D-completions of net convergence structures

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Abstract:

By extending Ershov's notion of a d-space from topological spaces to net convergence spaces, this paper details the d-completion of certain net convergence structures which are rich enough to support it. In particular, it is demonstrated that spaces which are embeddable into d-spaces which have iterated limits admit d-completions. The main result reported herein generalizes an existing procedure for d-completion of T_0 spaces.

Keywords:

net convergence, epitopological space, d-space, d-completion, dcpo.

1 Introduction

A directed subset D of a partially ordered set P is a non-empty set for which any finite subset $F \subseteq D$ has an upper bound in D. Being an obvious generalization of chains, directed sets capture succinctly the essence of convergence of sequences in general topology. Indeed, one can traced back to works like [9], where directed sets, together with nets, were used to determine topologies. Riding on its special affinity to convergence, directed sets were soon used to model phenomenon of approximations in computation. Consequently, partial orders which support the existence of suprema (in a way, viewed as limit points) of such sets emerge to become a salient notion in topology, domain theory and denotational semantics. These structures are called *directed complete posets*, abbreviated by *dcpo's*, have now become well-known in the community of theoretical computer science (particularly, programming semantics). One of the most important appearance of this concept is in the definition of the famous *Scott topology* invented by D.S. Scott in the late 1960's: "A subset U of a partially ordered set is open if it is upper and is inaccessible by *directed* suprema." (See [3, 5].)

Valued as an indispensable condition of completeness in partial orders, it sparked off an active line of research for order theorists to provide 'canonical' *dcpo-completions* in the event that a poset fails to be directed-complete. For instance, the pioneering work of G. Markowsky [8] details a procedure of dcpo-completion via chains. Directedness and directed-completeness continued their pervasion from the realm of posets to that of T_0 spaces – the key link here being the specialization order of the underlying topology: $x \sqsubseteq_\tau y$ if and only if $x \in \mathrm{cl}(\{y\})$, where cl is the closure operator with respect to the topology τ. Along this topological line of development, O. Wyler [12] and Y.L. Ershov [2] had already studied extensively the d-spaces which are topological spaces behaving like dcpo's (i.e., directed complete with respect to the specialization order and in which the limit points of directed subsets are exactly their suprema). In particular, Ershov showed that every T_0 space X has its *d-completion* (i.e., a universal completion parallel to that of dcpo-completions for posets) – the smallest ambient d-space into which X can be embedded. The inverse limits construction employed by [8] and [2] is essentially a 'bottom-up' approach. Later, a 'top-down' approach was carried out in recent works of T. Fan and D. Zhao [14], and that of K. Keimel and J.D. Lawson [6], exploiting a new topology called D-topology.

In this paper, we proceed to investigate the existence of d-completion for a net convergence space, which is one of the structures that generalizes topological spaces. More precisely, we introduce a suitable notion of d-space in the context of net convergence spaces and show that there exists a certain class $\mathcal{C}$ of (net) convergence spaces whose structures are rich enough to admit (the corresponding) d-completions. It turns out that such a class $\mathcal{C}$ contains all T_0 topological spaces, thus generalizing the existing results of d-completions of T_0 topologies spaces and dcpo-completions of posets.

The subsequent structure of this paper is as follows. Section 2 sets the appropriate categorical stage to facilitate many kinds of completions that will take place in the present context of convergence spaces, and this is supported by a few well-known examples. Section 3 then ushers in the notion of convergence space and its related constructs. Niceness conditions form the subject of discourse in Section 4; and spaces in which iterated limits exist for Section 5. These concepts, as it turns out, will be exploited in the development of the theory of d-spaces in the context of net convergence structures in Section 6. This section then culminates with the main theorem which states some sufficiency condition for the existence of d-completions with respect to the definition of d-spaces which we had stated in preceding section.

2 Universal K-fications in topological categories

Our main study centers around the several universal **K**-fications in the category of net convergence spaces. We choose to carry out this programme within the convenient framework of topological categories in the sense of [11]. The notion of a concrete category is an essential one with regards to this choice.

By a concrete category we mean a category $\mathbf{C}$ whose *objects* are structured sets, i.e., pairs (X, ξ) where X is a set and ξ is a $\mathbf{C}$-structure on X, whose *morphisms* $f : (X, \xi) \longrightarrow (Y, \eta)$ are suitable maps between X and Y and whose composition law is the usual composition of maps – in other words: a category $\mathbf{C}$ together with a faithful (forgetful) functor $U : \mathbf{C} \to \mathbf{Set}$ where $\mathbf{Set}$ denotes the category of sets and maps.

Definition 2.1 A concrete category $\mathbf{C}$ is called *topological* if it satisfies the following conditions:

1. *Existence of initial structures*:
   For any set X and any family $(X_i, \xi_i)_{i \in I}$ of $\mathbf{C}$-objects X_i's indexed by I and mappings $\xi_i : X \to X_i$ each indexed by I, there exists a unique $\mathbf{C}$-structure (X, ξ) which is initial with respect to $(X, f_i, (X_i, \xi_i), I)$, i.e., such that for any $\mathbf{C}$-object (Y, η) a map $g : (Y, \eta) \longrightarrow (X, \xi)$ is a $\mathbf{C}$-morphism if and only if for every $i \in I$ the composite map $f_i \circ g : (Y, \eta) \longrightarrow (X_i, \xi_i)$ is a $\mathbf{C}$-morphism.
2. *Fibre-smallness*:
   For any set X, the $\mathbf{C}$-fibre of X, i.e., the class of all $\mathbf{C}$-structures on X, is a set.
3. *Terminal separator property*:
   For any set X with cardinality one, there exists precisely one $\mathbf{C}$-structure on X.

The concept of a full reflective subcategory allows one to deal with, in a coherent manner, different kinds of completions which we are about to embark on.

Definition 2.2 A full subcategory $\mathbf{K}$ of a category $\mathbf{C}$ is called *reflective* if the inclusion functor Incl has a left adjoint R, which then is called a *reflector*.

Equivalently, this is realized in the following way: For each $\mathbf{C}$-object C, there exists a $\mathbf{K}$-object $\tilde{C}$ and a $\mathbf{C}$-morphism $r_C : C \longrightarrow \tilde{C}$ such that for each $\mathbf{K}$-object D and each $\mathbf{C}$-morphism $f : C \longrightarrow D$, there is a unique $\mathbf{K}$-morphism $\overline{f} : \tilde{C} \longrightarrow D$ such that

$$\overline{f} \circ r_C = f.$$

We call the object $\tilde{C}$ the $\mathbf{K}$-modification of C and the universal $\mathbf{C}$-morphism $r_C : C \longrightarrow \tilde{C}$ the (universal) reflection. Several subcategories $\mathbf{K}$ of the topological categories we consider in this paper are crucially of the following two kinds:

Definition 2.3 A *bireflective* subcategory $\mathbf{K}$ of a category $\mathbf{C}$ is a reflective subcategory in which for each $\mathbf{C}$-object C the reflection $r_C : C \longrightarrow \tilde{C}$ is a bimorphism, i.e., it is both a monomorphism and an epimorphism.

Definition 2.4 A full subcategory **K** of a category **C** is said to be *isomorphism-closed* if for any **C**-object C and any **K**-object K, whenever C is isomorphic to K then C is itself a **K**-object.

