen that $\downarrow g(C)$ is a Scott-closed set of L. So, it follows from the definition of $\prec_S$ that $x \leq g(c)$ for some $c \in C$. But f is left adjoint to g so that $f(x) \leq c$, i.e. $f(x) \in C$. So $f(x) \prec_S f(y)$. □
Let A be a subset of a dcpo P and

$$w(A) = \{\vee D : D \subseteq A \ D \text{ is an up-directed subset of } A\}.$$

Now for each ordinal α we define a subset $w^{\alpha}(A)$ by $w^{\alpha+1}(A) = w(w^{\alpha}(A))$ and

$$w^{\alpha}(A) = \bigcup\{w^{\beta}(A) : \beta < \alpha\}$$

if α is a limit ordinal. Then due to the cardinal reason there exists an ordinal α such that $w^{\alpha+1}(A) = w^{\alpha}(A)$, and $w^{\alpha}(A)$ is the smallest Scott closed set containing A, or the closure of A in the Scott space ΣP.
It is well known that if $f : P \longrightarrow Q$ is a morphism in **DCPO** then f is a continuous mapping from the Scott space ΣP to ΣQ, hence for each $E \in \mathbf{C}(Q)$, $f^{-1}(E) \in \mathbf{C}(P)$. Thus $f^{-1} : \mathbf{C}(Q) \longrightarrow \mathbf{C}(P)$ is a mapping which obviously is a right adjoint. We now show that the left adjoint of $f^{-1} : \mathbf{C}(Q) \longrightarrow \mathbf{C}(P)$ preserves $\prec_S$. We shall denote this left adjoint by h.

Lemma 5.5.0.1 *Let $f : P \longrightarrow Q$ be a morphism in* **DCPO**. *Then $h : \mathbf{C}(P) \longrightarrow \mathbf{C}(Q)$ preserves $\prec_S$.*

Proof
Suppose $A, B \in \mathbf{C}(P)$ such that $A \prec_S B$. Let $\mathcal{D} \in \mathbf{C}(\mathbf{C}(P))$ such that $\vee\mathcal{D} \geq h(B)$, that is $h(B) \subseteq \cup\mathcal{D}$. Then
$B \subseteq f^{-1}(\cup\mathcal{D}) = \bigcup\{f^{-1}(D) : D \in \mathcal{D}\} = \bigvee\{f^{-1}(D) : D \in \mathcal{D}\} = \bigvee\{\downarrow \{f^{-1}(D) : D \in \mathcal{D}\}\}$. It is easily shown that $\downarrow \{f^{-1}(D) : D \in \mathcal{D}\} \in \mathbf{C}(\mathbf{C}(Q))$. Thus as $A \prec_S B$, it follows that $A \subseteq f^{-1}(D)$ for some $D \in \mathcal{D}$, hence $h(A) \subseteq D$. This shows that $h(A) \prec_S h(B)$. □
Remark
Actually it is not difficult to see that the $f^{-1} : \mathbf{C}(Q) \longrightarrow \mathbf{C}(P)$ preserves the sups of Scott-closed subsets. To see this, let $\mathcal{B} = \{B_j : j \in J\}$ be a Scott-closed set of $\mathbf{C}(Q)$. By Proposition(3.1), $\vee\mathcal{B} = \cup\mathcal{B} = \cup_{j\in J}B_j$. But one readily sees that

