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FILTER CONVERGENCE STRUCTURES ON POSETS

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Abstract

Abstract. *In this paper, we introduce two convergence structures on each poset and thus embed the category of posets and Scott continuous maps into the category of convergence spaces which is cartesian-closed. More specifically, for each poset P, we define two convergence spaces $(P, \downarrow_d)$ and $(P, \downarrow_c)$. The convergence space $(P, \downarrow_c)$ was first constructed by Heckmann for directed complete posets. The main results of our investigation are: (i) $(P, \downarrow_d)$ induces the Scott topology on P; (ii) if P is a continuous poset, then $\downarrow_c = \downarrow_d$ but in general they are different; (iii) $(P, \downarrow_d)$ is topological iff P is a continuous poset; (iv) $(P, \downarrow_d)$ is topological iff it is pretopological; (v) if Y is a topological space then the function space $[P_d \to Y]$ is topological, where $P_d = (P, \downarrow_d)$; (vi) $(P \times Q)_d$ is homeomorphic to $P_d \times Q_d$ for any posets P and Q; (vii) the meet-continuity and strongly meet-continuity are equivalent for directed-complete meet semilattices; (viii) for a meet-semilattice P, $(P, \downarrow_d)$ is a convergence meet-semilattice iff $(P, \leq)$ is meet-continuous.*

1 Introduction

Directed complete posets are fundamental structures for domain theory, which was first introduced by Dana Scott to give the denotational semantics for untyped λ-calculus. Dag Normann [4] has recently shown that the fully abstract model for PCF of hereditarily-sequential functionals is not ω-complete and therefore not a domain in the traditional sense. Thus, the consideration of non directed complete continuous posets becomes a necessity.

For any poset P, there is a naturally defined topology $\sigma(P)$ on P, called the Scott topology on P. A mapping $f : P \to Q$ between two posets is continuous with respect to the Scott topologies on P and Q if and only if it preserves the existing suprema of directed subsets. Such mappings are called Scott continuous mappings. Let POS_d denote the category of all posets and Scott continuous mappings. The category POS_d contains a number of full subcategories which feature strongly in theoretical computer science, such as the category $DCPO$ of all directed complete posets, the category $CONPOS$ of all continuous posets and the category DOM of all domains. The assignment of (P, σ) to each poset P extends to a functor from POS_d to the category TOP_0 of T_0 topological spaces, thus embedding POS_d into TOP_0 as a full subcategory of TOP_0. However, as Heckmann [1] pointed out, for directed complete posets, there are at least two shortcomings for this embedding: (i) the category TOP_0 is not cartesian-closed; (ii) the embedding does not preserve products. Heckmann first considered this problem for the category $DCPO$. For each dcpo (directed complete poset) he defined a filter convergence space $(P, \downarrow_c)$, thus embedding $DCPO$ into the category $CONV$ of filter convergence spaces as a full subcategory. In the current paper, we define another convergence space $(P, \downarrow_d)$ for each poset, thus obtaining a functor from POS_d to $CONV$. In section 2, we give some preliminaries on Scott topology and convergence spaces. In section 3, we define $(P, \downarrow_d)$ and $(P, \downarrow_c)$ for any poset P and study their basic properties. In section 4, we investigate the relationship between the meet-continuity (strongly meet-continuity) of P and the property of $(P, \downarrow_d)$.

2 Scott topology on posets and convergence spaces

A subset A of a poset P is directed if it is nonempty and each pair of elements of A has an upper bound in A. A poset P is called a dcpo if every directed subset of P has a supremum (i.e. least upper bound). Let P be a poset. A subset $A \subseteq P$ is called a lower set if $A = \downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$. The upper sets in a poset are defined dually. Define $A^{\downarrow} = \{d \in P : d \leq a \text{ for all } a \in A\}$, which is a set of lower bounds of any set A. A subset $U \subseteq P$ is called a Scott open set if (i) U is an upper set, and (ii) for each directed set $D \subseteq P$ that has a supremum in P, $\bigvee D \in U$ implies $D \cap U \neq \emptyset$. All Scott open sets of P form a topology on P, called the Scott topology and is denoted by $\sigma(P)$. A subset

$A \subseteq P$ is Scott closed if $P \backslash A$ is Scott open. Thus A is Scott closed iff (i) A is a lower set and (ii) for any directed set $D \subseteq A$, $\bigvee D \in A$ whenever $\bigvee D$ exists.

A mapping $f : P \to Q$ between posets is Scott continuous if it is continuous with respect to the Scott topologies on P and Q. The following lemma can be proved in the same way as for dcpos. (See [3] for the proof for dcpos.)

Lemma 1. *A mapping $f : P \to Q$ is Scott continuous iff it preserves the existing suprema of directed sets, that is $f(\bigvee D) = \bigvee f(D)$ holds for any directed set $D \subseteq P$ with $\bigvee D$ exists.*

An element a of a poset P is way-below an element b, denoted by $a \ll b$, if for every directed set D that has a supremum and $b \leq \bigvee D$, there exists $d \in D$ such that $a \leq d$.

Definition 1. *A poset P is continuous iff for every $y \in P$, $\{x \in P : x \ll y\}$ is a directed set and $\bigvee\{x \in P : x \ll y\} = y$.*

A lattice is called completely distributive iff it is complete and for any family $\{x_{j,k} : j \in J, k \in K(j)\}$ in L the identity $\bigwedge_{j \in J} \bigvee_{k \in K(j)} x_{j,k} = \bigvee_{f \in M} \bigwedge_{j \in J} x_{j,f(j)}$ holds, where M is the set of choice functions defined on J with values $f(j) \in K(j)$. The following lemma can be proved in a similar way as for dcpos (see [3]).