The reason for singling out these kinds of subcategories is that:

Theorem 2.5 (Theorem 2.2.12, [11])
Every bireflective and isomorphism-closed subcategory **K** *of a topological category* **C** *is topological.*

The above theorem comes handy in ensuring the existence of initial structures for certain subcategories of net convergence spaces we are considering in this paper.

3 Net convergence spaces

In this section, we introduce our main character – net convergence spaces. A pre-ordered set I (i.e., one with a reflexive and transitive relation $\sqsubseteq$) is said to be *directed* if for every pair $i_1,\ i_2 \in I$, there is always an $i_3 \in I$ such that $i_1,\ i_2 \sqsubseteq i_3$. A *net* in a set X is a mapping from a directed pre-order I to X. If we need to be more explicit about the elements of a net, we use the notation $(x_i)_{i\in I}$. Otherwise, we use Greek letters $\mu,\ \nu$. In particular, for any pre-order I and any fixed element $x \in X$, the *constant net* $(x) : I \to X,\ i \mapsto x$ is a net in X. For any directed subset D of a partially ordered set P (poset, for short), D defines a net $(x_d)_{d\in D}$ where $x_d := d$ for each $d \in D$. Given a set X, we use $\Psi(X)$ to denote the set of all nets in X.

Throughout our discussion, we make a more than casual use of the phrase 'eventually' in the following sense: If $P(x)$ is a property of the elements $x \in X$, we say that $P(x_j)$ holds *eventually* in a net $(x_j)_{j\in J}$ if there is a $j_0 \in J$ such that $\varphi(x_k)$ holds whenever $k \sqsupseteq j_0$.

We say that $(x_i)_{i\in I}$ is a subnet of $(y_j)_{j\in J}$, denoted by $(x_i)_{i\in I} \leq (y_j)_{j\in J}$, if

$$\forall j_0 \in J, \exists i_0 \in I, \forall i \sqsupseteq i_0, x_i \in T^{(y_j)}_{j_0}.$$

Here, $T^{(y_j)}_{j_0}$ denotes the set $\{y_j \mid j \sqsupseteq j_0\}$ and is called the j_0^{th}-tail of the net $(y_j)_{j\in J}$. Equivalently, $(x_i)_{i\in I}$ is a subnet of $(y_j)_{j\in J}$ if and only if for each $j \in J$, the net $(x_i)_{i\in I}$ is eventually in the j^{th}-tail of $(y_j)_{j\in J}$. Then $\leq$ is a pre-order on $\Psi(X)$ which will be called the subnet pre-order.

Remark 3.1 (1) In the literature of net convergence structures, there are different definitions of a subnet. In particular, Kelley in [7] defines $(x_i)_{i\in I}$ to be a subnet of $(y_j)_{j\in J}$ if and only if there is a mapping $h : I \to J$ such that

1. $x_i = y_{h(i)}$ for all $i \in I$, and
2. for each $j \in J$, there is $i_0 \in I$ such that whenever $i \sqsupseteq i_0$ then $h(i) \sqsupseteq j$.

Several authors, such as [1], consider additionally that h is monotone. Our version, adopted from [10], is less general but yet natural enough to crucially allow the safe passage of certain arguments regarding subnets where other competing versions fail.

(2) Note that we may have two distinct nets μ_1 and μ_2 which are subnet of each other. We will call a net μ a proper subnet of net ν if $\mu \leq \nu$ and $\nu \not\leq \mu$.

Definition 3.2 A net is called an *ultranet* if it has no proper subnet.

In other words, it is minimal with respect to the subnet pre-order. More precisely, a net ν is an ultranet if for any subnet ν' of ν, ν is also a subnet of ν'. Given a net μ in a set X, by the Hausdorff maximality principle, there is always an ultranet ν which is a subnet of μ. The abridged version of saying this is that ν is an '*ultra-subnet*' of μ.

Example 3.3 Every constant net (x) is an ultranet since for every subnet $(y_j)_{j\in J}$ of (x), it holds that (x) is eventually in the i^{th}-tail of $(y_j)_{j\in I}$ as all of the y_j's are actually x.

Proposition 3.4 *The following conditions are equivalent for a net $(x_i)_{i\in I}$ in a set X.*

(i) $(x_i)_{i\in I}$ *is an ultranet.*
(ii) For any subset E of X, $(x_i)_{i\in I}$ is eventually in either E or $X\backslash E$.
(iii) For any subsets A and B, if $(x_i)_{i\in I}$ is eventually in $A\cup B$ then $(x_i)_{i\in I}$ is either eventually in A or in B.

Proof. (i) $\implies$ (ii): Suppose not. Then for each $i \in I$, there exist $i_1,\ i_2 \sqsupseteq i \in I$ such that $x_{i_1} \in E$ and $x_{i_2} \in X\backslash E$. This gives rise to a *proper* subnet of $(x_i)_{i\in I}$ whose elements are respectively in E. However, this contradicts the assumption that E has no proper subnets.

(ii) $\implies$ (iii): Consider $X = A \cup B$ with no loss of generality and set $E = A$. Then $X\backslash E = B\backslash A$. Applying (ii), the net $(x_i)_{i\in I}$ will eventually be in E or in $X\backslash E$. Equivalently, the net is eventually in A or in $B\backslash A$. Thus the net is eventually in A or in B.

(iii) $\implies$ (i): Suppose that $(x_i)_{i\in I}$ has a proper subnet $(y_j)_{j\in J}$. We assume, without lose of generality, that it is not true that $(x_i)_{i\in}$ is in $A = \{y_j \mid j \in J\}$ eventually. Then $B = \{x_i \mid i \in I\}\backslash A \neq \emptyset$. Since $(x_i)_{i\in I}$ is in $A\cup B$ eventually and not in A eventually, by (iii), the net $(x_i)_{i\in I}$ is in B eventually. Assume that $T^{(x_i)}_{i_0} \subseteq B$ holds for some $i_0 \in I$. Then $(y_j)_{j\in J}$ is in $T^{(x_i)}_{i_0}$, and hence in B eventually. But this is not possible because all y_j' are in A and A is disjoint from B. This contradiction shows that $(x_i)_{i\in I}$ must be an ultranet.

Having explained what nets are, we are ready for the following.

Definition 3.5 By a *net convergence space*, we mean a pair $(X, \to)$ where X is a non-empty set and $\to$ a relation between the set $\Psi(X)$ and X, i.e., $\to\subseteq \Psi(X) \times X$ such that the following axioms are satisfied:

1. (CONSTANT NET)
   For every $x \in X$, we always have $(x) \rightarrow x$.
2. (SUBNET)
   If $(y_j)_{j\in J} \leq (x_i)_{i\in I}$ and $(x_i)_{i\in I} \rightarrow x$, then $(y_j)_{j\in J} \rightarrow x$.

We write $(x_i)_{i\in I} \rightarrow x$ for $((x_i)_{i\in I}, x) \in\rightarrow$. Where there might be convergence structures derived from an existing one on a same set X, we always use $\rightarrow_X$ to denote the original one.

A function $f : (X, \rightarrow_X) \longrightarrow (Y, \rightarrow_Y)$ between net convergence spaces is said to be *continuous* if $(x_i)_{i\in I} \rightarrow_X x$ implies $(f(x_i))_{i\in I} \rightarrow_Y f(x)$. **NConv** denotes the category of net convergence spaces and continuous functions. A special kind of continuous map which is crucial in our theory of D-completion deserves special mention:

Definition 3.6 A *pre-embedding* of a convergence space X to another Y is a continuous mapping $e : X \longrightarrow Y$ such that

$$(x_i)_{i\in I} \rightarrow_X x \iff (e(x_i))_{i\in I} \rightarrow_Y e(x).$$

An *embedding* is an injective pre-embedding.

