# Characterising E-projectives via Co-monads

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Abstract. This paper demonstrates the usefulness of a comonadic approach to give previously unknown characterisation of projective objects in certain categories over particular subclasses of epimorphisms. This approach is a simple adaptation of a powerful technique due to M. Escardó which has been used extensively to characterise injective spaces and locales over various kinds of embeddings, but never previously for projective structures. Using some examples, we advertise the versatility of this approach – in particular, highlighting its advantage over existing methods on characterisation of projectives, which is that the comonadic machinery forces upon us the structural properties of projectives *without* relying on extraneous characterisations of the underlying object of the co-algebra arising from the comonad.

Keywords: E-projectives, KZ-comonads, Right U-quotients, Ordered monoids, Normal semi-rings, Semilattices, Z-frames

## 1. Introduction

The problem of characterizing projectives and their duals, injectives, in various categories has a long history in mathematics with its origin tracing back to module theory, e.g., characterizing projective modules in module theory. An object $P$ of a category $\mathbf{C}$ is *projective* if for every epimorphism $e : A \longrightarrow B$ and every morphism $f : P \longrightarrow B$, there is a $\mathbf{C}$-morphism (not necessarily unique) $f' : P \longrightarrow A$ such that $f = f' \circ e$. In the category of sets, every object is projective, while the only projectives in the category of groups are the free ones. In algebra, projective objects are viewed as a generalization of free objects.

Much later on, attention with regards to the study of projectives and injectives was shifted from algebraic structures to ordered structures and topological spaces. R. Sikorski showed that the injectives in the category of boolean algebras are precisely the complete boolean algebras ([**19**]). Later, R. Balbes extended this result to show that the injectives in the category of distributive lattices are also the complete boolean algebras ([**2**]). G. D. Crown in [**5**] characterised the projectives (and injectives) in the category of sup complete lattices as the completely distributive lattices. Another important example is D.S. Scott's discovery that the injective $T_0$-spaces are exactly the continuous lattices with their Scott-topology [**18**]. Notably, in their effort of generalizing Scott's result to frames, B. Banaschewski and S. B. Niefeld in [**4**] reported that the only projective frame is **2** the two-element chain. Indeed, a similar result has been already been reported in [**3**] for the category of distributive lattices. For situations like these where projectives are scarce, it is natural to consider more general types of projective objects, whence the notion of **E**-projectives. To achieve this, one takes not all the epimorphisms but only a certain subclass **E** of epimorphisms. More precisely, an object $P$ of a category **C** is **E**-projective or projective over the **E**-morphisms if for every **C**-morphism $f : P \longrightarrow A$ and every **E**-morphism $e : A \longrightarrow B$, there is a **C**-morphism $f' : P \longrightarrow A$ such that $f = f'e$. Dually, there is the notion of **E**-injectives.

In [**4**] B. Banaschewski and S. B. Niefeld showed that the projectives over **E** the collection of *regular epimorphisms* (i.e., morphisms which are co-equalisers of some parallel pair of morphisms) in the category of frames are exactly the stably completely distributive lattices; thereby relaxing on the condition of projectivity to admit a larger class of objects. In the literature, projectives over the regular epimorphisms are also called *regular projectives*. In certain categories such as the category **KHausSp** of compact Hausdorff spaces and continuous maps, every epimorphism is regular. Then for such categories the problem amounts to that of characterising the regular projectives; and this latter problem, in certain situations, may turn out to be easier. One such instance is Gleason's theorem, i.e., the projective objects in **KHausSp** (which coincide with the regular projective objects) are precisely the extremally disconnected spaces (refer to the proof found in [**13**, pp. 98–103]).

Recent years have seen a continued interest in the both the areas of characterisation of the **E**-injectives and projectives for poset-enriched categories. Along the 'injective' line, one important advancement was made by M. H. Escardó. His technique involves the use of KZ-monads to characterise injectives over certain embeddings in some poset-enriched categories. Escardó's method relies heavily on the following result:

THEOREM 1.1. *(*[**9**, Theorem 4.2.2, p.32]*)*
*Let $T$ be a KZ-monad on* **X**. *Then, the following are equivalent for any $A \in$* **X**:

(i) *$A$ is right injective over right $T$-embeddings.*
(ii) *$A$ is injective over right $T$-embeddings.*
(iii) *$A$ is a $T$-algebra.*

This method is particularly powerful since this monadic approach characterises the injectives over right $T$-embeddings (where $T$ is the given KZ-monad on the category) as precisely the underlying algebras of the monad $T$. The KZ-monadic machinery then forces upon us the characterisation of these underlying algebras by invoking certain reflective adjunctions (see condition (KZ1) on [**9**, Lemma 4.1.1, p.31]). This technique proves versatile in reproducing well-known characterisations of injectives in various poset-enriched categories (e.g., the injectives over subspace embeddings are the continuous lattices, see [**8, 9**]), as well as new ones (e.g., the injective locales over perfect sublocale embeddings are those that satisfy some new kind of stable continuity, see [**10**]).

