On topologies defined by irreducible sets

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Abstract. In this paper, we define and study a new topology constructed from any given topology on a set, using irreducible sets. The manner in which this derived topology is obtained is inspired by how the Scott topology on a poset is constructed from its Alexandroff topology. This derived topology leads us to a weak notion of sobriety called k-bounded sobriety. We investigate the properties of this derived topology and k-bounded sober spaces. A by-product of our theory is a novel type of compactness, which involves crucially the Scott irreducible families of open sets. Some related applications on posets are also given.

1. Introduction

The motivating ingredients for our present paper are the irreducible sets, and two well-known topologies on posets, namely, the Alexandroff and Scott topologies. A non-empty subset F of a topological space is *irreducible* if whenever $F \subseteq A \cup B$ for closed sets A and B then $F \subseteq A$ or $F \subseteq B$. The *Alexandroff topology* $\Upsilon(P)$ on a poset P is the topology consisting of all its upper subsets. Here, $A \subseteq P$ is *upper* if $A = \uparrow A = \{x \in P : x \geq y \text{ for some } y \text{ in } A\}$. It has long been known that the non-empty irreducible subsets of a poset with respect to the Alexandroff topology are exactly the *directed* sets (a non-empty set D is directed if every two elements in D have an upper bound in D). The *Scott topology* $\sigma(P)$ on a poset P is defined as all the upper sets U which are inaccessible by directed suprema. More precisely, $U \in \sigma(P)$ if and only if $U = \uparrow U$ and for any directed set D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists (see [**7**, p.134]).

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In the preceding discourse, we mentioned certain topologies on a given order. In the opposite direction, one can consider order structures on a given topology. Indeed every T_0 topology τ on a set X induces a partial order $\leq$, called the *specialization order*, i.e., $x \leq y$ if and only if $x \in \mathrm{cl}(\{y\})$. The specialization order of the Alexandroff topology on a poset coincides with the underlying order. Gathering from what has been said earlier, Scott openness can now be rephrased as follows: $U \subseteq P$ is Scott open if and only if U is open in the Alexandroff topology and inaccessible by suprema of Alexandroff-irreducible sets (i.e., for any irreducible set F, $\bigvee F \in U$ implies $F \cap U \neq \emptyset$ whenever $\bigvee F$ with respect to the specialization order of Alexandroff topology exists). Such a way of defining the Scott topology from the Alexandroff topology leads us to an idea of defining a new topology, called the irreducibly-derived topology, from any given T_0 topology on a set.

In this paper, we shall investigate the irreducibly-derived topologies systematically. In the ensuing development, we introduce a weak notion of sobriety: *k-bounded sobriety*. Here, we quickly advertise some of the major results obtained: (i) a space is k-bounded sober if and only if its irreducibly-derived topology coincides with the original topology; (ii) every partial order is induced by a k-bounded sober topology; (iii) for every complete lattice, there is a finest sober topology inducing the order on the lattice; (iv) every T_0 space can be embedded as a dense subset of a k-bounded sober space.

A "way-below"-like relation is also introduced and used to characterize those spaces whose irreducibly-derived topologies form completely distributive lattices. In the last section, a new kind of compactness is defined using certain irreducible sets.

2. Preliminaries

Given a topological space (X, τ), a non-empty subset F of X is called a τ-*irreducible* set (or simply *irreducible*) if whenever $F \subseteq A \cup B$ for closed sets $A,\ B \subseteq X$ one has either $F \subseteq A$ or $F \subseteq B$. The set of all τ-irreducible sets of X will be denoted by $\mathrm{Irr}_\tau(X)$. Below are some of the well-known properties of irreducible sets (see [**7**, p.46]), whose proofs we omit.

Proposition 2.1. (1) *$F \in \mathrm{Irr}_\tau(X)$ if and only if $\mathrm{cl}_X(F) \in \mathrm{Irr}_\tau(X)$.*

(2) *If $\mathcal{F}$ is a directed subset of $\mathrm{Irr}_\tau(X)$ (i.e., for any $F_1,\ F_2 \in \mathcal{F}$, there is $F_3 \in \mathcal{F}$ such that $F_i \subseteq F_3$ for $i = 1,\ 2$), then $\bigcup \mathcal{F} \in \mathrm{Irr}_\tau(X)$.*

(3) *If* $f : (X,\tau)\longrightarrow(Y,\nu)$ *is a continuous map between topological spaces and* $F \in \mathrm{Irr}_\tau(X)$, *then* $f(F) \in \mathrm{Irr}_\nu(Y)$.
(4) *If* υ *is a coarser topology than* τ *on* X, *then* $\mathrm{Irr}_\tau(X) \subseteq \mathrm{Irr}_\upsilon(X)$.
(5) *For every* $x \in X$, *both* $\{x\}$ *and* $\mathrm{cl}(\{x\})$ *are irreducible.*
(6) $F \in \mathrm{Irr}_\tau(X)$ *if and only if* $\bigcup\{\mathrm{cl}(\{x\}) \mid x \in F\} \in \mathrm{Irr}_\tau(X)$.
(7) *For any finite set* E, $E \in \mathrm{Irr}_\tau(X)$ *if and only if* $E \subseteq \mathrm{cl}(\{e\})$ *for some* $e \in E$.

For any T_0 space (X,τ), the *specialization order* $\leq$ on X is defined by $x \leq y$ if and only if $x \in \mathrm{cl}(\{y\})$. We may denote the specialization order on X by $\leq_\tau$ if there is a need to emphasize the topology. Sometimes, we also say that the topology τ on X induces the partial order $\leq_\tau$. Unless otherwise stated, throughout the paper, whenever an order-theoretic concept is mentioned in the context of a T_0 space X, it is to be interpreted with respect to the specialization order on X.

Remark 2.2. Let X be a T_0 space.

(1) For any $a \in X$, ${\downarrow} a := \{x \in X : x \leq a\} = \mathrm{cl}(\{a\})$.
(2) If $U \subseteq X$ is an open subset, then we have ${\uparrow} U = U$. That is, every open set is an upper set.
(3) For any subset $A \subseteq X$, $\bigvee A$ exists if and only if $\bigvee \mathrm{cl}(A)$ exists; moreover, $\bigvee A = \bigvee \mathrm{cl}(A)$ if they exist.
(4) If $D \subseteq X$ is a directed set with respect to the specialization order, then D is irreducible.