A convergence space $(X, \rightarrow)$ is said to be T_0 if for every $x,\ y \in X$, the following holds:

$$(x) \rightarrow y \wedge (y) \rightarrow x \iff x = y.$$

A few pathological examples must now be in place.

Example 3.7 1. If (X, τ) is a topological space, then $(X, \rightarrow_\tau)$ is a convergence space, where $(x_i)_{i\in I} \rightarrow_\tau x$ if and only if the net $(x_i)_{i\in I}$ converges to x in the topological sense, i.e., for every open set U of (X, τ), $x \in U$ implies that $x_i \in U$ eventually. Such a convergence space is called a *topological* convergence space. Denote by Ω the *Sierpinski space*, i.e., $\{0, 1\}$. Ω is the convergence space in which every net $(x_i)_{i\in I}$ converges to 0 and $(x_i)_{i\in I} \rightarrow 1$ if and only if $x_i = 1$ eventually. Then clearly $(\Omega, \rightarrow)$ is topological.

2. Let P be a poset. Define $(x_i)_{i\in I} \rightarrow_d x$ if and only if there is a directed subset $D \subseteq P$ such that $\bigsqcup D \sqsupseteq x$ and for each $d \in D$, $x_i \sqsupseteq d$ eventually. It is straightforward to verify that $(P, \rightarrow_d)$ is a convergence space. In [13], it is shown that P is topological if and only if P is a continuous poset.
   In general, let $\mathcal{M}$ be a collection of subsets of poset P such that $\{x\} \in \mathcal{M}$ for each $x \in P$. Define $(x_i) \rightarrow_{\mathcal{M}} x$ if and only if there is $A \in \mathcal{M}$ such that $\bigsqcup A \sqsupseteq x$ and $x_i \sqsupseteq a$ eventually for each $a \in A$. Then $(X, \rightarrow_{\mathcal{M}})$ is a convergence space.
3. Let (X, τ) be a topological space and $\sqsubseteq_\tau$ be the specialization order on X (i.e. $x \sqsubseteq_\tau y$ iff $x \in \mathrm{cl}(\{y\})$). Define $(x_i)_{i\in I} \rightarrow_c x$ if and only if $x \in \mathrm{cl}(\bigcup_{k\in I} T^{(x_i)}{}_k^{\downarrow})$, where $A^{\downarrow} := \{y \in X \mid \forall a \in A. y \sqsubseteq a\}$. Note that $A^{\downarrow}$ is in fact the set of all lower bounds of A in X. Then $(X, \rightarrow_c)$ is a convergence space, first defined in [4] for dcpos with their Scott topology.

4. A pre-metric space is a pair (X,ρ), where X is a non-empty set and $\rho : X \times X \longrightarrow [0,\infty)$ is a function satisfying $\rho(x,x) = 0$ for all $x \in X$. Now define $(x_i)_{i\in I} \to x$ if $(\rho(x_i,x))_{i\in I}$ converges to 0. Then $(X,\to)$ is a convergence space.
5. On $\mathbb{R}^2$ define $(x_i,y_i)_{i\in I} \to_1 (x,y)$ iff $(x_i)_{i\in I}$ converges to x in $\mathbb{R}$. Then $(\mathbb{R}^2,\to_1)$ is a convergence space. In general, if $f : X \longrightarrow Y$ is a mapping from a set X into a topological space Y, we define $(x_i)_{i\in I} \to_f x$, for net $(x_i)_{i\in I}$ and element x in X, if $(f(x_i))_{i\in I}$ converges to $f(x)$. Then $(X,\to_f)$ is a convergence space.
6. For any poset P and a net $(x_i)_{i\in I}$ in P, define $(x_i)_{i\in I} \to_D x$ if there is a directed subset $E \subseteq P$ such that $\bigsqcup E = x$ and for each $e \in E, x_i \in E\cap \uparrow e$ eventually. Then $(P,\to_D)$ is a convergence space.

Readers who are familiar with the theory of filter convergence would have realized the glaring similarities between the definitions of filter convergence spaces and net convergence spaces. In fact, one can show that there is a categorical adjunction between them. A given net $(x_i)_{i\in I}$ in X canonically induces a filter in X, denoted by $[(x_i)_{i\in I}]$, defined by

$$[(x_i)_{i\in I}] = \{A \subseteq X \mid \exists i_0 \in I.A \supseteq T_{i_0}^{(x_i)}\}$$

In other words, $[(x_i)_{i\in I}]$ is the filter generated by the filter base consisting of the tails of $(x_i)_{i\in I}$.

Remark 3.8 With this notation, it is easy to see that $(x_i)_{i\in I} \leq (y_j)_{j\in J}$ if and only if $[(y_j)_{j\in J}] \subseteq [(x_i)_{i\in I}]$.

Conversely, a given filter $\mathcal{F}$ on X induces a canonical net $\hat{\mathcal{F}}$ defined as follows. Define $I_{\mathcal{F}} := \{(a,A) \mid (a \in A) \wedge (A \in \mathcal{F})\}$ and impose the following pre-order:

$$(a,F) \leq (b,G) \iff G \subseteq F.$$

The canonical net in X induced by the filter $\mathcal{F}$ is given by the mapping:

$$\hat{\mathcal{F}} : I_{\mathcal{F}} \longrightarrow X,\ (a,A) \mapsto a.$$

Denote the set of all filters on X by $\Phi(X)$ and order it by the sub-filter order $\leq$ (in fact, it is the reverse inclusion). Experts in Galois connections would immediately recognize the adjunction of pre-orders between $\Phi(X)$ and $\Psi(X)$, i.e.,

$$\alpha_X : \Phi(X) \to \Psi(X),\ \mathcal{F} \to \hat{\mathcal{F}}$$

is left adjoint to

$$\beta_X : \Psi(X) \to \Phi(X),\ (x_i)_{i\in I} \to [(x_i)_{i\in I}].$$

Such a pair of adjoint maps, natural in X, is in fact an e-p (embedding-projection) pair, i.e., $\alpha_X \circ \beta_X \leq \mathrm{id}_{\Psi(X)}$ & $\beta_X \circ \alpha_X = \mathrm{id}_{\Phi(X)}$. It can be shown that such e-p pairs induce a categorical adjunction between the category of filter convergence spaces **FConv** and that of net convergence spaces **NConv**, i.e., **FConv** $\dashv$ **NConv**. At the moment of writing, it is not known to the authors whether these two categories are equivalent.