On the 'projective' side of the development, most contributions came from the Asian authors ([**23, 21, 16**]). Notably, Zhao in [**23**] set the path for the latter two to follow. Based only on adjunctions between categories (not necessarily poset-enriched ones), Zhao's method has a much lighter mathematical overhead as compared to Escardó's. Crucially, the preceding three works hinge on the following lemma:

LEMMA 1.2. *(*[**23**, Lemma 3.1, p.43]*)*
*Let* $G : \mathbf{C} \longrightarrow \mathbf{D}$ *and* $F : \mathbf{D} \longrightarrow \mathbf{C}$ *be a pair of functors such that* $F$ *is left adjoint to* $G$*, with co-unit denoted by* $\varepsilon$*. Further, let* $\mathbf{E}$ *denote the collection of all* $\mathbf{C}$*-morphisms* $f : A \longrightarrow B$ *such that* $G(f)$ *has a section, i.e., a right inverse in* $\mathbf{D}$*. Then, for any* $A \in \mathbf{C}$*, the following are equivalent:*

(i) $A$ *is* $\mathbf{E}$*-projective.*
(ii) $\varepsilon : FG(A) \longrightarrow A$ *has a right inverse.*
(iii) $A$ *is a retract of some* $FX$ *for some object* $X$ *in* $\mathbf{D}$*.*

Using the above lemma, [**23**] showed that the **E**-projective frames are precisely the stably $Z$-continuous ones, while [**21**] proved that the **E**-projective $Z$-quantales are precisely the stably $Z$-continuous ones. In a similar vein, [**16**] characterised the **E**-projective normal semi-rings as those stably $F$-continuous ones. All these three works share a common strategy in their use of Lemma 1.2: their characterisation of projectives relies heavily on condition (ii). In order to use (ii), one inevitably needs to obtain certain structural properties of $FG(A)$ and its stability under retracts in order to deduce that $A$ share those properties as $FG(A)$. These structural properties are derived *independent* of Lemma 1.2. In other words, for the more general adjunctions $F \vdash G$, their approach would work only if some salient structural properties of canonical structure $FX$ have been independently identified.

Owing to this particular constraint, Zhao's approach fails precisely in those cases where the structural properties of the underlying coalgebra $FG(A)$ of the comonad $FG$ are not easily available. For instance, with respect to the adjunction pair $G : \mathbf{Frm} \longrightarrow \mathbf{PreFrm}$ and $F : \mathbf{PreFrm} \longrightarrow \mathbf{Frm}$ between the categories of pre-frames and frames (where $G$ is the forgetful functor and $F$ is the Scott-closed lattice functor), the question of characterising the **E**-projectives in the sense of Zhao [**23**] still remains open. Even though some explicit lattice-theoretic properties of $F(P)$ have been worked out in [**12**], it is still not known how E-projectives may be characterised (see [**12**, p.313]). This question has drawn the attention of domain theorists, especially in the Asian[1] region.

Näively speaking, Escardó's method appears to be superior: the monadic machinery forces out the innate properties of the canonical structures (i.e., the algebras of the monad) without having to derive them elsewhere outside the main development. But this kind of comparison may not be meaningful or fair since the two methods were intended for different purposes: one is to characterise injectives and the other projectives; for different **E** morphism classes; and for different categorical environments: one requires poset-enriched categories and the other does not. Therefore, a natural question to ask is whether Escardó's method which is powerful in dealing with the injectives can be modified to deal with the projectives. In view of Theorem 1.1, an affirmative answer to this question might allow us to characterise the underlying coalgebra (of those comonads we are interested) as precisely the **E**-projectives for some class **E** of morphisms.

In this paper, we play the category-dual game by turning Escardó's set-up in the opposite direction. This involves carefully replacing all categorical concepts in the formulation of KZ-monads on [9, pp.31–34] by their duals, e.g., monomorphisms by epimorphisms, injectives by projectives, and choosing the correct inequalities.

[1] Private communications with some Asian domain-theorists revealed that they were unaware of Escardó's works [**10, 9, 10**].