Lemma 2. *A poset P is continuous iff $\sigma(P)$ is a completely distributive lattice.*

A filter $\mathscr{A}$ on a set X is a collection of subsets of X such that (i) $A, B \in \mathscr{A}$ implies $A \cap B \in \mathscr{A}$, and (ii) $A \in \mathscr{A}$ and $A \subseteq B$ implies $B \in \mathscr{A}$. A filter base $\mathscr{B}$ on a set X is a subset of the powerset $\mathscr{P}X$ satisfying the conditions:

(i) If $B \in \mathscr{B}$ then $B \neq \emptyset$.

(ii) If $B_1, B_2 \in \mathscr{B}$ then there exists $B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Let $\mathscr{B}$ be a filter base on X. Then the smallest filter containing $\mathscr{B}$ will be denoted by $[\mathscr{B}]$, called the filter generated by $\mathscr{B}$. Clearly, $[\mathscr{B}] = \{A \subseteq X : A \supseteq B \text{ for some } B \in \mathscr{B}\}$. We use $[x]$ to denote the filter generated by the base $\{\{x\}\}$ which is called a principal filter.

Definition 2. *A (filter) convergence space is a set X together with a relation '$\downarrow$' between ΦX (the collection of all filters on X) and X such that the following two axioms hold:*

(i) $[x] \downarrow x$ for all $x \in X$ (point filter axiom).

(ii) $\mathscr{A} \downarrow x$ and $\mathscr{B} \supseteq \mathscr{A}$ implies $\mathscr{B} \downarrow x$ (subfilter axiom).

For every topological space (X, τ), there is a corresponding convergence space $(X, \downarrow_\tau)$ defined by $\mathscr{A} \downarrow x$ iff $\mathscr{N}(x) \subseteq \mathscr{A}$, where $\mathscr{N}(x)$ denotes the collection of all neighbourhoods of x. A convergence space $(X, \downarrow)$ is topological if $\downarrow = \downarrow_\tau$ for some topology τ on X.

Conversely, given a convergence space $(X, \downarrow)$, we can define a topology $\tau_\downarrow$ on X. For any $O \subseteq X$, $O \in \tau_\downarrow$ iff for any filter $\mathscr{A}$, $\mathscr{A} \downarrow x$ and $x \in O$ imply $O \in \mathscr{A}$. The topological space $(X, \tau_\downarrow)$ is called the topological space induced by the convergence space $(X, \downarrow)$.

Definition 3. *For a poset P, define the relation $\downarrow_d$ between filters and points as follows: $\mathscr{A} \downarrow_d x$ if there exists a directed set $D \subseteq P$ such that (i) $x \leq \bigvee D$, and (ii) for each $d \in D$ there is $A \in \mathscr{A}$ with $d \in A^\downarrow$.*

For the sake of convenience, we shall use P_d to denote $(P, \downarrow_d)$ at times.

Obviously, condition (ii) in the above definition is equivalent to $D \subseteq \bigcup_{A \in \mathscr{A}} A^\downarrow$.

It is easy to verify that $(P, \downarrow_d)$ is a convergence space.

Lemma 3. *For any poset P, the following hold:*

(i) If $D \subseteq P$ is a directed set, then $\{\uparrow d : d \in D\}$ is a filter base on P.

(ii) For any directed set $D \subseteq P$ with $\bigvee D$ exists, we have $[\{\uparrow d : d \in D\}] \downarrow_d x$, where $x = \bigvee D$.

Lemma 4. *For each poset P, $(P, \downarrow_d)$ induces the Scott topology on P.*

Proof. Let $U \subseteq P$ be open in the induced topology. We show that U is Scott open. Let $x \in U$ and $x \leq y$, i.e. $y \in \uparrow x$. Note that $[\uparrow x] \downarrow_d x$, because there is a directed set $D = \{x\}$ with $\bigvee D = x$ and for $x \in D$, $x \in (\uparrow x)^\downarrow$. Thus $U \in [\uparrow x]$, i.e. $\uparrow x \subseteq U$, implying $y \in U$. Thus U is an upper set. Now assume that $D \subseteq P$ is a directed set with $\bigvee D$ exists and $\bigvee D \in U$. By Lemma 3 (ii), $[\{\uparrow d : d \in D\}] \downarrow_d \bigvee D$ and thus $U \in [\{\uparrow d : d \in D\}]$, i.e. $\uparrow d \subseteq U$ for some $d \in D$, which implies $D \cap U \neq \emptyset$. Therefore U is Scott open.

Conversely, let U be Scott open in P and $\mathscr{A} \downarrow_d x$, for some $x \in U$. Then there exists a directed set $D \subseteq P$ with $D \subseteq \bigcup_{A \in \mathscr{A}} A^\downarrow$ and $x \leq \bigvee D$. Since U is Scott open and $x \in U$, so $\bigvee D \in U$. Thus, there exists $d \in D \cap U$. Let $d \in A^\downarrow$ for some $A \in \mathscr{A}$. Then $d \leq a$ for all $a \in A$. Since $d \in U$ and U is an upper set, so $A \subseteq U$ and hence $U \in \mathscr{A}$. Therefore U is open in the induced topology. □

Let (X, τ) be a topological space. The specialisation preorder $\leq$ on X is the pre-partial order defined by: $x \leq y$ iff $x \in \text{cl}(\{y\})$, where $\text{cl}(\{y\})$ is the closure of $\{y\}$. This specialisation preorder is a partial order iff X is a T_0 space. The induced preorder of a convergence space X is defined to be the specialisation preorder of its induced topology.