A topological space is said to be *sober* (or equivalently, the topology is sober) if every closed irreducible set (i.e., a set that is both irreducible and closed) is the closure of a unique singleton. Every sober space is necessarily T_0. The first essential application of sobriety is probably in the characterizing of spaces X which satisfy the following condition: For any space Y, if $\Gamma(X) \cong \Gamma(Y)$ (as lattices), then X and Y are homeomorphic, where $\Gamma(X)$ and $\Gamma(Y)$ are the lattices of all closed sets of X and Y respectively (see [**15**]).

A space X is called *hyper-sober* if for any irreducible set F there is a unique $x \in F$ such that $F \subseteq \mathrm{cl}(\{x\})$. Of course, every hyper-sober space is sober. That the converse fails is witnessed by the following example.

Example 2.3. Let $X = [0,1]$ be the unit interval of real numbers with the upper topology (i.e., the open sets are $\emptyset$, $[0,1]$ and all sets of the form $(r,1]$, $r \in [0,1)$). Then X is sober. The set $F = [0,\frac{1}{2})$ is irreducible, but for any $x \in F$, $\mathrm{cl}(\{x\}) = [0,x]$ which certainly does not contain F. Hence X is not hyper-sober.

A poset P is said to be *Noetherian* if it satisfies the *ascending chain condition*: for any chain

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$$

in P, there exists n_0 such that $a_k = a_{n_0}$ holds for all $k \geq n_0$.

If P is Noetherian, then every directed set of P has a largest element.

PROPOSITION 2.4. *If X is a hyper-sober space, then the poset $(X, \leq)$ is Noetherian.*

PROOF. Let X be hyper-sober and $C = \{c_i : i \in \mathbb{N}\}$ be a chain in $(X, \leq)$, where $c_1 \leq c_2 \leq \cdots c_n \leq c_{n+1} \leq \cdots$.

Then the set C is irreducible by virtue of Remark 2.2(4). So, there exists $c_{i_0} \in C$ such that $C \subseteq \mathrm{cl}(\{c_{i_0}\})$. This then implies that $c_i = c_{i_0}$ for all $i \geq i_0$. Hence $(X, \leq)$ is Noetherian. $\square$

3. Irreducibly-derived topology

In this section we define a topology, called the irreducibly-derived topology, from a given topology τ on a set X using the irreducible sets of (X, τ). Some basic properties of this derived topology are proved.

DEFINITION 3.1. Let (X, τ) be a T_0 space. A subset U of X is called SI-open if the following conditions are satisfied:

(i) $U \in \tau$.
(ii) For any $F \in \mathrm{Irr}_\tau(X)$, $\bigvee F \in U$ implies $F \cap U \neq \emptyset$ whenever $\bigvee F$ exists.

The set of all SI-open sets of (X, τ) is denoted by τ_{SI}. Complements of SI-open sets are called SI-closed sets.

THEOREM 3.2. *For any T_0 space (X, τ), τ_{SI} is a topology on X.*

PROOF. We only need to prove that if U and V are SI-open sets then $U \cap V$ is SI-open. But this easily follows from the definition of irreducible sets. $\square$

We shall call τ_{SI} the *irreducibly-derived topology* of τ.

The space (X, τ_{SI}) will also be simply written as $SI(X)$.

PROPOSITION 3.3. *Let (X, τ) be a T_0 space.*

(1) *For any $x \in X$, it holds that $\mathrm{cl}_X(\{x\}) = \mathrm{cl}_{SI(X)}(\{x\})$.*
(2) *The specialization orders of spaces X and $SI(X)$ coincide.*
(3) *If $F \in \mathrm{Irr}(X)$, then $F \in \mathrm{Irr}(SI(X))$.*

PROOF. (1) Firstly, $\mathrm{cl}_X(\{x\}) \subseteq \mathrm{cl}_{SI(X)}(\{x\})$ because $\tau_{SI} \subseteq \tau$. To establish the reverse containment, it suffices to show that $X - \mathrm{cl}_X(\{x\}) \subseteq X - \mathrm{cl}_{SI(X)}(\{x\})$. Since $\mathrm{int}_{SI(X)}(X - \{x\}) = X - \mathrm{cl}_{SI(X)}(\{x\})$, to achieve the desired containment it amounts to showing that $X - \mathrm{cl}_X(\{x\})$ is itself an SI-open set (contained in $X - \{x\}$). For this purpose, suppose that $F \in \mathrm{Irr}(X)$ with $\bigvee F$ exists and $\bigvee F \in X - \mathrm{cl}_X(\{x\})$. If $F \cap X - \mathrm{cl}_X(\{x\}) = \emptyset$, then $F \subseteq \mathrm{cl}_X(\{x\})$, which implies that x is an upper bound of F. Thus $\bigvee F \in \mathrm{cl}_X(\{x\})$, a contradiction. Noting that $X - \mathrm{cl}_X(\{x\})$ is also an open set of X, so it is SI-open. Hence $\mathrm{cl}_X(\{x\}) \supseteq \mathrm{cl}_{SI(X)}(\{x\})$.

(2) A direct result of (1) and the definition of specialization order.

(3) This holds because every SI-closed set is closed in X.

□

Just like the characterization of Scott closed sets we also have the following.

PROPOSITION 3.4. *A closed set C of a T_0 space (X, τ) is SI-closed if and only if for any $F \in \mathrm{Irr}(X)$, $F \subseteq C$ implies $\bigvee F \in C$ whenever $\bigvee F$ exists.*

The following result shows that one can define the SI-open sets by using only closed irreducible sets.