Given any function $f : X \to Y$, one can define $f^+ : \Psi(X) \to \Psi(Y)$ by

$$f^+(x_i)_{i\in I} := (f(x_i))_{i\in I}$$

and also $f^- : \Psi(Y) \to \Psi(X)$ as follows:

$$f^-(y_j)_{j\in J} := \alpha_X \circ [f^{-1}(T_j^{(y_j)}) \mid j \in J].$$

Here, $[f^{-1}(T_j^{(y_j)}) \mid j \in J]$ denotes the filter generated by the filter base $\{f^{-1}(T_j^{(y_j)}) \mid j \in J\}$. For any $(x_i)_{i\in I} \in \Psi(X)$ and any $(y_j)_{j\in J} \in \Psi(Y)$, $f^+((x_i)_{i\in I}) \leq (y_j)_{j\in J} \iff (x_i)_{i\in I} \leq f^-((y_j)_{j\in J})$. This can be justified as follows:

$$\begin{aligned} & f^+((x_i)_{i\in I}) \leq (y_j)_{j\in J} \\ \iff & (f(x_i))_{i\in I} \leq (y_j)_{j\in J} \\ \iff & \forall j \in J. \exists i_0 \in I. T_{i_0}^{(f(x_i))} \subseteq T_j^{(y_j)} \\ \iff & \forall j \in J. \exists i_0 \in I. f(T_{i_0}^{(x_i)}) \subseteq T_j^{(y_j)} \\ \iff & \forall j \in J. \exists i_0 \in I. T_{i_0}^{(x_i)} \subseteq f^{-1}(T_j^{(y_j)}) \\ \iff & (x_i)_{i\in I} \leq f^-((y_j)_{j\in J}) \end{aligned}$$

NConv admits both initial and terminal structures for any given family of mappings. Hence products and coproducts of any collection of spaces exist. All of these properties are not surprising, thanks to the fact that **NConv** is a topological category.

The existence of initial structures guarantees the existence of subspaces. More precisely, a subspace X_0 of a space X is a subset of X equipped with the initial structure with respect to the set inclusion $\iota : X_0 \hookrightarrow X$. In particular, that $(X_0, \to_{X_0})$ is a subspace of $(X, \to_X)$ is equivalent to having:

$$(x_i)_{i\in I} \to_{X_0} x \iff (x_i)_{i\in I} \to_X x.$$

Of course, every subspace of a topological convergence space is topological.

Crucially, the category **NConv** also admits exponentials. Given two spaces X and Y, let $[X \to Y]$ be the set of all continuous mappings from X to Y. For a net $(f_i)_{i\in I}$ in $[X \to Y]$ and a $f \in [X \to Y]$, define $(f_i)_{i\in I} \to f$ iff for any $(x_k)_{k\in K} \to_X x$, one has

$$(f_i)_{i\in I} \cdot (x_k)_{k\in K} \to f(x),$$

where $(f_i)_{i\in I}\cdot(x_k)_{k\in K}$ is defined to be the net $(f_i(x_k))_{(i,k)\in I\times K}$. It turns out that $([X\to Y],\to)$ is a net convergence space, which we call the *function space* from X to Y. Further to the existence of products and exponentials, one of course expects nothing less than the fact that **NConv** is a cartesian closed category. This is one of the major reasons why convergence spaces are considered in preference to topological spaces since it is well known that the category of topological spaces is not Cartesian closed.

Regarding function spaces, the following property is frequently used.

Proposition 3.9 *Let X and Y be convergence spaces. Then Y is homeomorphic to a subspace of $[X\to Y]$.*

Proof. Define $k : Y\longrightarrow[X\to Y]$ by $k(y)=\overline{y}$, the constant mapping $x\to y$. Clearly, each $k(y)$ is in $[X\to Y]$ and k is injective. To show that k is a continuous map from Y to $[X\to Y]$, suppose $(y_j)_{j\in J}\to_Y y$. We aim to show that $(k(y_j))_{j\in J}\to_{[X\to Y]} k(y)$. To achieve this, take any $(x_i)_{i\in I}\to_X x$. Now for each $(j,i)\in J\times I$, the term $(k(y_j))(x_i)=\overline{y_j}(x_i)=y_j$. But $(y_j)_{j\in J}\to_Y y$ by our supposition. So $(k(y_j)(x_i))_{(j,i)\in J\times I}\to_Y y=\overline{y}(x)$. This is, of course, equivalent to saying $(k(y_j))_{j\in J}\to_{[X\to Y]} k(y)$.

To show that k is an embedding, suppose the net $(y_j)_{j\in J}$ in Y is such that $(k(y_j))_{j\in J}\to_{[X\to Y]} k(y)$. Now apply the net $(k(y_j))_{j\in J}$ to the constant net (x) which, we know, always converges to x in X. Thus by the definition of the convergence structure on $[X\to Y]$, it follows that

$$(k(y_j))_{j\in J}(x)=(\overline{y_j}(x))_{j\in J}=(y_j)_{j\in J}\to_Y k(y)(x)=y.$$

We conclude that $(y_j)_{j\in J}\to_Y y$ which implies that k is an embedding.

In what follows, by a *(convergence) space* we shall always mean a net convergence space unless otherwise stated. Given a space $(X,\to)$, there is a *induced topology* on X, making X to be a topological space denoted by $TX:=(X,\tau_\to)$ and is defined as follows: A subset $U\in\tau_\to$ if for any net $(x_i)_{i\in I}$ in X, $(x_i)_{i\in I}\to x$ and $x\in U$ always imply that $x_i\in U$ eventually.

It is straightforward to check that τ is indeed a topology on X. Moreover, if $(x_i)_{i\in I}\to x$, then $(x_i)_{i\in I}$ converges to x topologically. For any poset P, the induced topology on $(P,\to_d)$ is the Scott topology($U\subseteq P$ is Scott open iff $U=\uparrow U$ and if $D\subseteq P$ is a directed set with $\bigsqcup D\in U$ then $D\cap U\neq\emptyset$).

Such an induced topology can be viewed in terms of categorical adjunctions. Denoting the category of topological spaces by **Top**, one has the following sequence of adjunctions:

$$\mathbf{Top}\dashv\mathbf{FConv}\dashv\mathbf{NConv}.$$

In this perspective, one can see T as the right adjunction of the composition of adjunctions. Consequently, one views net convergence space as a generalization of filter convergence space inasmuch as filter convergence space is that of topological space.

Given a topological space (X, τ), the specialization order on X is the partial order $\sqsubseteq_\tau$ (or $\sqsubseteq$ for short) defined by $x \sqsubseteq_\tau y$ iff $x \in \mathrm{cl}(\{y\})$. For a convergence space $(X, \to)$, we call the specialization order of $(X, \tau_\to)$ the *specialization order* of $(X, \to)$.

Remark 3.10 1. Every continuous map $f : (X, \to_X) \longrightarrow (Y, \to_Y)$ is also continuous with respect to the induced topology and hence monotone with respect to the specialization order.
2. If Y is a topological space and X is a subspace of Y, then the specialization order on X coincides with the restricted order of that on Y, that is, $x \sqsubseteq x'$ holds in X if and only if $x \sqsubseteq x'$ holds in Y.

Just before we end this section, we want to make record of the following simple but useful result.

Proposition 3.11 *Every reflective subcategory of* **NConv** *whose reflector is the identity is isomorphism-closed.*

Proof. Let $\mathrm{id}_X : X \to \tilde{X}$ be the reflector on a convergence space $(X, \to_X)$. Given any **K**-object Y which is isomorphic to X via a homeomorphism $f : X \longrightarrow Y$, one can extend it to a unique continuous mapping whose underlying map is again $f : \tilde{X} \longrightarrow Y$. Thus for any convergent net $(x_i)_{i \in I} \to_{\tilde{X}} x$, it follows from the continuity of $f : \tilde{X} \longrightarrow Y$ that $(f(x_i))_{i \in I} \to_Y f(x)$. Passing this convergence through the continuous map $f^{-1} : Y \longrightarrow X$, one immediately has $(x_i)_{i \in I} \to_X x$. Since $\tilde{X}$ is coarser than X, it follows that $\to_X = \to_{\tilde{X}}$. Thus $(X, \to_X)$ is an object in **K**.