Remark 1. *For every poset P, the specialisation order of $(P, \sigma(P))$ coincides with the order on P [3].*

Corollary 1. *For every poset P, $(P, \downarrow_d)$ induces the order on P.*

3 Comparison between $(P, \downarrow_d)$ and $(P, \downarrow_c)$

In [1], Heckmann defined a convergence structure $(P, \downarrow_c)$ for every dcpo as follows: For every dcpo P and filter $\mathscr{A}$, $\mathscr{A} \downarrow_c x$ iff $x \in \mathrm{cl}(\bigcup_{A\in\mathscr{A}} A^{\downarrow})$ where 'cl' is the closure operator in the Scott topology. Obviously this definition can be applied to any poset P instead of just dcpos.

Lemma 5. *For any poset P, $\downarrow_d \subseteq \downarrow_c$ holds.*

Proof. Let $\mathscr{A} \downarrow_d x$. Then there is a directed set $D \subseteq \bigcup_{A\in\mathscr{A}} A^{\downarrow}$ with $x \leq \bigvee D$. Then $D \subseteq \mathrm{cl}(\bigcup_{A\in\mathscr{A}} A^{\downarrow})$ and hence $\bigvee D \in \mathrm{cl}(\bigcup_{A\in\mathscr{A}} A^{\downarrow})$. It follows that $x \in \mathrm{cl}(\bigcup_{A\in\mathscr{A}} A^{\downarrow})$ because every Scott closed set is a lower set. Hence $\mathscr{A} \downarrow_c x$. $\square$

A lower subset A of P has one step closure if $x \in \mathrm{cl}(A)$ implies $x \leq \bigvee D$ for a directed $D \subseteq A$. A poset P is said to have one step closure if every lower subset has one step closure.

A subset B of a poset P is called a lower cut if $B = A^{\downarrow}$ for some $A \subseteq P$.

Lemma 6. *For a poset P, $\downarrow_d = \downarrow_c$ iff every union of directed collection of lower cuts has one step closure.*

Proof. If $\downarrow_d = \downarrow_c$ then for any filter $\mathscr{A}$ and $x \in \mathrm{cl}(\bigcup_{A\in\mathscr{A}} A^{\downarrow})$, we have $\mathscr{A} \downarrow_c x$ and so $\mathscr{A} \downarrow_d x$. Then $x \leq \bigvee D$ for some directed set $D \subseteq \bigcup_{A\in\mathscr{A}} A^{\downarrow}$. Hence $\bigcup_{A\in\mathscr{A}} A^{\downarrow}$ has one step closure. Now if $\mathscr{F} = \{F_i : i \in I\}$ is a collection of directed lower cuts, then $\mathscr{B} = \{F_i^{\uparrow} : i \in I\}$ is a filter base. Note that for any $F_i \in \mathscr{F}$, $F_i = F_i^{\uparrow\downarrow}$ because F_i is a lower cut. So $\bigcup \mathscr{F} = \bigcup\{B^{\downarrow} : B \in \mathscr{B}\} = \bigcup\{A^{\downarrow} : A \in [\mathscr{B}]\}$ and hence $\bigcup \mathscr{F}$ has one step closure.

Conversely, if the union of every directed collection of lower cuts has one step closure, then $\bigcup_{A\in\mathscr{A}} A^{\downarrow}$ has one step closure for any filter $\mathscr{A}$. Now, if $\mathscr{A} \downarrow_c x$, then $x \in \mathrm{cl}(\bigcup_{A\in\mathscr{A}} A^{\downarrow})$ so $x \leq \bigvee D$ for some directed $D \subseteq \bigcup_{A\in\mathscr{A}} A^{\downarrow}$, which implies $\mathscr{A} \downarrow_d x$. By Lemma 5, $\downarrow_d = \downarrow_c$. $\square$

Thus, in particular, if P has one step closure, then $\downarrow_d = \downarrow_c$.

Lemma 7. *Every continuous poset has one step closure.*

Proof. Suppose P is a continuous poset and $A \subseteq P$, $A = \downarrow A$. Let $A' = \{x \in P : \text{there exists a directed } E \subseteq A, \bigvee E \geq x\}$. We show that A' is Scott closed. Obviously A' is a lower set, so we only need to show that it is closed under existing suprema of directed sets. Let $D \subseteq A'$ be directed for which $\bigvee D$ exists. For each $d \in D$, $\Downarrow d = \{x \in P : x \ll d\}$ is a directed set with $\bigvee \Downarrow d = d$, because P is continuous. Let $C = \bigcup\{\Downarrow d : d \in D\}$. Then C is a directed subset of A with $\bigvee C = \bigvee\{\bigvee(\Downarrow d) : d \in D\} = \bigvee D$. Now for each $d \in D$ and $x \in \Downarrow d \subseteq C$, there is a directed subset $K \subseteq A$ such that $d \leq \bigvee K$, which implies $x \leq k$ for some $k \in K$. Thus $x \in A$ for each $x \in C$. Therefore $\bigvee D = \bigvee C \in A'$ because $C \subseteq A$. $\square$

Corollary 2. *For any continuous poset P, $\downarrow_d = \downarrow_c$.*

The next lemma shows that for each complete lattice L, $\downarrow_c$ and $\downarrow_d$ coincide. Note that such an L does not necessarily have one step closure.

Lemma 8. *Let L be a complete lattice and $\mathscr{A}$ a filter on L. The following statements are equivalent for any $x \in L$:*
(1) $\mathscr{A} \downarrow_d x$.
(2) $x \leq \bigvee_{A\in\mathscr{A}} \bigwedge A$.
(3) $\mathscr{A} \downarrow_c x$.