The derived results are then applied to three different adjunctions between poset-enriched categories to yield new characterisations of projectives. To be do this, we introduce all these three examples in the preliminaries, and use them as running examples later in the paper. The preliminaries section also contains some essential definitions and results concerning comonads and Kan lifts. Our aim is to demonstrate that (the dual version of) Escardó's set-up is, in fact, just as powerful in characterising projectives as it is for injectives. In the ensuing sections, we assume the reader is familiar with basic category theory, order and lattice theory and domain theory. For category theory, we refer the reader to [**15, 17**]; order and lattice theory [**6**]; and domain theory [**1, 11**].

## 2. Preliminaries

### 2.1. Running examples.

2.1.1. *Ordered monoids and normal semirings.* Ordered algebraic structures not only yield a rich resource of poset-enriched categories but also numerous instances of categorical adjunctions. Here we focus on ordered monoids and normal semi-rings.

Recall that a *monoid* is a semi-group, i.e., almost a group except for the existence of inverses. The triple $(M, \cdot, \leq)$ is an *ordered monoid* if $M$ is a monoid with identity $1_M$, together with a partial order $\leq$ on it which is compatible with the monoid operation, i.e., for any $a,\ b$ and $c \in M$, $a \leq b$ implies $a \cdot c \leq b \cdot c$, and $c \cdot a \leq c \cdot b$, and the multiplicative identity $1_M$ is the top element of $M$. The set of natural numbers has two different well-known ordered monoid structures, namely, $(\mathbb{N}, +, \leq)$ and $(\mathbb{N}, \max, \leq)$.

Let $(M, \cdot, \leq)$ and $(K, \otimes, \leq)$ be any two ordered monoids, with identities $1_M$ and $1_K$ respectively. A mapping $f : M \longrightarrow K$ is an *ordered monoid morphism* if the following conditions hold: (i) $f(a \cdot b) = f(a) \otimes f(b)$ for any $a,\ b \in M$. (ii) $f(1_M) = 1_K$. (iii) $a \leq b$ implies $f(a) \leq f(b)$. Having defined the salient arrows, we now have **OrdMon** the category whose objects are the ordered monoids and whose morphisms the ordered monoid morphisms.

We now turn to look at semi-rings. Semi-rings are a notion intended to generalize rings. A *semi-ring* is a non-empty set $R$ on which operations of addition and multiplication have been defined such that: (i) $(R, +)$ is a commutative monoid with identity element 0. (ii) $(R, \cdot)$ is a monoid with identity element $1_R$ (or simply 1). (iii)For all $a,\ b$ and $c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$. (iv) For all $r \in R$, $0 \cdot r = 0 = r \cdot 0$. (v) $1 \neq 0$.

Some examples of semi-rings are given below:

(i) The set of natural numbers, together with the usual addition and multiplication, $(\mathbb{N}, +, \cdot)$, is a commutative semi-ring.
(ii) A bounded distributive lattice $(L, \vee, \wedge)$ is a commutative idempotent semi-ring. Here, idempotence of a semi-ring refers to the idempotence of both addition and multiplication.
(iii) Let $R$ be a ring. The set of ideals of $R$, denoted by $\mathrm{Id}(R)$, with the usual addition $I + J := \{i + j \mid i \in I,\ j \in J\}$ and and multiplication of ideals $I \cdot J := \{i \cdot j \mid i \in I,\ j \in J\}$, is a semi-ring.
(iv) Let $R$ be a commutative ring and $S$ be the set of all elements of $R$ which are not zero-divisors. Let $T := S^{-1}R$ be the total ring of quotients of $R$. A *fractional ideal* $K$ of $R$ is an $R$-submodule of $T$ satisfying the condition

that $aK \subseteq R$ for some $a \in R$. The set fract$(R)$ of all fractional ideals of $R$ is closed under taking intersections, sums and products. Moreover, $(\text{fract}(R), +, \cdot)$ is a commutative semi-ring with additive identity $(0)$ and multiplicative identity $R$.

(v) A commutative integral domain $R$ is a *Prüfer domain* if every finitely generated fractional ideal of $R$ has a multiplicative inverse in fract$(R)$. This condition is equivalent to the condition that, in Id$(R)$, the intersection distributes over addition, i.e., $(Id(R), +, \bigcap)$ is a semi-ring.