4 Niceness conditions

Niceness is a salient aspect of net convergence spaces. Various niceness conditions, introduced in this section, correspond to existing one for filter convergence structures already considered by R. Heckmann in [4].

In the following, the order on a convergence space refers to the specialization order unless otherwise stated.

Definition 4.1 A space $(X, \to)$ is *up-nice* if whenever $(x_i)_{i \in I} \to x$ and $(y_j)_{j \in J}$ is a net that satisfies the following condition (†):

$$\forall i_0 \in I . \exists j_0 \in J . \forall j \sqsupseteq j_0 . y_j \in \uparrow T_{i_0}^{(x_i)}. \qquad (\dagger)$$

then $(y_j)_{j \in J} \to x$.

A space is *down-nice* if $(x_i)_{i \in I} \to x$ and $y \sqsubseteq x$ then $(x_i)_{i \in I} \to y$. A space is *order-nice* if it is both up-nice and down-nice. The objects of order-nice spaces and morphisms of continuous maps together constitute a full subcategory of convergence space which is denoted by **OnConv**.

Example 4.2 1. Every topological convergence space is order-nice.
2. For any poset P, the space $(P, \to_d)$ is order-nice.
3. For a poset P, in general, the space $(P, \to_D)$ mentioned in Example 3.7(6) is neither up-nice nor down-nice.

Theorem 4.3 *For a given space $(X, \to)$, define a coarser space $N(X)$ as follows:*

$$(x_i)_{i\in I} \to_{N(X)} x \iff \exists (z_k)_{k\in K} \in \Psi(X).\forall k_0 \in K.\exists i_0 \in I.\forall i \sqsupseteq i_0.x_i \in \uparrow T_{k_0}^{(z_k)} \qquad (\ddagger)$$

and $(z_k)_{k\in K} \to z$ for some $z \sqsupseteq x$. Then $N(X)$ carries the finest order-nice convergence structure on the same set which is coarser than X.

Proof. We now verify that $N(X)$ is an order-nice convergence space in stages:

1. From the definition of $\to_{N(X)}$, it's clear that if $(x_i)_{i\in I} \to x$ then $(x_i)_{i\in I} \to_{N(X)} x$. Thus the CONSTANT NET axiom is satisfied.
2. Assume that $(x_i)_{i\in I} \to_{N(X)} x$ and $(y_j)_{j\in J} \leq (x_i)_{i\in I}$. We want to show that $(y_j)_{j\in I} \to_{N(X)} x$. Since $(x_i)_{i\in I} \to_{N(X)} x$, one has a net $(z_k)_{k\in K}$ which satisfies the above condition ($\ddagger$) at one's disposal. Now for each $k_0 \in K$, one can find $i_0 \in I$ such that whenever $i \sqsupseteq i_0$ it holds that $x_i \in \uparrow T_{k_0}^{(z_k)}$. Corresponding to this i_0, there is $j_0 \in J$ such that whenever $j \sqsupseteq j_0$ we have $y_j \in T_{i_0}^{(x_i)}$. Consequently, this means that if $j \sqsupseteq j_0$ then $y_j \in \uparrow T_{k_0}^{(z_k)}$. Thus $(y_j)_{j\in J} \to_{N(X)} x$.
3. We show now that $N(X)$ is an order-nice space. That $N(X)$ is down-nice follows trivially from the transitivity of $\sqsubseteq$. It suffices to show that it is up-nice. For that purpose, suppose that $(x_i)_{i\in I} \to_{N(X)} x$ and $(y_j)_{j\in J}$ is a net that satisfies ($\dagger$). We aim to show that $(y_j)_{j\in J} \to_{N(X)} x$, i.e., there is a net $(z_k)_{k\in K}$ satisfying the property that for each $k_0 \in K$, there exists $j_0 \in J$ such that whenever $j \sqsupseteq j_0$ one has $y_j \in \uparrow T_{k_0}^{(z_k)}$. Since $(x_i)_{i\in I} \to_{N(X)} x$, there is already a net $(z_k)_{k\in K}$ such that

$$\forall k_0 \in K.\exists i_0 \in I.\forall i \sqsupseteq i_0.x_i \in \uparrow T_{k_0}^{(z_k)}$$

and $(z_k)_{k\in K} \to z$ with $z \sqsupseteq x$. Because of ($\dagger$), for such an $i_0 \in I$, one can find $j_0 \in J$ such that whenever $j \sqsupseteq j_0$ then $y_j \sqsupseteq x_{i'}$ for some $i' \sqsupseteq i_0$. Now since $i' \sqsupseteq i_0$, it follows that $x_{i'} \sqsupseteq z_{k'}$ for some $k' \sqsupseteq k_0$. Consequently, this means that for each $k_0 \in K$, one can find $j_0 \in J$ such that whenever $j \sqsupseteq j_0$ then $y_j \sqsupseteq z_{k'}$ with $k' \sqsupseteq k_0$, i.e., $y_j \in \uparrow T_{k_0}^{(z_k)}$.
4. Finally, we show that $N(X)$ carries the finest convergence structure on the same set X which is coarser than X. Suppose that $(X, \to')$ is order-nice and coarser than $(X, \to)$. We want to show that $N(X)$ is finer than $(X, \to')$. To this end, assume that $(x_i)_{i\in I} \to_{N(X)} x$. By definition, there exists $(z_k)_{k\in K}$ such that for each $k_0 \in K$ there exists $i_0 \in I$ such that whenever $i \sqsupseteq i_0$ we have $x_i \in \uparrow T_{k_0}^{(z_k)}$ and and $(z_k)_{k\in K} \to z$ for some $z \sqsupseteq x$. Since $(X, \to')$ is coarser than $(X, \to)$ and $(X, \to')$ is down-nice, it follows

that $(z_k)_{k\in K} \to' x$. Since $(X, \to')$ is up-nice, it follows that $(x_i)_{i\in I} \to' x$ by definition. Thus we can conclude that $N(X)$ is finer than $(X, \to')$ and the proof is complete.

Definition 4.4 For a given space $(X, \to)$, we define $N(X)$ to be the *order-nice modification* of X.

Proposition 4.5 **OnConv** *is a bireflective and isomorphism-closed subcategory of* **NConv**.

Proof. We extend N to a functor, assigning to each convergence space X its order-nice modification $N(X)$ and leaving each continuous map $f : X \to Y$ as it is. That $f : (X, \to_{N(X)}) \longrightarrow (Y, \to_{N(Y)})$ is continuous follows from the fact that f is monotone. Clearly, the identity mapping $\mathrm{id}_X : X \to N(X)$ is continuous. We aim to show that for each order-nice space $(Y, \to_Y)$ and continuous mapping $f : (X, \to_X) \longrightarrow (Y, \to_Y)$, the same mapping f is also a continuous mapping from $(X, \to_{N(X)})$ to $(Y, \to_Y)$. To this end, let $(x_i)_{i\in I} \to_{N(X)} x$. By definition, there is a net $(z_k)_{k\in K} \to_X z$ for some $z \sqsupseteq x$ such that

$$\forall k_0 \in K. \exists i_0 \in I. \forall i \sqsupseteq i_0. x_i \in \uparrow T_{k_0}^{(z_k)}.$$

Since $f : X \longrightarrow Y$ is continuous, we have $(f(z_k))_{k\in K} \to_Y f(z)$. Also f being continuous is monotone with respect to the specialization order so that $f(z) \sqsupseteq f(x)$. Moreover, for each $k_0 \in K$, there is $i_0 \in I$ such that whenever $i \sqsupseteq i_0$ we have $f(x_i) \in \uparrow T_{k_0}^{(f(z_k))}$. But Y is up-nice so that one can now conclude that $(f(x_i))_{i\in I} \to_Y f(z)$. Since $f(z) \sqsupseteq f(x)$ and Y is down-nice, we have that $(f(x_i))_{i\in I} \to_Y f(x)$ as desired.