Proof. (1) implies (3): This follows from Lemma 5.

(3) implies (2): Suppose $\mathscr{A} \downarrow_c x$. Then $x \in \mathrm{cl}(\bigcup_{A\in\mathscr{A}} A^{\downarrow})$. Since L is a complete lattice, for any subset $A \in \mathscr{A}$, $A^{\downarrow} = \downarrow \bigwedge A \subseteq \downarrow (\bigvee_{A\in\mathscr{A}} \bigwedge A)$. Thus $\mathrm{cl}(\bigcup_{A\in\mathscr{A}} A^{\downarrow}) \subseteq \downarrow (\bigvee_{A\in\mathscr{A}} \bigwedge A)$ because $\downarrow (\bigvee_{A\in\mathscr{A}} \bigwedge A)$ is a Scott closed set. Hence $x \leq \bigvee_{A\in\mathscr{A}} \bigwedge A$.

(2) implies (1): Suppose $x \leq \bigvee_{A\in\mathscr{A}} \bigwedge A$. Then $x \leq \bigvee D$ where $D = \{\bigwedge A : A \in \mathscr{A}\}$. As D is obviously a directed set and $D \subseteq \bigcup_{A\in\mathscr{A}} A^{\downarrow}$, so $\mathscr{A} \downarrow_d x$. $\square$

We now give an example of a dcpo for which $\downarrow_c \neq \downarrow_d$. First, note that for any filter base $\mathscr{B}$, $\bigcup_{A\in[\mathscr{B}]} A^{\downarrow} = \bigcup_{B\in\mathscr{B}} B^{\downarrow}$.

Example 1. *Let $P = \{x^i_n : i, n \in \mathbb{N}\} \cup \{x^i : i \in \mathbb{N}\} \cup \{a_n : n \in \mathbb{N}\} \cup \{\top\}$. Let the order on P be defined by:*
$\forall i \in \mathbb{N}.\ x^i_1 \leq x^i_2 \leq ... \leq x^i_k \leq ... \leq x^i \leq x^{i+1} \leq ... \leq \top$,
and
$\forall i, k \in \mathbb{N}.\ x^i_k \leq a_k \leq a_{k+1} \leq ... \leq \top$.
Then P is a dcpo and

(a) $\bigvee\{x_k^i : k \in \mathbb{N}\} = x^i, \forall i \in \mathbb{N}$*;*

(b) $\bigvee\{a_k : k \in \mathbb{N}\} = \bigvee\{x^k : k \in \mathbb{N}\} = \top$*.*

Now for each k, let $A_k = \{a_i : i \geq k\}$*. Then* $\{A_k : k \in \mathbb{N}\}$ *is a filter base. Let* $\mathscr{A}$ *be the filter generated by* $\{A_k : k \in \mathbb{N}\}$*.*

We note that ${A_1}^{\downarrow} = \{x_1^1, x_1^2, ..., a_1\}, {A_2}^{\downarrow} = \{x_2^1, x_2^2, ..., a_2\}, ..., {A_k}^{\downarrow} = \{x_k^1, x_k^2, ..., a_k\}$ *and* $\top \in cl(\bigcup_{A\in\mathscr{A}} A^{\downarrow})$*.*

Therefore, in P_c*, the filter* $\mathscr{A}$ *converges to* $\top$*. On the other hand, we see that the elements in each of the sets* ${A_1}^{\downarrow}, ..., {A_k}^{\downarrow}$ *are incomparable. Moreover, the only kind of directed subsets that can be formed from the union of these* ${A_k}^{\downarrow}$*'s are subsets of exactly one of the chains* $\{x_k^i : k \in \mathbb{N}\}$ *for some* $i \in \mathbb{N}$*. Thus, these directed subsets cannot have suprema equal to* $\top$ *and so* $\mathscr{A}$ *does not converge to* $\top$ *in* P_d*.*

Lemma 9. *For any convergence space* $(X, \downarrow)$*, the following conditions are equivalent:*

(1) $(X, \downarrow)$ *is topological.*

(2) $\downarrow = \downarrow_\tau$*, where* τ *is the topology induced by* $(X, \downarrow)$*.*

Proof. (1) implies (2). Assume $(X, \downarrow)$ is topological. Then, by definition, $\downarrow = \downarrow_\sigma$ for some topology σ on X, i.e. $\mathscr{A} \downarrow x$ iff $\mathscr{N}_\sigma(x) \subseteq \mathscr{A}$. We claim that $\mathscr{N}_{\tau_\downarrow}(x) \subseteq \mathscr{N}_\sigma(x)$ for any $x \in X$. To this end, take any open set $U \in \mathscr{N}_{\tau_\downarrow}(x)$. Since $(X, \downarrow)$ is topological, $\mathscr{N}_\sigma(x) \downarrow x$ always holds. Together with the fact that $x \in U$, this implies that $U \in \mathscr{N}_\sigma(x)$ by the definition of the induced topology $\tau_\downarrow$. Thus $\mathscr{N}_{\tau_\downarrow}(x) \subseteq \mathscr{N}_\sigma(x)$.

(2) implies (1). This is trivial. □

Remark 2. *By Lemmata 4 and 9, it follows that* $(P, \downarrow_d)$ *and* $(P, \downarrow_c)$ *are topological iff* $\downarrow_d = \downarrow_\tau$ *and* $\downarrow_c = \downarrow_\tau$ *respectively, where* $\tau = \sigma(P)$*.*

A convergence space X is pretopological if the neighbourhood filter of each point converges to this point [5].