$$f^{-1}(\cup_{j\in J}B_j) = \cup_{j\in J}f^{-1}(B_j).$$

So $\cup_{j\in J}f^{-1}(B_j)$ is a Scott-closed set and thus $f^{-1}(\vee\mathcal{B}) = \vee f^{-1}(\mathcal{B})$. Invoking Proposition(5.5.0.13), we can also deduce that h preserves $\prec_S$.
It follows from above lemma that the function $P \longrightarrow C(P)$ can be extended to a functor $\mathbf{C} : \mathbf{DCPO} \longrightarrow \mathbf{Sup}^*$, where for each dcpo P, $\mathbf{C}(P) = \sigma^{op}(P)$ and for each morphism $f : P \longrightarrow Q$, $\mathbf{C}(f) : \mathbf{C}(P) \longrightarrow \mathbf{C}(Q)$ is the left adjoint of f^{-1}.
Now for each complete lattice A, let $\Gamma(A)$ be the dcpo of all Scott compact elements of A. If $g : A \longrightarrow B$ is a morphism in $\mathbf{Sup}^*$ then g restricts to a morphism $\Gamma(g) : \Gamma(A) \longrightarrow \Gamma(B)$. Thus we have a functor $\mathbf{\Gamma} : \mathbf{Sup}^* \longrightarrow \mathbf{DCPO}$.

Theorem 5.5.0.3 *The functor* $\mathbf{\Gamma}$ *is right adjoint to the functor* $\mathbf{C}$.

Proof
Let P be a dcpo. Let $\eta_P : P \longrightarrow \Gamma(\mathbf{C}(P))$ be defined by $\eta(x) = \downarrow x, \forall x \in P$. Then it is clear that η_P is a morphism in **DCPO**. Suppose L is any complete lattice and $h : P \longrightarrow \Gamma(L)$ is a morphism in **DCPO**. Define $\overline{h} : \mathbf{C}(P) \longrightarrow L$ by

$$\overline{h}(E) = \vee\{h(x) : x \in E\}, \forall E \in \mathbf{C}(P).$$

Then $h(x) = \overline{h}(\eta_P(x))$ holds for every $x \in P$. By the earlier remark (2), it remains to prove that $\overline{h}$ is a lower adjoint that preserves the $\prec_S$ relation.
Let $\mathcal{D} \in \mathbf{C}(\mathbf{C}(P))$. We must show that $\overline{h}(\vee\mathcal{D}) = \vee\overline{h}(\mathcal{D})$. It suffices to prove $\leq$. But $\overline{h}(\vee\mathcal{D}) = \overline{h}(\cup\mathcal{D}) = \vee_{x\in\cup\mathcal{D}}h(x) \leq \vee\{\vee_{d\in D}h(d) : D \in \mathcal{D}\} = \vee\{\overline{h}(D) : D \in \mathcal{D}\} = \vee\overline{h}(\mathcal{D})$.
Next we show that $\overline{h}$ preserves $\prec_S$. We shall resort to Proposition(5.5.0.13). Denote the right adjoint of $\overline{h}$ by g. Let $C \in \mathbf{C}(L)$. It suffices to prove that $g(\vee C) \leq \vee g(C)$. But

$$\begin{aligned} g(\vee C) &= \vee\{A \in \mathbf{C}(P) : \overline{h}(A) \leq \vee C\} \\ &= \vee\{\downarrow p, p \in P : \overline{h}(\downarrow x) \leq \vee C\} \\ &= \vee\{\downarrow p, p \in P : h(x) \leq \vee C\}. \end{aligned}$$

But $h(x) \prec_S h(x)$ (because $h(x) \in \Gamma(L)$) so that $h(x) \leq \vee C$ implies that $h(x) \in C$ and thus $x \leq g(c)$ for some $c \in C$. Thus $g(\vee C) \leq \vee g(C)$. □

Let **CLat** be the full subcategory of **DCPO** consisting of all complete lattices and let **BCDCPO** be the full subcategory of consisting of all up-bounded complete dcpo's. Let **SASLat** (respectively **WSASLat**) of **Sup*** consisting of Scott-stable (respectively weakly Scott-stable) and algebraically smooth lattices as objects.