PROPOSITION 3.5. *An open set U of space (X, τ) is SI-open if and only if for any closed irreducible set F, $\bigvee F \in U$ implies $F \cap U \neq \emptyset$ whenever $\bigvee F$ exists.*

PROOF. Only the sufficiency needs proof. Assume that U is an open subset of X satisfying the stated condition and G is an irreducible set (not necessarily closed) such that $\bigvee G$ exists and $\bigvee G \in U$. Then by Remark 2.2(3), $\bigvee \mathrm{cl}(G) = \bigvee G \in U$, implying $\mathrm{cl}(G) \cap U \neq \emptyset$. But the latter then implies $G \cap U \neq \emptyset$. Hence U is SI-open. □

The following is a straightforward consequence of Definition 3.1 and Proposition 2.1(4):

PROPOSITION 3.6. *If υ and τ are T_0 topologies on X such that $\upsilon \subseteq \tau$ and their specialization orders coincide, then $\upsilon_{SI} \subseteq \tau_{SI}$.*

REMARK 3.7. The condition of coincidence of specialization orders in the above proposition is essential as the following example shows. We first recall that the on a given poset P, the Scott topology $\sigma(P)$ is the derived topology of the Alexandroff topology, $\Upsilon(P)$. Now, consider

Υ on the set $X := \{0, 1, 2, \ldots, \top, \top_1, \top_2, \ldots, \top_\omega\}$ with the following distinct orders $\leq_1$ and $\leq_2$:

$$0 \leq_1 1 \leq_1 2 \leq_1 \ldots \leq_1 \top \leq_1 \top_1 \leq_1 \top_2 \leq_1 \ldots \top_\omega$$

and

$$0 \leq_2 1 \leq_2 2 \leq_2 \ldots \leq_2 \top_1 \leq_2 \top_2 \leq_2 \ldots \top_\omega, \quad 0 \leq_2 \top.$$

Clearly, $\Upsilon(X, \leq_2)$ is finer than that on $\Upsilon(X, \leq_1)$. But the set $\{0, 1, 2, \ldots\}$ is Scott-open in $(X, \leq_1)$ and not Scott open in $(X, \leq_2)$.

PROPOSITION 3.8. *Let X be a T_0 space and $U \subseteq X$. Then the following are equivalent:*

(i) *U is clopen in X.*
(ii) *U is clopen in $SI(X)$.*

In particular, if X is zero-dimensional (i.e., the clopen sets form a base for X), then so is $SI(X)$.

PROOF. (i) $\implies$ (ii): Assume, without loss of generality, that U is a non-trivial clopen subset of X. Let $G \in \mathrm{Irr}(X)$ with $\bigvee G \in U$. If $G \cap U = \emptyset$, then $G \subseteq X - U$, which is an upper set. Thus $\bigvee G \in X - U$, a contradiction. Hence $G \cap U \neq \emptyset$ and so U is SI-open. Similarly we can deduce that $X - U$ is also SI-open, U is SI-clopen.

(ii) $\implies$ (i): Trivial.

The last statement is immediate from the definition of zero-dimensional spaces. $\square$

Recall that a space X is *connected* if whenever $X = U \cup V$ for two disjoint clopen sets U and V, then either $U = \emptyset$ or $U = X$.

COROLLARY 3.9. *The following are equivalent for a T_0-space X:*

(i) *X is connected.*
(ii) *$SI(X)$ is connected.*

If a space X is compact, then obviously $SI(X)$ is compact. The following is another property preserved under the construction of the derived topology.

A topological space X is called a *d-space* (or *monotone convergence space*) if (i) X is a directed complete poset (dcpo, for short) with respect to the specialization order, and (ii) for any directed set D of X, D converges to $\bigvee D$ as a net (see for e.g., [**7**, p.133]). Equivalently, X is a d-space if and only if $(X, \leq)$ is a dcpo and all open sets are Scott open with respect to the specialization order. Since all SI-open sets of X are open sets and the specialization order of derived topology coincides with that of the original topology on X, we have the following:

Proposition 3.10. *If a space X is a d-space, then so is $SI(X)$.*

Let $X = [0, 1]$ be the usual chain of real numbers between 0 and 1. Then the Alexandroff topology on this poset, $(X, \Upsilon(X))$, is not a d-space. However, $SI((X, \Upsilon(X)) = \Sigma([0, 1])$ is a d-space, where $\Sigma([0, 1])$ is the Scott space of $[0, 1]$ (see Section 5). Thus, the converse of Proposition 3.10 need not hold.

4. k-bounded sober spaces

In this section, we introduce a new variant of sobriety, called *k-bounded sobriety*. The upshot of this section is to define an ordinal $\kappa(X)$ for each space X, called its *κ-index*, which somewhat reflects the degree of k-bounded sobriety of the space.

Definition 4.1. A space X is called *k-bounded sober* if for any non-empty irreducible closed set F whose supremum exists, there is a unique point $x \in X$ such that $F = \mathrm{cl}(\{x\})$.

Remark 4.2.
(1) Every sober space X is clearly k-bounded sober and every k-bounded sober space is T_0.
(2) A T_0 space is said to be *bounded sober* if every upper bounded closed irreducible set is the closure of a unique singleton. This weaker notion of sobriety was recently introduced and studied by the first author and T. Fan in [**17**], in which it is established that the subcategory of bounded sober spaces is reflective in the category of T_0-spaces, i.e., the inclusion functor has a left adjoint. It is now clear that the following implications hold:

$$\begin{array}{c}\text{hyper-}\\\text{sobriety}\end{array} \Longrightarrow \text{sobriety} \Longrightarrow \begin{array}{c}\text{bounded}\\\text{sobriety}\end{array} \Longrightarrow \begin{array}{c}\text{k-bounded}\\\text{sobriety}\end{array}.$$

(3) If the specialization order of X is a sup-complete order (i.e., in which every non-empty subset has a supremum) and X is k-sober, then X is sober.

Given a T_0 space which is not k-bounded sober, can one perform a sort of completion of it to a larger space which is k-bounded sober? We develop some machinery to perform just this.

For a T_0 space X, denote by $\mathrm{KB}(X)$ the set of all closed irreducible sets of X whose suprema exist. Thus, all sets of the form $\mathrm{cl}(\{x\})$ (for some $x \in X$) are members of $\mathrm{KB}(X)$.

For any closed set F of a T_0 space X, we write

$$K_F := \{A \in \mathrm{KB}(X) : A \subseteq F\}.$$

It is then straightforward to verify that $\{K_F : F \in \Gamma(X)\}$ indeed defines a co-topology on $\mathrm{KB}(X)$. We shall use $\mathrm{KB}(X)$ to denote the corresponding topological space.

REMARK 4.3. (1) For any T_0 space X and $F \in \Gamma(X)$, F is irreducible in X if and only if K_F is irreducible in $\mathrm{KB}(X)$.