Theorem 1. *For a poset* P*, the following are equivalent:*

(1) $(P, \downarrow_d)$ *is topological*

(2) $(P, \downarrow_d)$ *is pretopological*

(3) P *is a continuous poset.*

(4) For any $x \in P$*, there is a smallest filter* $\mathscr{A}$ *with* $\mathscr{A} \downarrow_d x$*.*

Proof. (1) implies (2) by [5].

(2) implies (3). Suppose $(P, \downarrow_d)$ is pretopological. The induced topology on P by $\downarrow_d$ is the Scott topology. By the assumption, $\mathscr{N}_\sigma(x) \downarrow_d x$, where $\mathscr{N}_\sigma(x)$ denotes the set of Scott neighbourhoods of x, so there exists a directed set $D \subseteq \bigcup_{U\in\mathscr{N}_\sigma(x)} U^{\downarrow}$ with $\bigvee D \geq x$. It is now enough to show that for each $u \in D, u \ll x$. If W is a directed set with $\bigvee W$ exists and $\bigvee W \geq x$, then $W \cap U \neq \emptyset$ for every $U \in \mathscr{N}_\sigma(x)$. Since $u \in D, u \in U^{\downarrow}$ for some $U \in \mathscr{N}_\sigma(x)$. Choose $y \in W \cap U$, then $y \geq u$. Therefore $u \ll x$. Thus P is continuous.

(3) implies (4). To show this, it is enough to check that $\mathscr{N}_\sigma(x) \downarrow_d x$ because every filter $\mathscr{A}$ with $\mathscr{A} \downarrow_d x$ must contain $\mathscr{N}_\sigma(x)$. Take $D = \{y \in P : y \ll x\}$. Then D is directed with $\bigvee D = x$. Furthermore, for each $y \in D, U = \{w : y \ll w\}$ is Scott open with $x \in U$, and $y \in U^{\downarrow}$. All these show that $\mathscr{N}_\sigma(x) \downarrow_d x$.

(4) implies (3). Let $a \in P$ and $\mathscr{A}$ be the smallest filter with $\mathscr{A} \downarrow_d a$. By definition, there is a directed set D with $\bigvee D \geq a$ and $D \subseteq \bigcup\{B^{\downarrow} : B \in \mathscr{A}\}$. Now we show that $D \subseteq \{x \in P : x \ll a\}$. For any $e \in D$ and directed set W with $\bigvee W \geq a$, the filter $[\uparrow w : w \in W]$ converges to a under $\downarrow_d$, so $[\uparrow w : w \in W] \supseteq \mathscr{A}$. There is $B \in [\uparrow w : w \in W]$ such that $e \in B^{\downarrow}$ and $B \supseteq \uparrow w$ for some $w \in W$, which implies $e \leq w$. It follows that $e \ll a$.

(3) implies (1). Let P be continuous. If $\mathscr{A} \downarrow_d x$ and U is Scott open containing x, then there exists a directed set $D \subseteq \bigcup_{A\in\mathscr{A}} A^{\downarrow}$ with $\bigvee D \geq x$, and $e \in D \cap U$. Suppose $e \in A^{\downarrow}$ for some $A \in \mathscr{A}$. Then $A \subseteq \uparrow e \subseteq U$, implying $U \in \mathscr{A}$. Now for each point $y \in P$, $F = \{x \in P : x \ll y\}$ is directed with $\bigvee F = y$ and each $\uparrow x$ $(x \in F)$ is a neighbourhood of y with $x \in (\uparrow x)^{\downarrow}$. This shows that $\mathscr{N}_\sigma(x) \downarrow_d x$. Hence $(P, \downarrow_d)$ is topological. □

Definition 4. *[1] A function* $f : X \to Y$ *between two convergence spaces* $(X, \downarrow_X)$ *and* $(Y, \downarrow_Y)$ *is continuous if* $\mathscr{A} \downarrow_X x$ *implies* $f^+(\mathscr{A}) \downarrow_Y f(x)$*.*

Definition 5. *[1] For two convergence spaces* X *and* Y*, the function space* $[X \to Y]$ *is the set of continuous functions from* X *to* Y *with* $\mathscr{F} \downarrow f$ *iff for all* $\mathscr{A} \downarrow x$ *in* X*,* $\mathscr{F}.\mathscr{A} \downarrow f(x)$ *in* Y*. Here,* $\mathscr{F}.\mathscr{A} = [F.A : F \in \mathscr{F}, A \in \mathscr{A}]$ *and* $F.A = \{f(a) : f \in F, a \in A\}$*.*

A convergence space $(X, \downarrow)$ is called locally finitary if for all $\mathscr{A} \downarrow x$ and induced open neighbourhoods U of x, there is a finite subset $F \subseteq U$ with $\uparrow F \in \mathscr{A}$.

Proposition 1. *If* Y *is a topological space then* $[P_d \to Y]$ *is topological, where* $[P_d \to Y]$ *refers to the set of continuous functions from* P_d *to* Y *and* P_d *denotes* $(P, \downarrow_d)$*.*

Proof. It is clear that P_c is locally finitary for any poset P. By Lemma 5, P_d is locally finitary also. By Theorem 7.20 [1], we conclude that $[P_d \to Y]$ is topological. □

Let POS_d denote the category of posets and Scott continuous mappings.

For any function $f : X \to Y$, define $f^+ : \Phi X \to \Phi Y$ by $f^+(\mathscr{A}) = [f^+(A) : A \in \mathscr{A}]$.