Corollary 5.5.0.5 *The functor* $\mathbf{C} : \mathbf{DCPO} \longrightarrow \mathbf{Sup}^*$ *restricts to an equivalence between* **CLat** *(respectively* **BCDCPO***) and* **SASLat** *(respectively* **WSASLat***).*

Let **Conpos** be the full subcategory of **DCPO** consists of all continuous posets and **CDL** be the full subcategory of **Sup*** consists of all completely distributive lattices. From the classical result that a dcpo P is continuous if and only if $\mathbf{C}(P)$ is a completely distributive lattice and $\mathbf{C}(Spec(L)) \cong L$ holds for every completely distributive lattice L, where $Spec(L)$ is the set of all irreducible elements of L, which are the same as all the Scott-compact elements of L, i.e. $Spec(L) = \Gamma(L)$.

Corollary 5.5.0.6 *The category* **Conpos** *is equivalent to the category* **CDL**.

5.6 Extension to Z-theory

Following [5], we may generalize the concept of smoothness to Z-smoothness.

Definition 5.6.0.6 ***(Z-closed-complete)***
A Z-complete poset A is said to be Z-closed-complete if for every Z-closed set C of A, the supremum of C exists in A.

Remark
In the case when $Z = Id$ (selection of directed subsets), Z-closed-completeness is actually equivalent to completeness.

Proposition 5.6.0.14

(i) For each Z-complete poset A, the map $\downarrow: A \to A \to C_z(A)$ which sends $x \in A$ to $\downarrow x$ is an order-preserving map.

(ii) A is Z-closed-complete if and only if the map $\downarrow$ has a left adjoint. Moreover, the left adjoint, if it exists, must be the sup map $\vee : C_z(A) \to A$.
(See Definition(4.2.3.2) for the definition of $C_z(A)$.)

Proof
(i) Let $x \leq y$ in A. Since $w \leq x$ and $x \leq y$ implies $w \leq y$ by transitivity. So $\downarrow x = \{w \in A : w \leq x\} \subseteq \{w \in A : w \leq y\} = \downarrow y$.
(ii) Given that A is Z-closed-complete. Then for any $C \in C_z(A), \vee C$ exists in A. Define $\vee : C_z(A) \to A$ by $C \mapsto \vee C, C \in C_z(A)$. Now $\vee C \leq a$ implies $\downarrow \vee C \subseteq \downarrow a$ by (i).
Let $c \in C$. Then $c \leq \vee C$. This implies that $C \subseteq \downarrow a$.
Conversely, if $C \subseteq \downarrow a$, then $\vee C \leq a$. Thus the map $\downarrow$ has a left adjoint, i.e. $\vee$.
Conversely, if the map $\downarrow$ has a left adjoint $h : C_z(A) \to A$, i.e. $h \dashv \downarrow$.
Let $C \in C_z(A)$ and $a \in A$ such that $h(C) \leq a \Leftrightarrow C \subseteq \downarrow a$. Since $h \dashv \downarrow, C \subseteq h(C)$ and $h(\downarrow u) \subseteq u$ for all $C \in C_z(A)$ and $u \in A$. Thus, $c \in C$ implies $c \in \downarrow h(C)$, which implies that $h(C)$ is an upper bound of C. Since h is order-preserving, $h(C) \leq h(\downarrow u) \leq u$. Thus $h(C) = \vee C$. It follows that A is Z-closed-complete. □
Hereafter, all the posets in this section are Z-closed-complete if there is no ambiguity.

Definition 5.6.0.7 ***(Z-Scott-below)***
Let A be a Z-closed-complete poset and $x, y \in A$. x is Z-Scott-below y, denoted by $x \prec_z y$, if for each Z-closed sets C of A, $\vee C \geq y$ implies $x \in C$.

Remark
It is easy to see that $x \prec_z y$ if and only if for each z-set C whose supremum is greater than or equal to y, there is an element $c \in C$ such that $x \leq c$. This is because z-closed sets are lower sets.
Similar to the $\prec_S$ relation, one has the following:

Proposition 5.6.0.15 *In a Z-closed-complete poset A one has the following statements for all $u, v, x, y \in A$:*

(i) $x \prec_z y$ implies $x \leq y$.