(2) For each $A \in \mathrm{KB}(X)$, it holds that

$$\mathrm{cl}_{\mathrm{KB}(X)}(\{A\}) = K_A.$$

Hence the specialization order of $\mathrm{KB}(X)$ is the inclusion order.

THEOREM 4.4. *Let X be a T_0-space.*

(1) $\mathrm{KB}(X)$ *is k-bounded sober.*
(2) *The mapping* $\iota = (x \mapsto \mathrm{cl}(\{x\}) : X \hookrightarrow \mathrm{KB}(X)$ *is a dense embedding.*

PROOF. (1) Take an arbitrary irreducible closed set K_F of $\mathrm{KB}(X)$, whose supremum exists. Denote this supremum $\bigvee K_F$ by some $E \in KB(X)$. Then $F \subseteq E$ since for any $x \in F$, $\mathrm{cl}(\{x\}) \in K_F$. As $E \in \mathrm{KB}(X)$, $\bigvee E$ exists and we denote it by e. Now we show that $e = \bigvee F$. First, e is an upper bound of F since $F \subseteq E$. Now if a is an upper bound of F, then $\mathrm{cl}(\{a\}) \in \mathrm{KB}(X)$ and is an upper bound of K_F in $\mathrm{KB}(X)$. Thus, $E \subseteq \mathrm{cl}(\{a\})$ which implies that $e \leq a$. Hence $\bigvee F = e$, and so $F \in \mathrm{KB}(X)$. Now $K_F = \{A \in \mathrm{KB}(X) : A \subseteq F\} = \mathrm{cl}_{\mathrm{KB}(X)}(\{F\})$. It then follows that $\mathrm{KB}(X)$ is k-bounded sober.

(2) The mapping $\iota : X \longrightarrow \mathrm{KB}(X)$ that sends $x \in X$ to $\mathrm{cl}(\{x\})$ is easily seen to be an embedding. Now for any closed set K_F of $\mathrm{KB}(X)$, if $\mathrm{KB}(X) - K_F \neq \emptyset$ then there exists $H \in \mathrm{KB}(X)$ with $H \not\subseteq F$. Now choose an element $h \in H - F$. So, $\iota(h) \in \mathrm{KB}(X) - K_F$. Therefore, $\iota(X)$ is dense in $\mathrm{KB}(X)$. □

The result below connects the irreducibly derived topology with k-bounded sobriety.

THEOREM 4.5. *Let (X, τ) be a T_0 space. Then X is k-bounded sober if and only if $\tau = \tau_{SI}$.*

PROOF. Let (X, τ) be a k-bounded sober space and U be any τ-open set of X. If F is a closed irreducible set whose supremum, $\bigvee F$, exists and belongs to U, then $F = \mathrm{cl}(\{u\})$ where $u = \bigvee F$. Since $u \in F$ and $u \in U$, we have $F \cap U \neq \emptyset$. Thus, U is SI-open. It then follows that $\tau = \tau_{SI}$.

Now assume that $\tau = \tau_{SI}$. Let F be an irreducible closed set of (X, τ) whose supremum exists (which we denote by x). Then, as $\tau = \tau_{SI}$, F is also SI-closed. By Proposition 3.4, $x = \bigvee F \in F$. This then implies that $F = \mathrm{cl}(\{x\})$. Such an element x is unique because X is T_0. Hence X is k-bounded sober. $\square$

Let X be a T_0 space. For each ordinal α, we define a space $X^{(\alpha)}$ by transfinite induction as follows:

$$X^{(0)} = X, \quad X^{(\alpha+1)} = SI(X^{(\alpha)}),$$

and if α is a limit ordinal, then $X^{(\alpha)}$ is the space X with the topology that is the intersection of the topologies of $X^{(\beta)}$ with $\beta < \alpha$.

Since for any $\beta \leq \alpha$, $X^{(\alpha)} \subseteq X^{(\beta)} \subseteq X^{(0)}$, there is a smallest ordinal λ such that $X^{(\lambda)} = X^{(\lambda+1)}$.

Definition 4.6. For each T_0 space X, define the *κ-index* of X to be the smallest ordinal λ such that $X^{(\lambda)} = X^{(\lambda+1)}$. We denote this smallest ordinal λ by $\kappa(X)$.

Lemma 4.7. *Let X be a T_0 space. Then for any $x \in X$ and any ordinal α, the closures of $\{x\}$ taken respectively in X and in $X^{(\alpha)}$ coincide, i.e.,*

$$\mathrm{cl}_X(\{x\}) = \mathrm{cl}_{X^{(\alpha)}}(\{x\}).$$

Proof. We prove this by transfinite induction. The statement is true for $\alpha = 1$ by Lemma 3.3. Also, if it is true for α then again by Lemma 3.3 it is true for $\alpha + 1$.

Now assume that the statement is true for all $\alpha < \beta$ where β is a limit ordinal. Then $\mathrm{cl}(\{x\})$ is a closed set in all $X^{(\alpha)}$ where $\alpha < \beta$. Hence $\mathrm{cl}(\{x\})$ is closed in $X^{(\beta)}$. Furthermore, the topology of X is finer than that of $X^{(\gamma)}$ so that $\mathrm{cl}_X(\{x\}) \subseteq \mathrm{cl}_{X^{(\beta)}}(\{x\}) \subseteq \mathrm{cl}_X(\{x\})$, implying that $\mathrm{cl}_X(\{x\}) = \mathrm{cl}_{X^{(\beta)}}(\{x\})$. The proof is thus complete. $\square$

Corollary 4.8. *For any T_0 space X, the following conditions hold:*

1. *$X^{(\kappa(X))}$ is k-bounded sober.*
2. *The specialization order on the underlying set X induced by the topology of $X^{(\kappa(X))}$ coincides with that induced by the topology of X.*

For any poset $(P, \leq)$, the upper topology $\nu(P)$ on P is the smallest topology generated by sets of the form $X - \downarrow x$ for $x \in X$ (see Definition O-5.4 of [**7**]). In general, if τ is an *order compatible* topology on a poset P (i.e., the specialization order induced by τ agrees with the original order on P), then for any $x \in P, \mathrm{cl}_\tau(\{x\}) = \downarrow x$. Therefore, $\nu(P)$ is the coarsest order compatible topology on P. Additionally, if an order-compatible topology τ on a poset P is sober, then τ is also contained in the Scott topology $\sigma(P)$ of P (see 1.9 of [**13**]).