If $R$ and $S$ are semi-rings, then a function $\gamma : R \longrightarrow S$ is a *semi-ring morphism* if the following conditions hold: (i) $\gamma(0_R) = 0_S$. (ii) $\gamma(1_R) = 1_S$. (iii) $\gamma(r + r') = \gamma(r) + \gamma(r')$ and $\gamma(r \cdot r') = \gamma(r) \cdot \gamma(r')$ for all $r,\ r' \in R$.

Here, we deal with a special kind of semi-rings. A semi-ring $R$ is *normal* if $x \cdot y + x = x = y \cdot x + x$ for all $x$ and $y \in R$. It is immediate that normal semi-rings are additively idempotent. Denote by **SRng** the category of semi-rings and semiring morphisms. The full subcategory, **NSRng**, of **SRng** consists of all normal semi-rings as objects.

For a poset $(P, \leq)$ and $X \subseteq P$, we denote the set $\{p \in P \mid \exists x \in X.\ p \leq x\}$ by $\downarrow X$. In case $X = \{x\}$, a singleton, we unambiguously write $\downarrow x$ for $\downarrow X$. Dually, one defines $\uparrow$. If $X$ is a finite subset of $Y$ , we use the notation $X \in \mathcal{P}_{\text{fin}}(Y)$ or $X \subseteq_{\text{fin}} Y$.

Let $(M, \cdot, \leq)$ be an ordered monoid. Define $D_0(M) := \{\downarrow A \mid A \in \mathcal{P}_{\text{fin}}(M)\}$. Because $M$ is an ordered monoid, $M = \downarrow 1_M$, and so $M \in D_0(M)$. On $D_0(M)$, define the semi-ring addition as binary union. As for the semi-ring multiplication $\otimes$, define as follows: for any $\downarrow A,\ \downarrow B \in D_0(M)$, $\downarrow A \otimes \downarrow B := \downarrow \{a \cdot b \mid a \in A,\ b \in B\}$. Crucially, for an ordered monoid $(M, \cdot, \leq)$, the triple $(D_0(M), \bigcup, \otimes)$ is a normal semi-ring.

By virtue of this result, in order to extend the assignment $(M, \cdot, \leq) \mapsto (D_0(M), \bigcup, \otimes)$ (where $M \in \text{obj}(\mathbf{OrdMon})$) to a functor $F : \mathbf{OrdMon} \longrightarrow \mathbf{NSRng}$, it remains to assign to an arbitrarily given **OrdMon**-morphism $f : M \longrightarrow N$ a **NSRng**-morphism $F(f) : D_0(M) \longrightarrow D_0(N)$, defined by $Ff(\downarrow A) = \downarrow f(A)$, for any $A \in D_0(M)$. It is routine to verify that $F$ is indeed a functor. In the opposite direction, it is easy to see that any normal semi-ring $(S, +, \cdot)$ can be given an ordered monoid structure, namely, $(S, \cdot, \leq)$, where $a \leq b \iff a + b = b$ for any $a,\ b \in S$. Thus, we have the forgetful functor $G : \mathbf{NSRng} \longrightarrow \mathbf{OrdMon}$.

Regarding the order structure of a semi-ring, more can be said.

Proposition 2.1. *Let $(R, +, \cdot)$ be a normal semi-ring and $A \subseteq_{\text{fin}} R$. Then*

$$\bigvee A = \sum_{a \in A} a.$$

Proof. Follows from the additive idempotence of $R$. □

Corollary 2.2. *Let $R$ be a normal semi-ring and $A,\ B \subseteq_{\text{fin}} R$ with $\downarrow A = \downarrow B$. Then*

$$\sum_{a \in A} a = \sum_{b \in B} b.$$

Corollary 2.3. *Let $R$ be a normal semi-ring, $x \in R$ and $A \subseteq R$. Then*

$$x \cdot \bigvee_{a \in A} a = \bigvee_{a \in A} x \cdot a$$

Using these preceding properties, it is easy to establish the following:

THEOREM 2.4. *Let* $F : \mathbf{OrdMon} \longrightarrow \mathbf{NSRng}$ *and* $G : \mathbf{NSRng} \longrightarrow \mathbf{OrdMon}$ *be the aforementioned functors. Then* $F$ *is left adjoint to* $G$ *with unit given by*

$$\eta_M : M \longrightarrow GF(M),\ x \mapsto \downarrow x \quad (M \in \mathrm{obj}(\mathbf{OrdMon}))$$

*and co-unit given by*

$$\varepsilon_R : FG(R) \longrightarrow R,\ \downarrow A \mapsto \sum_{a \in A} a \quad (R \in \mathrm{obj}(\mathbf{NSRng})).$$

In the ensuing sections, whenever an example mentions the adjunction between **OrdMon** and **NSRng**, we are implicitly referring to the above adjunction, and the symbols $F$, $G$ then refer to the above pair.