Proposition 2. *A mapping $f : P_d \to Q_d$ is continuous iff f is a Scott continuous function from P to Q.*

Proof. Assume that $f : P_d \to Q_d$ is continuous. If $x \le y$, then $[y] \downarrow_d x$ and $f^+([y]) \downarrow_d f(x)$ hold. This implies $f(x) \le f(y)$ and so f is monotone. If $D \subseteq P$ is a directed set with $\bigvee D$ exists, then $[\uparrow z : z \in D] \downarrow_d \bigvee D$ holds. Thus $f^+([\uparrow z : z \in D]) \downarrow_d f(\bigvee D)$. So there exists a directed set $E \subseteq Q$ such that each $e \in E$ is below some $f(z), z \in D$ and $\bigvee E \ge f(\bigvee D)$. Since f is monotone, $\bigvee\{f(z) : z \in D\}$ exists and equals $f(\bigvee D)$.

Conversely, let $f : P_d \to Q_d$ be Scott continuous and $\mathscr{A} \downarrow_d x$ in $(P, \downarrow_d)$. Then there exists a directed set $D \subseteq P$ such that $\bigvee D \ge x$ and for each $e \in D$ there is $A \in \mathscr{A}$ with $e \in A^{\downarrow}$. Since f is Scott continuous, $\bigvee f(D) = f(\bigvee D) \ge f(x)$. As f is monotone, for all $a \in A$, $f(e) \le f(a)$ implies $f(e) \in (f(A))^{\downarrow}$. Therefore $f^+(\mathscr{A}) \downarrow_d f(x)$ as required. □

Thus, the assignment $P \to P_d$ extends to a functor $F : POS_d \to CONV$ which is full and faithful. So, POS_d is equivalent to a full subcategory of $CONV$.

Definition 6. *[1] Let $\prod_{i\in I} X_i$ be the product of a family $(X_i)_{i\in I}$ of convergence spaces with the initial structure for the projections $\pi_i : \prod_{i\in I} X_i \to X_i$. Then $\mathscr{A} \downarrow x$ in the product iff $\pi_i^+(\mathscr{A}) \downarrow x_i$ for all $i \in I$.*

Lemma 10. *For any collection $(P_i)_{i\in I}$ of posets, the identity mapping $\left(\prod_{i\in I} P_i\right)_d \to \prod_{i\in I}(P_i)_d$ is continuous.*

Proof. By Proposition 2, the projections $\prod_{i\in I} P_i \to P_i$ are Scott continuous, hence continuous $\left(\prod_{i\in I} P_i\right)_d \to (P_i)_d$. Thus, the identity mapping from $\left(\prod_{i\in I} P_i\right)_d$ to $\prod_{i\in I}(P_i)_d$ is continuous. □

Proposition 3. *For any two posets P and Q, $(P \times Q)_d = P_d \times Q_d$.*

Proof. By Lemma 10, we have $\mathscr{A} \downarrow_d (x, y)$ in $(P \times Q)_d \implies \mathscr{A} \downarrow_d (x, y)$ in $P_d \times Q_d$. We just need to show that if $\mathscr{A} \downarrow_d (x, y)$ in $P_d \times Q_d$, then $\mathscr{A} \downarrow_d (x, y)$ in $(P \times Q)_d$. Since $\mathscr{A} \downarrow_d (x, y)$ in $P_d \times Q_d$, $\pi_1^+(\mathscr{A}) \downarrow_d x$ and $\pi_2^+(\mathscr{A}) \downarrow_d y$. Then there are directed subsets D and E of P and Q respectively, such that $\bigvee D \ge x$, $\bigvee E \ge y$ and for each $u \in D$ there exists $A \in \mathscr{A}$ with $u \in (\pi_1^+(A))^{\downarrow}$ and for each $v \in E$ there exists $B \in \mathscr{A}$ with $v \in (\pi_2^+(B))^{\downarrow}$. Let $K = D \times E$. Then

(a) K is a directed subset of $P \times Q$.

(b) $\bigvee K = (\bigvee D, \bigvee E) \ge (x, y)$.

(c) For each $(u, v) \in K$, there are $A \in \mathscr{A}, B \in \mathscr{A}$ with $u \in (\pi_1^+(A))^{\downarrow}$ and $v \in (\pi_2^+(B))^{\downarrow}$. So $(u, v) \le (p, q)$ for all $(p, q) \in (\pi_1^+(A), \pi_2^+(B))$.

Now $A \cap B \in \mathscr{A}$. Let $(r, s) \in A \cap B$. Then $(r, s) \in (\pi_1^+(A \cap B), \pi_2^+(A \cap B)) \subset (\pi_1^+(A), \pi_2^+(B))$, so that $(r, s) \ge (u, v)$. Thus $\mathscr{A} \downarrow_d (x, y)$ in $(P \times Q)_d$. □

4 Meet-continuous posets

In this section we study the posets P for which the convergence space $(P, \downarrow_d)$ satisfies an additional axiom [5]:

(CONV3) $\mathscr{A} \downarrow_d x$ and $\mathscr{B} \downarrow_d x$ imply $\mathscr{A} \cap \mathscr{B} \downarrow_d x$.

Definition 7. *(i) A poset P is called meet-continuous if for any directed subset D with $\bigvee D \ge x$, x is in the Scott closure of $\downarrow D \cap \downarrow x$ [2].*

(ii) A poset P is called strongly meet-continuous if for any directed subsets D and E with $\bigvee D, \bigvee E$ exist and $\bigvee D \ge \bigvee E$, there is a directed subset $H \subseteq \downarrow D \cap \downarrow E$ such that $\bigvee H = \bigvee E$.

The poset in Example 1 is a dcpo which is not meet-continuous. To see this, take $D = \{x_1^2, x_2^2, ..., x_k^2\}$. We note that $\bigvee D = x^2 \ge x^1$. However $\downarrow D \cap \downarrow x^1 = \emptyset$.