Since the irreducibly-derived topology of $(P, \nu(P))$ is an order compatible topology (by Proposition 3.3) which is coarser than $\nu(P)$ (by Definition 3.1), thus $SI(P, \nu(P)) = (P, \nu(P))$.

COROLLARY 4.9. *For any poset P, $(P, \nu(P))$ is k-bounded sober.*

If L is a sup-complete poset (every non-empty subset has a supremum), then every non-empty irreducible set has a supremum. Thus, if τ is an order-compatible topology on L, then τ is k-bounded sober if and only if it is sober. So, we obtain the following result:

COROLLARY 4.10. *If L is a sup-complete poset (in particular, if it is a complete lattice), then $(L, \nu(L))$ is sober.*

For any finite discrete poset, the upper-topology is discrete, and is thus sober. This shows that the converse of the above corollary does not hold in general.

THEOREM 4.11. *Let (X, τ) be a T_0 space. If υ is a k-bounded sober topology which is coarser than τ and induces the same specialization order as τ, then υ is coarser than τ_{SI}.*

PROOF. The desired result follows immediately from the set inclusions

$$\upsilon = \upsilon_{SI} \subseteq \tau_{SI}$$

by Theorem 4.5 and Proposition 3.6. □

COROLLARY 4.12. (1) *For any poset P, the topology of $(P, \Upsilon(P))^{(\lambda)}$ is the finest k-bounded sober order compatible topology on P, where $\lambda = \kappa(P, \Upsilon(P))$.*

(2) *If L is a sup-complete poset, then there is a finest sober order compatible topology on L.*

REMARK 4.13. The notion of k-bounded sobriety stands in stark contrast to sobriety: any poset P admits a k-bounded sober order compatible topology by Corollary 4.9, while Johnstone's famous counterexample in [**12**] is a dcpo which can never admit a sober order compatible topology. The Scott space of the dcpo constructed in [**13**] is not sober, but it is bounded sober (Example 4 of [**17**]). In [**11**], Isbell constructed a complete lattice L whose Scott topology is not sober, thus the Scott topology on this complete lattice is not k-bounded sober either. Note that since Isbell's construction, L, is a complete lattice, there is a finest sober order compatible topology on L by Corollary 4.12. This complete lattice thus contrasts against Johnstone's dcpo because the latter can never admit any sober order compatible topology.

EXAMPLE 4.14. Let $\mathbb{Q}$ be the poset of all rational numbers with the conventional order $\leq$. The set $B = \{x \in \mathbb{Q} : x < \sqrt{2}\}$ is an upper bounded irreducible closed set in the upper topology. However, there is no point $a \in \mathbb{Q}$ such that $B = \downarrow a$. Thus $(\mathbb{Q}, \nu(\mathbb{Q}))$ is k-bounded sober but not bounded sober.

Now let X be a T_0 space and $\lambda = \kappa(X)$. If F is an irreducible set of X such that $\bigvee F$ exists, then F is also such a type of set of $X^{(\lambda)}$. So, there is a unique point $x \in X$ such that $\mathrm{cl}_{X^{(\lambda)}}(\{x\}) = \mathrm{cl}_{X^{(\lambda)}}(F)$. But $\mathrm{cl}_{X^{(\lambda)}}(\{x\}) = \mathrm{cl}_X(\{x\})$, which leads us to the following:

THEOREM 4.15. *Let X be a T_0 space. Then for any irreducible set F of X whose supremum, $\bigvee F$, exists, there is a unique point $x \in X$ such that $\mathrm{cl}_X(\{x\}) = \mathrm{cl}_{X^{(\lambda)}}(F)$, where $\lambda = \kappa(X)$.*

5. Scott topology and SI-topology

Recall that a subset U of a poset P is called *Scott open* if (i) $U = \uparrow U$ and (ii) for any directed subset D, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$ whenever $\bigvee D$ exists. The Scott open sets on each poset P form the *Scott topology* $\sigma(P)$ (see [**7**, p.134]). The space $(P, \sigma(P))$ is denoted by $\Sigma(P)$. The Scott topology of P is order compatible.

REMARK 5.1.
(1) For certain classes of dcpos, such as the continuous dcpo's, the Scott topology is sober (see [**7**, p.141]). A poset P is *continuous* if for any $a \in P$, the set $\{y \in P : y \ll a\}$ is directed and $\bigvee\{y \in P : y \ll a\} = a$, where $y \ll a$ (read as y is *way-below* a) is defined by the condition that for any directed set D if $\bigvee D \geq a$ then $d \geq y$ for some $d \in D$.
(2) The chain $I = [0, 1]$ of real numbers between 0 and 1 is a continuous dcpo, so the Scott topology on I is sober. But I is obviously not Noetherian. Thus, the hyper-sobriety condition in Proposition 2.4 cannot be replaced by sobriety.

Let P be a poset. A non-empty subset $F \subseteq P$ is irreducible with respect to the Alexandroff topology $\Upsilon(P)$ iff it is a directed set. Thus, we have the following:

LEMMA 5.2. *For any poset P, $SI(P, \Upsilon(P)) = (P, \sigma(P)) = \Sigma(P)$.*

Thus, the irreducibly-derived topology of a T_0 space can be seen as a generalization of the Scott topology on a poset.

It is well known that a mapping $f : P \longrightarrow Q$ between two posets P and Q is a continuous mapping with respect to the Scott topologies on P and Q if and only if $f(\bigvee D) = \bigvee f(D)$ holds for any directed subset

$D \subseteq P$ whose supremum, $\bigvee D$, exists. A similar result holds for the SI-topology.

LEMMA 5.3. *A continuous mapping $f : (X, \tau) \longrightarrow (Y, \delta)$ is a continuous mapping between (X, τ_{SI}) and (Y, δ_{SI}) if and only if $f(\bigvee F) = \bigvee f(F)$ holds for any $F \in \mathrm{Irr}_\tau(X)$ whenever $\bigvee F$ exists.*

PROOF. Note that the continuous image of an irreducible set is irreducible. Thus, necessity can be shown in a rather straightforward manner.

