A Family of Identities via Arbitrary Polynomials

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Two conjectures were posed in 2007 by Thomas Dence in this JOURNAL [2].

Conjecture 1. *Let n and k be odd positive integers with $k \le n$. Then*

$$\sum_{j=0}^{(n-1)/2} \binom{n}{j} (-1)^j (n-2j)^k = \begin{cases} 0, & \text{if } k < n; \\ 2^{n-1} n!, & \text{if } k = n. \end{cases}$$

Conjecture 2. *Let n and k be even positive integers with $k \le n$. Then*

$$\sum_{j=0}^{(n-2)/2} \binom{n}{j} (-1)^j (n-2j)^k = \begin{cases} 0, & \text{if } k < n; \\ 2^{n-1} n!, & \text{if } k = n. \end{cases}$$

In 2009, Hidefumi Katsuura established the following result from which he deduced the above conjectures.

Theorem (Katsuura [3]) *For any complex numbers x and y, and any positive integer n, we have*

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j (xj+y)^k = \begin{cases} 0 & \text{if } k = 0, 1, \cdots, n-1 \\ (-x)^n \times n! & \text{if } k = n. \end{cases}$$

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This is actually a special case of a much more general result which we prove and thus reprove Katsuura's result and conjectures 1 and 2.

First, for any complex number z and non-negative integer k, define

$$(z)_k = \begin{cases} 1, & \text{if } k = 0; \\ z(z-1)\cdots(z-k+1), & \text{otherwise.} \end{cases}$$

Theorem *Assume that $(z-1)^n$ divides the complex polynomial $f(z) = \sum_{i=0}^{m} a_i z^i$, where n is a positive integer. Let $P(z)$ be any complex polynomial of degree at most n.*

(i) *If the degree of $P(z)$ is less than n, then*

$$\sum_{i=0}^{m} a_i P(i) = 0. \tag{1}$$

(ii) *If the degree of $P(z)$ is n, then*

$$\sum_{i=0}^{m} a_i P(i) = c \sum_{i=n}^{m} a_i (i)_n,$$

where c is the coefficient of z^n in $P(z)$.

Proof. (i) It suffices to show that (1) holds for $P(z) = (z)_k$ for $k = 0, 1, 2, \cdots, n-1$, because every polynomial of degree at most $n-1$ is a linear combination of the polynomials $1, z, (z)_2, \cdots, (z)_{n-1}$.

Since $(z-1)^n$ is a factor of $f(z)$, we have $f(1) = 0$ and $f^{(k)}(1) = 0$ for $k = 1, 2, \cdots, n-1$, where $f^{(k)}(z)$ is the kth derivative of $f(z)$, i.e.,

$$0 = f^{(k)}(1) = \sum_{i=0}^{m} a_i (i)_k.$$

Thus (1) holds for $P(z) = (z)_k$.

(ii) If $P(z)$ has degree n, then $P(z) - c(z)_n$ is a polynomial of degree at most $n-1$. By (i), we have

$$\sum_{i=0}^{m} a_i(P(i) - c(i)_n) = 0,$$

or equivalently,

$$\sum_{i=0}^{m} a_i P(i) = c\sum_{i=0}^{m} a_i(i)_n = c\sum_{i=n}^{m} a_i(i)_n,$$

where the last equality follows from the fact that $(i)_n = 0$ when $i \in \{0, 1, \cdots, n-1\}$. $\square$

Corollary *Let n be any positive integer and $P(z)$ be any complex polynomial of degree at most n.*

(i) *If the degree of $P(z)$ is less than n, then*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}P(i) = 0.$$

(ii) *If the degree of $P(z)$ is n, then*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}P(i) = c(-1)^n n!.$$

where c is the coefficient of the term z^n in $P(z)$.

This follows directly by taking $f(z) = (-1)^n(z-1)^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}z^i$.

Katsuura's theorem now follows directly by letting $P(z) = (xz+y)^k$, where x and y are complex numbers. We can also apply the corollary to prove conjectures 1 and 2 directly.