It is easy to verify that every strongly meet-continuous poset is also meet-continuous.

Remark 3. *If P is a continuous poset, then it is strongly meet-continuous. As a matter of fact, if D and E are directed subsets of P such that $\bigvee D \ge \bigvee E$, then for any $x \in E$, $\Downarrow x = \{y \in P : y \ll x\} \subseteq \downarrow D$. Take $H = \bigcup_{x\in E} \Downarrow x$. Then $\bigvee H = \bigvee_{x\in E} x = \bigvee E$ and $H \subseteq \downarrow D \cap \downarrow E$.*

A meet-semilattice is a poset P in which for any two elements x and y, $x \wedge y$ exists.

Lemma 11. *Let L be a meet-semilattice and a dcpo. Then the following statements are equivalent:*

(1) For any $x \in L$ and directed $D \subseteq L$, $x \wedge \bigvee D = \bigvee\{x \wedge y : y \in D\}$.

(2) L is meet-continuous.

(3) L is strongly meet-continuous.

(4) For any directed set $D \subseteq L$ with $\bigvee D \ge x$ there is a directed $E \subseteq \downarrow D \cap \downarrow x$ such that $\bigvee E = x$.

Proof. (1) implies (2). Let $D \subseteq L$ be any directed set with $\bigvee D \geq x$. Let $E = \{u \wedge x : u \in D\}$. Then E is a directed set and $E \subseteq \downarrow D \cap \downarrow x$. $\downarrow D \cap \downarrow x$ is a lower set and $\bigvee E = \bigvee\{u \wedge x : u \in D\}$. By (1), $x \wedge \bigvee D = \bigvee\{u \wedge x : u \in D\} = \bigvee E$. So $x \wedge \bigvee D = x = \bigvee E$. Thus, $x \in \text{cl}(\downarrow D \cap \downarrow x)$.

(2) implies (1). Let $D \subseteq L$ be a directed set and $x \in L$. Let $y = \bigvee D \wedge x$, so $y \leq \bigvee D$ and $y \leq x$. By (2), $y \in \text{cl}(\downarrow D \cap \downarrow y) \subseteq \text{cl}(\downarrow D \cap \downarrow x)$. Consider $\downarrow \bigvee\{x \wedge u : u \in D\} \supseteq \text{cl}(\downarrow D \cap \downarrow x)$. Now $y \in \downarrow \bigvee\{x \wedge u : u \in D\}$, so $y \leq \bigvee\{x \wedge u : u \in D\}$. Also, $y \geq \bigvee\{x \wedge u : u \in D\}$. Therefore, $y = \bigvee D \wedge x = \bigvee\{x \wedge u : u \in D\}$.

(1) implies (3). Let $D \subseteq L$ and $E \subseteq L$ be directed sets with $\bigvee D \geq \bigvee E$. Then $\bigvee D \wedge \bigvee E = \bigvee E$. By (1), $\bigvee\{u \wedge \bigvee E : u \in D\} = \bigvee E$. Applying (1) again, $\bigvee\{\bigvee\{u \wedge e : u \in D\} : v \in E\} = \bigvee E$. Let $D' = \{u \wedge e : u \in D, v \in E\}$. Then D' is directed and is contained in $\downarrow D \cap \downarrow E$ with $\bigvee D' = \bigvee E$. So (3) is proven.

(1) implies (4). Let $E = \{x \wedge y : y \in D\}$. Then E is directed and $E \subseteq \downarrow D \cap \downarrow x$. By (1), $\bigvee E = \bigvee\{x \wedge y : y \in D\} = x \wedge \bigvee D$. Since $\bigvee D \geq x$, $\bigvee E = x \wedge \bigvee D = x$.

(4) implies (1). By (4), $\bigvee D \geq x$ and so $x \wedge \bigvee D = x = \bigvee E \in \downarrow D \cap \downarrow x$ which implies that $x \in \text{cl}(\downarrow D \cap \downarrow x)$. Consider $\downarrow \bigvee\{x \wedge u : u \in D\} \supseteq \text{cl}(\downarrow D \cap \downarrow x) \ni x$. So $x \leq \bigvee\{x \wedge u : u \in D\}$. Also, $x \geq \bigvee\{x \wedge u : u \in D\}$. Therefore, $x = \bigvee D \wedge x = \bigvee\{x \wedge u : u \in D\}$. □

Proposition 4. *Let P be a poset. The following statements are equivalent:*

(1) P is strongly meet-continuous.

(2) $\mathscr{A} \downarrow_d x$ and $\mathscr{B} \downarrow_d x$ imply $\mathscr{A} \cap \mathscr{B} \downarrow_d x$.

Proof. Let $(P, \downarrow_d)$ satisfy the condition (CONV3). If $D \subseteq P$ and $E \subseteq P$ are directed sets with $\bigvee D \geq \bigvee E$, then $[\uparrow u : u \in D] \downarrow_d e$ and $[\uparrow v : v \in E] \downarrow_d e$, where $e = \bigvee E$. Thus $[\uparrow u : u \in D] \cap [\uparrow v : v \in E] \downarrow_d e$. Thus there is a directed set H with $\bigvee H \geq e$ and $H \cup \{B^\downarrow : B \in [\uparrow u : u \in D] \cap [\uparrow v : v \in E]\} \subseteq \downarrow D \cap \downarrow E$. Obviously $\bigvee H = e = \bigvee E$.