Now assume that $f : (X, \tau_{SI}) \longrightarrow (Y, \delta_{SI})$ is continuous and F is an irreducible set of space X such that $\bigvee F$ exists. Note that every continuous mapping is a monotone mapping (with respect to the specialization orders). Thus $f(\bigvee F)$ is an upper bound of $f(F)$. Next, let $b \in Y$ be an upper bound of $f(F)$. The set $\downarrow b$ is δ_{SI}-closed, and so $F \subseteq f^{-1}(\downarrow b)$ and $f^{-1}(\downarrow b)$ is τ_{SI}-closed. Thus, $\bigvee F \in f^{-1}(\downarrow b)$, implying $f(\bigvee F) \in \downarrow b$. Hence $f(\bigvee F) = \bigvee f(F)$. $\square$

PROPOSITION 5.4. *The following statements are equivalent for a poset P:*

(1) *The Scott topology $\Sigma(P)$ is hyper-sober.*
(2) *P is Noetherian.*

PROOF. The necessity follows from Proposition 2.4 and the fact that $\Sigma(P)$ is order compatible.
Assume now that P is Noetherian. Then P is a dcpo as every directed set has a largest element. Also $x \ll x$ holds for all $x \in P$, forcing P to be continuous. Therefore, $(P, \sigma(P))$ is sober. If F is an irreducible set in P, then the closure $\mathrm{cl}(F)$ is an irreducible closed set. Also as P is Noetherian, we deduce that $\mathrm{cl}(F) = \downarrow F$. Now, there is $b \in \mathrm{cl}(F)$ such that $\mathrm{cl}(F) = \mathrm{cl}(\{b\})$ by the sobriety of $\sigma(P)$. Thus $\downarrow F = \downarrow b$, implying $b \in F$. Now $F \subseteq \mathrm{cl}(F) = \mathrm{cl}(\{b\})$ and $b \in F$. All these show that $(P, \sigma(P))$ is hyper-sober. $\square$

REMARK 5.5. Consider the chain $I = [0, 1]$ of real numbers in the unit interval. The Alexandroff topology on I consists of all the upper sets, which is different from the Scott topology $\sigma(I) = \{(r, 1] : r \in [0, 1]\} \cup \{I\}$. Thus, by Theorem 4.5 and Lemma 5.2 the Alexandroff topology on I is not k-bounded sober.

Let $E = [0, 1] \cap \mathbb{Q}$ with the usual order of numbers. The Scott topology on E is not sober because P is not a dcpo. However, if P is a chain and F is a Scott closed set with supremum given by $\bigvee F$, then as F is a directed set we have $\bigvee F \in F$. Thus, $F = \downarrow \bigvee F = \mathrm{cl}(\{\bigvee F\})$. Thus, we have the following:

Proposition 5.6. *For any chain P, $\Sigma(P)$ is k-bounded sober.*

The preceding result will be extended to continuous posets in Section 7.

Theorem 5.7. *The following statements are equivalent for a dcpo P:*

(1) *The Alexandroff topology is hyper-sober.*
(2) *The Alexandroff topology is sober.*
(3) *The Alexandroff topology on P is k-bounded sober.*
(4) *The Scott topology on P is hyper-sober.*
(5) *P is Noetherian.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) and (5) $\Longrightarrow$ (1) are trivially true. That (4) $\Longleftrightarrow$ (5) follows from Proposition 5.4. It remains to show that (3) $\Longrightarrow$ (5). Now assume that (3) holds. Then, by Lemma 5.2, $\sigma(P) = \Upsilon(P)$. Thus, every upper set is Scott open. Since the supremum of every directed subset exists, therefore every directed set must have a maximum element (if D is a directed set and $\bigvee D \notin D$, then the upper set $P- \downarrow D$ is not Scott open). Hence the poset P is Noetherian. $\square$

6. SI-continuous spaces

A complete lattice L is said to be *completely distributive* if for any set family $(u_j^i)_{j\in J_i}$, one for each $i \in I$, $\bigwedge_{i\in I} \bigvee_{j\in J_i} u_j^i = \bigvee_{f\in\prod_{i\in I} J_i} \bigwedge_{i\in I} u_{f(i)}^i$ holds. For any complete lattice L, define a binary relation $\lhd$ on L as follows: for $x, y \in L$, $x \lhd y$ if for any subset $A \subseteq L$, $\bigvee A \geq y$ implies $x \in \downarrow A$. A complete lattice L is *super-continuous* if $a = \bigvee\{x \in L : x \lhd a\}$ holds for any $a \in L$. Raney's theorem asserts that L is completely distributive if and only if L is super-continuous [**16**].

For any topological space X, let $(\mathcal{O}(X), \subseteq)$ denote the lattice of open sets of X. The open set lattice $(\mathcal{O}(X), \subseteq)$ is complete and distributive. For two elements U and V of $(\mathcal{O}(X), \subseteq)$, $U \lhd V$ is thus taken to mean that for any family $\mathcal{A} \subseteq \mathcal{O}(X)$, $\bigcup \mathcal{A} \supseteq V$ implies the existence of $A \in \mathcal{A}$ such that $U \subseteq A$.

A T_0 space X is called a *C-space* [**3**] if

$$\forall U \in \mathcal{O}X.\ \forall a \in U.\ \exists b \in U.\ a \in \mathrm{int}_X(\uparrow b),$$

It was proved in [**3**] (also see [**1, 10**]) that a T_0 space is a C-space if and only if the lattice $(\mathcal{O}(X), \subseteq)$ is completely distributive. The following result is well-known (see [**3**]):

LEMMA 6.1. *Let X be a T_0 space. Consider the following conditions on the elements U and V of the open set lattice $(\mathcal{O}(X), \subseteq)$:*

(1) $U \lhd V$.
(2) *U has a lower bound in V.*
(3) *$U \subseteq \mathrm{int}_X(\uparrow u)$ for some $u \in V$.*

Then (2) and (3) are equivalent and (2) implies (1).
If X is a C-space, then all the three conditions are equivalent.

The following is an important result in domain theory.

PROPOSITION 6.2. [**8**, Proposition 5.1.37] *Let P be a poset. Then P is continuous if and only if $(\sigma(P), \subseteq)$ is a completely distributive lattice.*

Let X be a T_0 space. For $x,\ y \in X$, define $x \ll_{SI} y$ if for any irreducible set F, $y \leq \bigvee F$ implies $x \in \downarrow F$ whenever $\bigvee F$ exists. We denote the set $\{x \in X : x \ll_{SI} a\}$ by $\twoheaddownarrow_{SI} a$, and the set $\{x \in X : a \ll_{SI} x\}$ by $\twoheaduparrow_{SI} a$.