(ii) $u \leq x \prec_z y \leq v$ implies $u \prec_z v$.

(iii) If $\phi \in Z(A)$, we have a bottom element 0 of A. Then $0 \prec_z x$ for all $x \in A$.

Generalizing in the light of Z-theory, we define Z-smoothness as follows:

Definition 5.6.0.8 ***(Z-smooth poset)***
A poset A is Z-smooth if the following conditions hold:

(i) A is a Z-closed-complete.

(ii) For each $a \in A$, the set $\alpha(a)$ defined by

$$\alpha(a) = \{x \in A : x \prec_z a\},$$

is a Z-closed set of A and $\vee\alpha(a) = a$.

Similarly, one can show that:

Proposition 5.6.0.16 *For a Z-closed complete poset A, A is Z-smooth implies that for each $a \in A$, there is a smallest Z-closed set C such that $\vee C = a$.*

In fact, there are many more similar propositions and theorems that hold true in the light of such a generalization. In this way, the results in the previous sections are strengthened to include other subset systems (besides the directed subsets).

Chapter 6

Conclusion

6.1 Survey of my work and contributions

A large portion of my time has been spent on understanding many insightful papers and books. Nuclearity is one topic that has driven me a distance in my research. With minimal assumptions, nuclearity implies projectivity. It turns out that the base object I in an autonomous category passes some properties (which are related to limits and colimits) to the other nuclear objects. The implication of this result is that: On one hand, while characterising nuclearity in a given category, the conditions we are looking for are more well-defined. On the other hand, we know which properties are inherited to the nuclear objects by the base object, so that our understanding of nuclearity is not obscured by these properties.

Some work has been devoted to the restatements of theorems and results mainly due to D.Zhao, with regard to z-frames. This covers the basic content of unifying the concepts of frames, preframes, σ-frames and κ-frames. This portion essentially is on the Theory of Frames which was my initial objective in this academic exercise. I have made it a point to include details of proofs (only in the section of Z-frame) as exercises to familiarise myself with these new concepts. So, my lengthy proofs are by no means the fault of the original authors.

A small but crucial observation was made while I studied **E**-projectivity of frames (in the adjunction context between **PreFrm** and **Frm**). This observation sparked off an entirely new concept of smooth lattices. It turned out that smoothness characterizes lattices of Scott-closed sets. Moreover, it runs parallel to continuity so naturally that the Spectral Theory of Continuous Lattices becomes more transparent. The topic of smooth lattices forms the last but most original work of my research in the entire three years.

6.2 Prospects for future work

Firstly, I still have not yet succeeded in charcterizing the nuclear preframes. In the more general setting of Z-theory, I also have not obtained the necessary and sufficient conditions of the type of Z-selection for which **zFrm** has an autonomous structure. Much work remains to be done. I guess there might be a need to employ Universal Algebra in tackling the second problem.

Secondly, the concept of smooth lattices is still new to me. I am sure there are many things worth looking at. Due to time factor and my own handicap in mathematics, I was unable to look at all the properties regarding smoothness. Nonetheless, some work has been done in the direction of generalising smoothness using Z-theory. Indeed many of the theorems can be strengthened in the light of Z-theory. However, I have not included these materials as the arguments involved are essentially the same as in Chapter 5. There remains some unsolved questions.

(i) Is every irreducible element of a distributive continuous lattice Scott-compact?

(ii) What are the Scott-compact open sets of $\mathbb{R}$?

(iii) How can one characterize the **E**-projective frames (in adjunction to **PreFrm**)?

In fact, (iii) was the original question in Chapter 4 that sets the road of investigation in Chapter 5. However, there remains a difficulty: Although retracts preserve continuity, they do not seem to preserve smoothness. More work needs to be done to investigate the stability of the $\prec_S$ relation under retraction.

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