Lastly, since the reflector is the identity, it follows from Proposition 3.11 that **OnConv** is isomorphism-closed.

The above proposition, together with Theorem 2.5, establishes the following fact:

Corollary 4.6 *Let $\{X_i \mid i \in I\}$ be a family of order-nice (resp. up-nice, down-nice) spaces and X a convergence space which carries the initial convergence structure with respect to a collection of functions $\{f_i : X \to X_i \mid i \in I\}$. Then X is order-nice (resp. up-nice, down-nice). In particular, all subspaces and products of order-nice (resp. up-nice, down-nice) spaces are themselves order-nice (resp. up-nice, down-nice).*

Proposition 4.7 *Let X and Y be convergence spaces. Then the following are equivalent:*

(i) Y is order-nice (resp. up-nice, down-nice).
(ii) $[X \to Y]$ is order-nice (resp. up-nice, down-nice).

Proof. We prove the above only for up-niceness. Down-niceness is similar.
(i) $\implies$ (ii): Suppose that $(f_i)_{i\in I} \to_{[X\to Y]} f$ and there is a net $(g_j)_{j\in J}$ such that

$$\forall i_0 \in I.\exists j_0 \in J.\forall j \sqsupseteq j_0.g_j \in\uparrow T_{i_0}^{(f_i)}.$$

We want to show that $(g_j)_{j\in J} \to_{[X\to Y]} f$. To achieve this, take an arbitrary $(x_k)_{k\in K} \to_X x$. Since $(f_i)_{i\in I} \to_{[X\to Y]} f$, it follows that $(f_i(x_k))_{(i,k)\in I\times K} \to_Y f(x)$. By the continuity of f_i and g_j, and their monotonicity, it follows that

$$\forall k_0 \in K.\forall i_0 \in I.\exists j_0 \in J.\forall j \sqsupseteq j_0.g_j(x_k) \in\uparrow T_{(i_0,k_0)}^{(f_i(x_k))}.$$

So this means that

$$\forall (i_0, k_0) \in I \times K.\exists (j_0, k_0) \in J \times K.\forall (j, k) \sqsupseteq (j_0, k_0).g_j(x_k) \in\uparrow T_{(i_0,k_0)}^{(f_i(x_k))}.$$

Now we appeal to the up-niceness of Y to conclude that $(g_j(x_k))_{(j,k)\in J\times K} \to_Y f(x)$. Thus $(g_j)_{j\in J} \to_{[X\to Y]} f$.
(ii) $\implies$ (i): Since Y may be regarded as a subspace of the order-nice space $[X \to Y]$, it immediately follows from Proposition 3.9 that Y is up-nice.

Corollary 4.8 *For any space X, the space $\Omega X := [X \to \Omega]$ is always order-nice (resp. up-nice, down-nice).*

Theorem 4.9 OnConv *is a Cartesian closed category.*

Proof. Because (1) the subcategory **OnConv** is closed under products and function spaces formed in **NConv**, and (2) the mappings are the same under the reflector N, the result then follows immediately from the Cartesian closedness of **NConv**.

5 Iterated-limit spaces

Kelley [7] characterized all topological convergence spaces to be exactly those convergence spaces which satisfy the following axioms:

1. (ITERATED LIMIT) A space $(X, \to)$ satisfies the ITERATED LIMIT (IL for short) axiom, if $(x_i)_{i\in I} \to x$ and $(x_{i,j})_{j\in J(i)} \to x_i$ for each $i \in I$, then $(x_{i,f(i)})_{(i,f)\in I\times M} \to x$, where $M = \Pi_{i\in I} J(i)$ is a product of directed sets.
2. (DIVERGENCE) If $(x_i)_{i\in I} \not\to x$, then $(x_i)_{i\in I}$ has a subnet $(y_j)_{j\in J}$ no subnet $(z_k)_{k\in K}$ of which ever has $(z_k)_{k\in K} \to x$.

In this section, we study the properties of spaces which satisfy the ITERATED LIMIT (IL, for short) axiom. We call such spaces *IL spaces*.

Remark 5.1 It follows immediately that every topological space is an IL space.

Proposition 5.2 *1. Let $\{X_i \mid i \in I\}$ be a family of IL spaces and X a convergence space which carries the initial convergence structure with respect to a collection of functions $\{f_i : X \to X_i \mid i \in I\}$. Then X is an IL space. In particular, all subspaces and products of IL spaces are themselves IL spaces.*

2. Every retract of an IL space is an IL space.

3. For any convergence space Y, the function space ΩY is an IL space.

Proof. For (1), let $(x_k)_{k\in K} \to_X x$ with respect to the initial convergence and suppose for each $k \in K$, there is a directed set $J(k)$ such that $(x_{k,j})_{j\in J(k)} \to_X x_k$. We want to show that $(x_{k,g})_{(k,g)\in K\times M} \to_X x$ where $M = \prod_{k\in K} J(k)$. This means that we have to show that for each $i \in I$, it holds that

$$(f_i(x_{k,g(k)}))_{(k,g)\in K\times M} \to_{X_i} f_i(x).$$

Because for each $k \in K$ it holds that $(x_{k,j})_{j\in J(k)} \to_X x_k$, it holds that for each $k \in K$ and for each $i \in I$, one has $(f_i(x_{k,j}))_{j\in J(k)} \to_{X_i} f_i(x_k)$. Since X_i's are all IL spaces, this implies that for each $i \in I$, the following convergence holds:

$$(f_i(x_{k,g(k)}))_{(k,g)\in K\times M} \to_{X_i} f_i(x)$$

where $M = \prod_{k\in K} J(k)$, as desired.

For (2), assume that that $f : Y \longrightarrow X,\ g : X \to Y$ are continuous maps such that $f \circ g = \mathrm{id}_X$ and Y is an IL space. We proceed to show that X is also an IL space. For that purpose, assume that $(x_i)_{i\in I} \to_X x$ and for each $i \in I$, there is a directed set $J(i)$ such that $(x_{j,i})_{j\in J(i)} \to_X x_i$. Since g is continuous, one has for each $i \in I$ that $(g(x_{j,i}))_{j\in J(i)} \to_Y g(x_i)$. As Y is an IL space, it follows that $(g(x_{i,f(i)}))_{(i,f)\in I\times M} \to_Y g(x)$ where $M = \prod_{i\in I} J(i)$. Now the proof will be complete by applying the continuous f to this net and then invoking the fact that $f \circ g = \mathrm{id}_X$. Thus X is an IL space.