REMARK 6.3. Let X be a T_0 space.

(1) If $x \ll_{SI} y$, then $x \leq y$. Also, $x \leq y \ll_{SI} z \leq w$ implies $x \ll_{SI} w$.
(2) Note that every open set is an upper set. Thus, for any $x,\ y \in X$, if $y \in \mathrm{int}_{SI(X)}(\uparrow x)$ then $x \ll_{SI} y$.

A T_0 space X is called *SI-continuous* if for any $a \in X$, the following conditions hold:

(i) $\twoheaduparrow_{SI} a$ is open in X.
(ii) $\twoheaddownarrow_{SI} a$ is a directed set.
(iii) $\bigvee \twoheaddownarrow_{SI} a = a$.

THEOREM 6.4. *Let X be a T_0 space. Then the following statements are equivalent:*

(1) *$SI(X)$ is a C-space.*
(2) *X is SI-continuous.*

PROOF. (1) $\implies$ (2): Let $SI(X)$ be a C-space. First, note that if $a \in \mathrm{int}_{SI(X)}(\uparrow x)$ then $x \ll_{SI} a$. If $a,\ b \in X$ such that $a \not\leq b$, then $a \in X- \downarrow b$, which is SI-open. Since $SI(X)$ is a C-space, there is $x \in X- \downarrow b$ such that $a \in \mathrm{int}_{SI(X)}(\uparrow x)$. From this, it follows that for any $a \in X$, $\bigvee\{x \in X : a \in \mathrm{int}_{SI(X)}(\uparrow x)\} = a$.

If $a \in \mathrm{int}_{SI(X)}(\uparrow x_1), a \in \mathrm{int}_{SI(X)}(\uparrow x_2)$, then $a \in \mathrm{int}_{SI(X)}(\uparrow x_1) \cap \mathrm{int}_{SI(X)}(\uparrow x_2)$, which is SI-open, so there is $x_3 \in \mathrm{int}_{SI(X)}(\uparrow x_1) \cap \mathrm{int}_{SI(X)}(\uparrow x_2)$ such that $a \in \mathrm{int}_{SI(X)}(\uparrow x_3)$. Thus the set $H_a = \{x \in$

$X : a \in \mathrm{int}_{SI(X)}(\uparrow x)\}$ is a directed set and $H_a \subseteq \twoheaddownarrow_{SI} a$. Now H_a is irreducible and $\bigvee H_a = a$, thus we also have $\twoheaddownarrow_{SI} a \subseteq \downarrow H_a$. Hence $\twoheaddownarrow_{SI} a = \{x \in X : a \in \mathrm{int}_{SI(X)}(\uparrow x)\} = \downarrow H_a$ is a directed set and has a as its supremum.

In addition, from the above argument we see that

$$x \in \twoheaduparrow_{SI} b \iff \exists y \in \uparrow b.\ x \in \mathrm{int}_{SI(X)}(\uparrow y).$$

Thus, for any $b \in X$, it holds that $\twoheaduparrow_{SI} b = \bigcup_{y \in \uparrow b} \mathrm{int}_{SI(X)}(\uparrow y)$, which is open in X. Therefore, X is SI-continuous.

(2) $\implies$ (1): Let X be SI-continuous. Let $a \ll_{SI} b$. We show that $b \in \mathrm{int}_{SI(X)}(\uparrow a)$. The set $\twoheaddownarrow_{SI} b$ is a directed set whose supremum equals b. Then, as the union of a directed family of directed subsets, $\bigcup\{\twoheaddownarrow_{SI} z : z \in \twoheaddownarrow_{SI} b\}$ is a directed set (hence an irreducible set) whose supremum is b. Thus, there is $z_1 \ll_{SI} b$ such that $a \leq x \ll_{SI} z_1$ for some $x \in X$. So, $a \ll_{SI} z_1 \ll_{SI} b$. Inductively, we can obtain an infinite chain

$$a \ll_{SI} \cdots z_n \ll_{SI} z_{n-1} \ll_{SI} \cdots \ll_{SI} z_1 \ll_{SI} b.$$

For each $n \in \mathbb{N}$, the set $\twoheaduparrow_{SI} z_n$ is open in X by the first condition of the SI-continuity of X. Thus, $V := \bigcup_{n \in \mathbb{N}} \twoheaduparrow_{SI} z_n$ is open in X. Moreover, U is inaccessible by the suprema of irreducible sets (if F is irreducible and $\bigvee F \in U$, then $\bigvee F \in \twoheaduparrow_{SI} z_n$ for some n, thus there is $y \in F$ such that $y \geq z_n$, implying $y \in \twoheaduparrow_{SI} z_{n+1} \subseteq U$), so V is an SI-open set with $b \in V \subseteq \mathrm{int}_{SI(X)}(\uparrow a)$.

Now if U is SI-open and $b \in U$, then $\bigvee\{x \in X : x \ll_{SI} b\} = b \in U$ implying $x \in U$ for some $x \ll_{SI} b$ because $\{x \in X : x \ll_{SI} b\}$ is directed and hence irreducible. By the above argument, $b \in \mathrm{int}_{SI(X)}(\uparrow x)$. Therefore $SI(X)$ is a C-space. $\square$

Note that for a poset P, $x \ll y$ if and only if $x \ll_{SI} y$, where the topology on P is taken to be the Alexandroff topology. Thus, the above result is a generalization of Proposition 6.2.

7. SI-compactness

Given a topological space X, X is compact if and only if for any directed open cover $\mathcal{U}$ for X, there is already an open set $U \in \mathcal{U}$ with $X = U$. As noted above, each directed family $\mathcal{U}$ of open sets is an irreducible subset of $\Sigma\mathcal{O}(X)$, where $\mathcal{O}$ is the lattice of opens of X. We conclude the present paper with this section in which we consider the following variant of compactness.

Definition 7.1. A topological space X is called *SI-compact* if for any Scott irreducible subset $\mathcal{U}$ of $\mathcal{O}(X)$, $\bigcup \mathcal{U} = X$ implies $X = U$ for some $U \in \mathcal{U}$.