Conversely, suppose that P is strongly meet-continuous. Let $\mathscr{A} \downarrow_d x, \mathscr{B} \downarrow_d x$. Then there exist directed sets D, E such that $D \subseteq \bigcup\{A^\downarrow : A \in \mathscr{A}\}, E \subseteq \bigcup\{B^\downarrow : B \in \mathscr{B}\}$ and $\bigvee D \geq x, \bigvee E \geq x$. Since P is strongly meet-continuous, there is a directed set $H_1 \subseteq \downarrow D \cap \downarrow x$ such that $\bigvee H_1 = x$. Now $\bigvee E \geq \bigvee H_1$ so there is a directed set $H_2 \subseteq \downarrow H_1 \cap \downarrow E$ such that $\bigvee H_2 \geq \bigvee H_1 = x$. Now $[\uparrow w : w \in H_2] \downarrow x$ and $[\uparrow w : w \in H_2] \subseteq \mathscr{A} \cap \mathscr{B}$. It follows that $\mathscr{A} \cap \mathscr{B} \downarrow_d x$. □

Let P be a meet-semilattice. Then $\wedge : P \times P \to P$ is a mapping sending (x, y) to $x \wedge y$. A convergence semilattice is a convergence space $(X, \downarrow)$ such that X is a meet-semilattice and $\wedge : X \times X \to X$ is continuous.

Proposition 5. *For a meet-semilattice P, $(P, \downarrow_d)$ is a convergence meet-semilattice iff $(P, \leq)$ is meet-continuous.*

Proof. Suppose that $(P, \downarrow_d)$ is a convergence meet-semilattice. By Lemma 11, we only need to show that for any $x \in P$ and directed $D \subseteq P$, $x \wedge \bigvee D = \bigvee\{x \wedge y : y \in D\}$. Since $\wedge : X \times X \to X$ is continuous, it is Scott continuous. So $\bigwedge(\bigvee\{(x, u) : u \in D\}) = \wedge(x, \bigvee D) = x \wedge \bigvee D = \bigvee\{\wedge(x, u) : u \in D\} = \bigvee\{x \wedge u : u \in D\}$.

Conversely, assume that $(P, \leq)$ is meet-continuous and that $\mathscr{A} \downarrow_d (x, y)$. Now $\pi_1^+(\mathscr{A}) \downarrow_d x$ and $\pi_2^+(\mathscr{A}) \downarrow_d y$. Since $\pi_1^+(\mathscr{A}) \downarrow_d x$, there exists a directed subset $D_1 \subseteq P$ such that $\bigvee D_1 \geq x$ and $D_1 \subseteq \bigcup_{A_1 \in \pi_1^+(\mathscr{A})} {A_1}^\downarrow$. Since $\pi_2^+(\mathscr{A}) \downarrow_d y$, there exists a directed subset $D_2 \subseteq P$ such that $\bigvee D_2 \geq y$ and $D_2 \subseteq \bigcup_{A_2 \in \pi_2^+(\mathscr{A})} {A_2}^\downarrow$. Now $\bigvee D_1 \wedge \bigvee D_2 \geq x \wedge y$, so $\bigvee\{u \wedge v : u \in D_1, v \in D_2\} \geq x \wedge y$. We see that $D = \{u \wedge v : u \in D_1, v \in D_2\}$ is a directed set. For all $u \in D_1$ and all $v \in D_2$, there exists $A_1 \in \pi_1(\mathscr{A})$ with $u \in {A_1}^\downarrow$ and there exists $A_2 \in \pi_2(\mathscr{A})$ with $v \in {A_2}^\downarrow$. There exists $\overline{A_1} \in \mathscr{A}$, $A_1 \supseteq \pi_1^+(\overline{A_1})$ and there exists $\overline{A_2} \in \mathscr{A}$, $A_2 \supseteq \pi_2^+(\overline{A_2})$. Take $\overline{A} = \overline{A_1} \cap \overline{A_2}$. Then $u \wedge v \in \wedge(\overline{A})^\downarrow$. Since for all $(x_1, y_1) \in \overline{A}, x_1 \in \pi_1(\overline{A}) \subseteq \pi_1(\overline{A_1}) \subseteq A_1$, so $u \leq x_1$. In a similar approach, $v \leq y_1$. □

In conclusion, we have demonstrated that the definition of $\downarrow_d$ for any poset P has numerous desirable characteristics. This can be considered as preliminary work that may lead to applications in other fields such as theoretical computer science.

References

[1] R. Heckmann. A non-topological view of dcpos as convergence spaces. *Theoretical Computer Science*, 305:159-186, 2003.

[2] H. Kou, Y.-M. Liu, and M.-K. Luo. On meet-continuous dcpos. *Domain Theory, Logic and Computation*, 117-136, 2004.

[3] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, and D. S. Scott. *Continuous Lattices and Domains*. Cambridge University Press, 2003.

[4] D. Normann. On sequential functionals of type 3. *Mathematical Structures in Computer Science*, 16(2) (2006) 279-289.

[5] R. Beattie and H.-P. Butzmann. *Convergence Structures and Applications to Functional Analysis*. Kluwer Academic Publishers, 2002.

[6] Bärbel M. R. Stadler, Peter F. Stadler. *Basic Properties of Filter Convergence Spaces*. Technical report. URL: http://www.tbi.univie.ac.at/papers/Abstracts/01-pfs-007-subl1.pdf, 2002.

[7] D. Zhao. Generalization of frames and continuous lattices. *Ph. D. Thesis*, 1993, Cambridge University.

[8] G. N. Raney. Completely distributive lattices. *Proc. Amer. Math. Soc.*, 3(1952), 677-680.

[9] R.-E Hoffman. Continuous posets, prime spectra of completely distributive complete lattices, and Hausdorff compactifications. *Bremen*, (1979), 159-208.