For (3), assume that $(f_i)_{i\in I} \to f$ and for each $i \in I$ it holds that $(f_{i,j})_{j\in J(i)} \to f_i$. We aim to show that $(f_{i,g(i)})_{(i,g)\in I\times M} \to f$ where $M = \prod_{i\in I} J(i)$. To achieve this, we must show that for any given convergence $(x_k)_{k\in K} \to x$ it holds that $(f_{i,g(i)}(x_k))_{((i,g),k)\in(I\times M)\times K} \to f(x)$. Without loss of generality, assume that $f(x) = 1$. Since $(f_i(x_k))_{(i,k)\in I\times K} \to f(x)$, it follows that there are $i_0 \in I$ and $k_0 \in K$ such that $T^{(f_i(x_k))}_{(i_0,k_0)} \subseteq \{1\}$. It follows from $(f_{i,j})_{j\in J(i)} \to f_i$ for every $i \in I$ that in particular for each of those $i \sqsupseteq i_0$, there are $j_i \in J(i)$ and $k_i \in K$ such that for all $j \sqsupseteq j_i,\ k \sqsupseteq k_i$ one has $f_{i,j}(x_k) = 1$. Since K is directed, we may assume that $k_i \sqsupseteq k_0$. Given that $k \sqsupseteq k_0$, we can now define $g_0 \in M$ as follows:

$$g_0 : I \to \dot{\bigcup}_{i\in I} J(i),\ g_0(i) = j_i$$

where j_i is as above. Now for any $g \sqsupseteq g_0$, one has $g(i) \sqsupseteq g_0(i)$ for any $i \in I$, i.e., $g(i) \sqsupseteq j_i$. Thus for any $k \sqsupseteq k_0,\ i \sqsupseteq i_0$ and $g \sqsupseteq g_0$, it holds that $f_{i,g(i)}(x_k) = 1$. Hence $(f_{i,g(i)}(x_k))_{((i,g),k)\in(I\times M)\times K} \to f(x) = 1$.

6 d-spaces

For any net $(x_i)_{i\in I}$ in a space $(X, \rightarrow)$, denote $lim(x_i)_{i\in I} = \{x \in X : (x_i)_{i\in I} \rightarrow x\}$.

Definition 6.1 A convergence space $(X, \rightarrow)$ is called a d-space if

1. it is order-nice and T_0,
2. for any directed subset $(x_i)_{i\in I}$ of X, $x = \bigsqcup_{i\in I} x_i$ exists and $(x_i)_{i\in I} \rightarrow x$, and
3. for every net $(x_i)_{i\in I}$, the limit set $\lim(x_i)_{i\in I}$ is closed under directed suprema.

Example 6.2 1. A topological space is a d-space if and only if it is a monotone convergence space.
2. For each poset P, $(P, \rightarrow_D)$ is a d-space if and only if $(P, \rightarrow_d)$ is a d-space, and in turn, if and only if P is a dcpo.

Proposition 6.3 *1. If X is a d-space, then every closed set F of X is closed under taking supremum of directed sets.*
2. If X is a d-space and $f : X \longrightarrow Y$ is a continuous mapping, then for any directed set D of X,

$$f(\bigsqcup D) = \bigsqcup f(D).$$

3. d-space is stable under retract.
4. Let X and Y be convergence spaces. Then $[X \rightarrow Y]$ is a d-space if and only if Y is a d-space. In particular, ΩX is a d-space for any convergence space X.
5. The product of any collection of d-spaces is again a d-space.

Proof. 1. This follows from that every directed subset, as a net, converges to its supremum.
2. For any directed set $D \subseteq X$, $x = \bigsqcup D$ exists. Since f is monotone, it follows that $f(\bigsqcup D)$ is an upper bound of $f(D)$. If $y \in Y$ is an upper bound of $f(D)$, then $D \subseteq f^{-1}(\downarrow y)$. However, $\downarrow y = cl(\{y\})$ is a closed set, thus by part one, $\bigsqcup D \in f^{-1}(\downarrow y)$ and thus $f(\bigsqcup D) \leq y$. Hence $f(\bigsqcup D) = \bigsqcup f(D)$.
3. Let $s : X \longrightarrow Y$ and $r : Y \longrightarrow X$ be continuous mappings, where $r \circ s = \mathrm{id}_X$, X is a convergence space and Y a d-space. Since order-niceness has already been shown to be stable under retracts and trivially X is T_0, it remains to show that the convergence criteria are satisfied. Given any directed net $(x_i)_{i\in I}$ we wish to show that the supremum of $\{x_i : i \in I\}$ exists and $(x_i)_{i\in I} \rightarrow_X x$. Owing to the fact that Y is a d-space, it follows that the directed net $s(x_i)_{i\in I} \rightarrow_Y y$, where $y = \bigsqcup s(x_i)_{i\in I}$. Then by 2, $r(y) = \bigsqcup\{r(s(x_i)) : i \in I\} = \{x_i : i \in I\}$. Also $(x_i)_{i\in I} = (r(s(x_i)))_{i\in I} \rightarrow_X r(y)$. That $\lim(x_i)_{i\in I}$ is closed under directed sups follows from the continuity of r and s and the fact that for any directed set D of X, $\bigsqcup D = r(\bigsqcup s(D))$ proved just now.

4. It's easily shown that $[X \to Y]$ is T_0. By virtue of (3) and Proposition 3.9, it suffices to prove the sufficiency condition. So assume that Y is a d-space. Let $(f_i)_{i\in I}$ be a directed family in $[X \to Y]$. We propose that the function

$$f : X \longrightarrow Y, f(x) = \bigsqcup_{i\in I} f_i(x)$$

is the supremum of $(f_i)_{i\in I}$. Firstly, f is well-defined since $(f_i)_{i\in I}$ is directed with respect to the pointwise order and Y is a d-space so that the directed set $\{f_i(x) \mid i \in I\}$ in Y has a supremum. Next, we show that f is continuous. For any net $(x_k)_{k\in K}$ in X which converges to x, we aim to prove that $(f(x_k))_{k\in K} \to_Y f(x)$. Notice that $f(x_k) \sqsupseteq f_i(x_k)$ for each $i \in I$ so that by the order-niceness of Y and the continuity of the f_i's, one has $(f(x_k))_{k\in K} \to_Y f_i(x)$. Since $\lim(f(x_k))_{k\in K}$ is closed under directed suprema, it holds that $(f(x_k))_{k\in K} \to_Y \bigsqcup_{k\in K} f_k(x)$, i.e., $(f(x_k))_{k\in K} \to_Y f(x)$. Thus f is continuous. That f is indeed the supremum of the continuous mappings f_i's is obvious. Following the definition of the convergence in $[X \to Y]$ one can also verify straight forwardly that for any net $(h_i)_{i\in I}$ in $[X \to Y]$, $\lim(h_i)_{i\in I}$ is closed under directed suprema. Thus $[X \to Y]$ is a d-space.
5. We just show the second condition of d-space is satisfied by product of d-spaces. Let $(m_j)_{j\in J}$ be a directed family in $\prod_{i\in I} X_i$ where X_i's are d-spaces. Then for each $i \in I$, the image of the i-th projection $(\pi_i(m_j))_{j\in J}$ is a directed family in X_i since π_i is continuous. So $\bigsqcup_{j\in J} \pi_i(m_j)$ exists as X_i is a d-space. By putting $x_i := \bigsqcup_{j\in J} \pi_i(m_j)$ and define the net $x = (x_i)_{i\in I}$, then one clearly has $\bigsqcup_{j\in J} m_j = x$ and $(m_j)_{j\in J} \to x$.