Since every directed subset of $\mathcal{O}(X)$ is Alexandroff irreducible in $\mathcal{O}(X)$ and hence Scott irreducible, it follows that every SI-compact space is compact.

Proposition 7.2. *If $f : X \longrightarrow Y$ is a surjective continuous map and X is SI-compact, then Y is SI-compact.*

Proof. Let $f^{-1} : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ be the map that sends $U \in \mathcal{O}(Y)$ to $f^{-1}(U)$. Because f^{-1} preserves arbitrary suprema, it is Scott continuous mapping (see the remark preceding Lemma 5.3). If $\mathcal{A} \subseteq \mathcal{O}(Y)$ is a Scott irreducible family of open sets of Y (i.e., a Scott irreducible subset of $\mathcal{O}(Y)$) such that $\bigcup \mathcal{A} = Y$, then $f^{-1}(\mathcal{A})$ is a Scott irreducible subset of $\mathcal{O}(X)$ by Proposition 2.1(2). In other words, $f^{-1}(\mathcal{A})$ is a Scott irreducible family of open sets of X with $\bigcup f^{-1}(\mathcal{A}) = X$. Thus, there is a $A \in \mathcal{A}$ such that $f^{-1}(A) = X$, which then implies $A = Y$. So Y is SI-compact, as desired. $\square$

Remark 7.3.

1. If r is a join-irreducible element of a completely distributive lattice L (i.e., $r \leq x \vee y$ implies $r \leq x$ or $r \leq y$), then $\beta(r) := \{x \in L : x \lhd r\}$ is a directed set (by Lemma 1 of [**18**]). In addition, if $E \subseteq \beta(r)$ and $\bigvee E = r$ then E is also directed.
2. Let P be a continuous poset. Then the lattice $\sigma(P)$ of Scott opens is a completely distributive lattice. The lattice $\Gamma(L)$ of all Scott closed sets is dually isomorphic to the lattice $\sigma(P)$, so it is also completely distributive lattice (by Lemma VII 1.10 of [**13**]).

Lemma 7.4. *If F is an irreducible subset of a C-space X, then $\downarrow F$ contains a directed subset D with the same set of upper bounds of F. In particular, $\bigvee D = \bigvee F$, if either exists.*

Proof. Set $D = \{x \in X : \mathrm{int}(\uparrow x) \cap F \neq \emptyset\}$. If $y,\ z \in D$, then $\mathrm{int}(\uparrow y) \cap \mathrm{int}(\uparrow z) \cap F \neq \emptyset$ since F is irreducible. Pick x in this intersection. Since X is a C-space, there exists $w \in \mathrm{int}(\uparrow y) \cap \mathrm{int}(\uparrow z)$ such that $x \in \mathrm{int}(\uparrow w)$. Then $w \in D$ and $y,\ z \leq w$. It follows from the definition of a C-space that every element of F is the directed supremum of members of D. Thus, every upper bound of D will be an upper bound of F, and the converse is immediate. $\square$

Corollary 7.5. *The irreducible sets and the directed sets induce the same derived topology for C-spaces.*

Corollary 7.6. *Let P be a continuous dcpo. For any Scott irreducible subset F of $\Sigma(P) = (P, \sigma(P))$, there is a directed D such that $D \subseteq \downarrow F$ and $\bigvee D = \bigvee F$. In particular, the supremum of a Scott irreducible set always exists.*

Theorem 7.7. *If X is T_0 locally compact space, then it is compact if and only if it is SI-compact.*

Proof. The sufficiency is true as every SI-compact space is compact. It remains to prove the necessity. For this purpose, let X be a T_0 space which is both locally compact and compact. The complete lattice $\mathcal{O}(X)$ is then a continuous lattice (see Example I-1.7(5) of [**7**] or Lemma 4.2 of [**13**]). If $\mathcal{U}$ is a Scott irreducible subset of $\Sigma(\mathcal{O}(X))$ and $\bigcup \mathcal{U} = X$, then by Corollary 7.6 there is a directed set $\{V_i : i \in I\}$ of open sets of X such that (i) each V_i is contained in some $U \in \mathcal{U}$, and (ii) $\bigvee \{V_i : i \in I\} = \bigcup\{V_i : i \in I\} = \bigcup \mathcal{U} = X$. Since X is compact, there is $i_0 \in I$ such that $X \subseteq V_{i_0}$. Surely, $V_{i_0} \subseteq U$ for some $U \in \mathcal{U}$, and thus, $X = U$. Therefore, X is SI-compact. □

Remark 7.8. In proving the above Theorem, the essential condition we make use of is that the lattice $\mathcal{O}(X)$ of opens of X is a continuous lattice. A space with this property is called a *core compact* space. Note that a core compact space need not be locally compact (see VII 4.2 of [**13**]).

It is well-known that the Scott topology $\Sigma(P)$ for continuous posets need not be sober. The following result further distinguished k-bounded sobriety from sobriety.

Theorem 7.9. *For any continuous poset P, $\Sigma(P)$ is k-bounded sober.*

Proof. By Proposition 6.2, the Scott topological space, $\Sigma(P)$, on a continuous poset P is a C-space. Let F be any non-empty irreducible Scott closed subset of P whose supremum, $\bigvee F$, exists. Since F is Scott closed, $F = \downarrow F$. By Lemma 7.4, there exists a directed subset D of F such that D and F have a common set of upper bounds, and in particular, since $\bigvee F$ exists by assumption, $\bigvee D$ exists and $\bigvee D = \bigvee F$. Since F is Scott closed and D is a directed subset of F, it follows that $\bigvee D \in F$. Hence $\bigvee F \in F$ implies that $F = \downarrow \bigvee F$, and so $\Sigma(P)$ is k-bounded sober. □

8. Concluding remarks

This paper introduces a method of deriving a new topology out of a given one, which is motivated by the definition of the Scott topology.

The results we report here can be seen as a first step towards investigating this new topology (i.e., the SI-topology), and indeed there are several basic questions to which we possess no answers. For instance, it is not known whether KB(X) is the canonical k-bounded sobrification of X in the sense of [**14**]. Also, we do not know how to define $L^{(\lambda)}(X)$ without using the ordinals. We are also quite ignorant about the properties of SI-compact spaces at the moment: (1) Is there a compact space which is not SI-compact? (2) Is the product of two SI-compact spaces is SI-compact?

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