Let $P(x) = (n-2x)^k$. From the corollary, if $k < n$,

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-2j)^k = 0,$$

and if $k = n$,

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-2j)^k = (-2)^n(-1)^n n! = 2^n n!.$$

Observe that if both k and n are odd, then

$$\begin{aligned}
&\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-2j)^k \\
= &\sum_{j=0}^{(n-1)/2}(-1)^j\binom{n}{j}(n-2j)^k + \sum_{j=(n+1)/2}^{n}(-1)^j\binom{n}{j}(n-2j)^k \\
= &\sum_{j=0}^{(n-1)/2}(-1)^j\binom{n}{j}(n-2j)^k + \sum_{i=0}^{(n-1)/2}(-1)^{n-i}\binom{n}{n-i}(2i-n)^k \\
= &\ 2\sum_{j=0}^{(n-1)/2}(-1)^j\binom{n}{j}(n-2j)^k.
\end{aligned}$$

This is Conjecture 1. If both k and n are even, then

$$\begin{aligned}
&\sum_{j=0}^{n}(-1)^j\binom{n}{j}(n-2j)^k \\
= &\sum_{j=0}^{(n-2)/2}(-1)^j\binom{n}{j}(n-2j)^k + (-1)^{n/2}\binom{n}{n/2}\left(n-2\times\frac{n}{2}\right)^k + \sum_{j=(n+2)/2}^{n}(-1)^j\binom{n}{j}(n-2j)^k \\
= &\sum_{j=0}^{(n-2)/2}(-1)^j\binom{n}{j}(n-2j)^k + \sum_{i=0}^{(n-2)/2}(-1)^{n-i}\binom{n}{n-i}(2i-n)^k \\
= &\ 2\sum_{j=0}^{(n-2)/2}(-1)^j\binom{n}{j}(n-2j)^k.
\end{aligned}$$

This is Conjecture 2.

Many well-known combinatorial identities (see, for example, [1 pp106, 159]) are instances of our corollary, i.e., for special choices of the polynomial $P(z)$. Here are some examples.

(a) The following identity can be obtained by choosing $P(z)$ to be z^k:

$$\sum_{r=0}^{n}(-1)^r r^k\binom{n}{r} = \begin{cases} 0, & \text{if } 0 \le k < n; \\ (-1)^n n!, & \text{if } 0 < k = n. \end{cases}$$

(b) The following result can be obtained from our Corollary by choosing $P(z)$ to be $\binom{z}{k} = \frac{(z)_k}{k!}$:

$$\sum_{r=k}^{n}(-1)^r\binom{n}{r}\binom{r}{k} = \begin{cases} 0, & \text{if } 0 \le k < n; \\ (-1)^n, & \text{if } 0 < k = n. \end{cases}$$

(c) The following result follows by choosing $P(z)$ to be $(n-z)^k$:

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(n-r)^k = \begin{cases} 0, & \text{if } 0 \le k < n; \\ n!, & \text{if } 0 < k = n. \end{cases}$$

(d) For any positive integer q, the following result can be from our theorem by choosing $f(z)$ to be $(1-z^q)^n$ and $P(z)$ to be z^k:

$$\sum_{r=0}^{n}(-1)^r\binom{n}{r}(qr)^k = \begin{cases} 0, & \text{if } 0 \le k < n; \\ (-1)^n q^n n!, & \text{if } 0 < k = n. \end{cases}$$

Summary

In this short article, we prove an identity from which a theorem of Katsuura and two conjectures previously posed in this JOURNAL follow directly.

References

1. C. C. Chen and K.M. Koh, *Principles and Techniques in Combinatorics*, World Scientific, 1992.

2. T. P. Dence, Some Half-Row Sums from Pascal's Triangle via Laplace Transforms, this JOURNAL **38** (2007), 205-209.

3. H. Katsuura, Summations Involving Binomial Coefficients, this JOURNAL **40** (2009), 275-278.