Theorem 6.4 *The category of d-spaces is Cartesian closed.*

A subspace X_0 of a space X is called a d-base of X if for any $x \in X$, there is a directed family $\{y_i\}_{i\in I} \subseteq X_0$, such that $(y_i)_{i\in I} \to_X x = \bigsqcup\{y_i : i \in I\}$.

Central to the theory of D-completion of convergence spaces is the following fundamental lemma.

Lemma 6.5 *Let X be an order-nice space satisfying the IL axiom and X_0 be a d-base of X. Then every continuous mapping f_0 from the subspace X_0 into a d-space Y has a unique continuous extension f on X.*

Proof. For each $x \in X$, let $(y_i)_{i\in I}$ be a directed family in X_0 such that $(y_i)_{i\in I} \to_X \bigsqcup_{i\in I} y_i = x$. Define $f(x) := \bigsqcup_{i\in I} f_0(y_i)$. It is not clear whether f is well-defined, let alone continuous. So we must first prove that f is well-defined. To do this, we take any two directed nets $(y_i)_{i\in I}$ and $(z_k)_{k\in K}$ in X_0 such that $(y_i)_{i\in I} \to_X \bigsqcup_{i\in I} y_i = x$ and $(z_k)_{k\in K} \to_X \bigsqcup_{k\in K} z_k = x$. We shall show that

$$\bigsqcup_{i\in I} f_0(y_i) = \bigsqcup_{k\in K} f_0(z_k).$$

Now since $(y_i)_{i\in I} \to_X x$ and $x \sqsupseteq z_k$ for each $k \in K$, so $(y_i)_{i\in I} \to_X z_k$ for each $k \in K$ by the down-niceness of X. Then by the continuity of $f_0 : X_0 \longrightarrow Y$, we have $(f_0(y_i))_{i\in I} \to_Y f_0(z_k)$ for each $k \in K$ and hence the set $\{f_0(z_k) \mid k \in K\} \subseteq \lim_Y(f_0(y_i))_{i\in I}$. As the latter set is Scott-closed (since Y is a d-space), it follows that $\bigsqcup_{k\in K} f_0(z_k) \in \lim_Y(f_0(y_i))_{i\in I}$. So $(f_0(y_i))_{i\in I} \to_Y \bigsqcup_{k\in K} f_0(z_k)$. Also $\bigsqcup_{i\in I} f_0(y_i) \sqsupseteq f_0(y_i)$, the constant net $(\bigsqcup_{i\in I} f_0(y_i)) \to_Y \bigsqcup_{k\in K} f_0(z_k)$ by the up-niceness of Y. Similarly, the constant net $(\bigsqcup_{k\in K} f_0(z_k)) \to_Y \bigsqcup_{i\in I} f_0(y_i)$. Because Y is a T_0-space, we have that $\bigsqcup_{i\in I} f_0(y_i) = \bigsqcup_{k\in K} f_0(z_k)$ as desired. So f is well-defined.

We now show that f is continuous. Let $(x_k)_{k\in K} \to_X x$. We want to show that $(f(x_k))_{k\in K} \to_Y f(x)$. Since X_0 is a d-base for X, it follows that

1. for each $k \in K$, there exists a directed net $(y^k_j)_{j\in J(k)} \to_X \bigsqcup_{j\in J(k)} y^k_j = x_k$, and
2. there is a directed net $(y_j)_{j\in J} \to_X \bigsqcup_{j\in J} y_j = x$.

It is enough to show that $f(x_k) \sqsupseteq f_0(y_j)$ eventually for each $j \in J$, relying on the up-niceness of Y. For this purpose, we suppose that there exists $j_0 \in J$ such that for all $k' \in K$, there is $k \in K$ with $k \sqsupseteq k'$ such that $f(x_k) \not\sqsupseteq f_0(y_{j_0})$. Then $f_0(y_{j_0})$ belongs to the open set $Y\setminus\downarrow f(x_k) = Y\setminus\downarrow \bigsqcup_{j\in J(k)} f_0(y^k_j)$. Since X is an IL space, it follows that $(y^k_g(k))_{(k,g)\in K\times M} \to_X x$ where $M = \prod_{i\in I} J(i)$. Then by the down-niceness of X, this net will converge to y_{j_0}. Invoking the continuity of f_0, it follows that $(f_0(y^k_g(k)))_{(k,g)\in K\times M} \to_Y f_0(y_{j_0})$. Because $Y\setminus\downarrow \bigsqcup_{j\in J(k)} f_0(y^k_j)$ is an open set, one deduces that $f_0(y^k_{g(k)})$ is in the same open set eventually, implying that $f_0(y^k_{g(k)}) \not\sqsubseteq \bigsqcup_{j\in J(k)} f_0(y^k_j)$ for some g, which is impossible. So the proof is completed.

Definition 6.6 Given a space $(X,\to)$, a D-completion of $(X,\to)$ is a pair (Y,η) where Y is a d-space and $\eta : X\longrightarrow Y$ is a continuous mapping such that for each continuous mapping $f : X\longrightarrow Z$ into a d-space Z there is a unique continuous mapping $\hat{f} : Y\longrightarrow Z$ such that $f = \hat{f}\circ\eta$.

By [2, 6], every T_0 topological space has a D-completion. In [14], every poset has a dcpo-completion. Our main concern in this section is which net convergence spaces, apart from the topological ones, have a D-completions.

Theorem 6.7 *If X can be embedded into a IL d-space, then X has a D-completion.*

Proof. For the sake of convenience, we assume X is a subspace of an IL d-space Y. For each subspace A of Y, define

$$d(A) = \{y \in Y \mid \exists \text{ a directed family } x_i \subseteq A.(x_i)_{i\in I} \to_Y \bigsqcup_{i\in I} x_i = y\}.$$

Now for each ordinal α, we define a subset X^α by transfinite induction as follows:

$$X^1 = d(X),\ X^{\beta+1} = d(X^\beta) \text{ and } X^\alpha = \bigcup_{\gamma<\alpha} X^\gamma \text{ if } \alpha \text{ is a limit ordinal.}$$

By the usual ordinal reason, there is a smallest ordinal α satisfying the equation

$$X^{\alpha+1} = X^\alpha.$$

For any continuous mapping $g : X \longrightarrow Z$ from X into a d-space Z, there is a unique extension of g over X^1 by virtue of Lemma 6.7. By transfinite induction, it follows that f has a unique continuous extension over X^α.

Since every T_0 topological space is embedded into some Ω^M (c.f. Lemma II-3.4 of [3]), where Ω is the Sierpinski space and since Ω is clearly a d-space, so we have:

Corollary 6.8 *Every T_0 topological space has a D-completion which is also topological.*

7 Conclusion

In this paper, we have suggested a way of generalizing the concept of a d-space from topological spaces to net convergence spaces. In addition, we have obtained a sufficient condition that translates to a particular class of net convergence spaces that admits a d-completion to exist for each member of this class. Crucially, this class of net convergence spaces contains as members all T_0 topological spaces, and thus we have generalized existing results concerning d-completions of T_0-spaces and dcpo-completions of posets. However, the authors have not justified at the moment of writing whether such a class of net convergence spaces *properly* contains the category of T_0 spaces. Thus any future works leading from here must crucially include the construction of an example of a non-topological net convergence space which can be embedded into an d-space which has all iterated limits.

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