2.1.2. *Z-frames and frames.* By a *semilattice*, we mean a (finite) meet semilattice. Thus, every semilattice has a top element. For semilattices $S$ and $T$, a *semilattice homomorphism* $f : S \longrightarrow T$ is a mapping that preserves finite meets (and hence the top element). Denote by **SLat** the category of semilattices and semilattice homomorphisms. A subset $D$ of a poset is a *lower set* if $D = \downarrow D := \{p \in P \mid \exists d \in D.\ p \leq d\}$. We denote the collection of lower sets by $\mathcal{D}(P)$. For any semilattice $S$, $\mathcal{D}(S)$ is a complete lattice with respect to inclusion, and hence a semilattice. Following the subset-systems approach first proposed by [**20**] and used later by [**23, 7**], we define a subset system $Z$ on **SLat** to be a function that assigns to each semilattice $S$ a collection $Z(S)$ of subsets of $S$, such that the following are satisfied:

(Z1) $Z(S)$ is a subsemilattice of $\mathcal{D}(S)$ containing all $\downarrow x$ for $x \in S$.
(Z2) For any $\mathcal{D} \in Z(Z(S))$, $\bigcup \mathcal{D} \in Z(S)$.
(Z3) For any **SLat**-morphism $f : S \longrightarrow T$ and any $D \in Z(S)$, $\downarrow f(D) \in Z(T)$.

We call the elements of $Z(S)$ the $Z$*-ideals* of S. A subset $A$ of $S$ is called a $Z$*-set* if $\downarrow A \in Z(S)$.

A semilattice $S$ is said to be $Z$*-complete* if $\bigvee D$ exists for all $D \in Z(S)$. Given two $Z$-complete semilattices $S$ and $T$, a $Z$*-complete homomorphism* $f : S \longrightarrow T$ is a semilattice homomorphism that preserves the suprema of $Z$-ideals. A $Z$-complete semilattice $A$ is called a $Z$*-frame* if for each $a \in A$ and $D \in Z(A)$ it holds that $a \wedge \bigvee D = \bigvee (a \wedge D)$ where $a \wedge D := \{a \wedge d \mid d \in D\}$. It can be shown that a $Z$-complete semilattice $A$ is a $Z$-frame if and only if $\bigvee : Z(A) \longrightarrow A$ is a $Z$-complete semilattice homomorphism. The notion of $Z$-frame is to generalise the notion of a frame, and this can be seen by taking $Z$ to be the selection $\mathcal{P}$ of all subsets. Also, if $Z$ selects all the directed subsets, then one recovers the notion of preframe. A $Z$-complete homomorphism between two $Z$-frames is also called a $Z$*-frame homomorphism*. We denote by **ZFrm** the category of all $Z$-frames and $Z$-frame homomorphisms. It can be shown that the categories **SLat** and **ZFrm** are connected by adjoint functors; more precisely:

THEOREM 2.5. *(*[**23**, Lemma 1.4, p.40]*)*
*Suppose* $Z$ *is a subset system on* **SLat**. *Let* $F_Z : \mathbf{SLat} \longrightarrow \mathbf{ZFrm}$ *be the functor defined by* $F_Z(S) = (Z(S), \subseteq)$ *and*

$$F_Z(f : S \longrightarrow T) = (F_Z(f) : F_Z(S) \longrightarrow F_Z(T),\ D \mapsto \downarrow f(D))$$

*and* $G_Z : \mathbf{ZFrm} \longrightarrow \mathbf{SLat}$ *the forgetful functor. Then* $F_Z$ *is left adjoint to* $G_Z$ *with unit given by*

$$\eta_S : S \longrightarrow Z(S),\ x \mapsto \downarrow x \quad (S \in \mathrm{obj}(\mathbf{SLat}))$$

*and co-unit given by*

$$\varepsilon_S : Z(S) \longrightarrow S,\ D \mapsto \bigvee D \quad (S \in \mathrm{obj}(\mathbf{SLat})).$$

A subset $X$ of a $Z$-complete semilattice $A$ is *$Z$-closed* if (i) $X = \downarrow X$, and (ii) for any $Z$-set $D$, $D \subseteq X$ implies $\bigvee D \in X$. The collection of all $Z$-closed subsets of $A$ is denoted by $\Gamma_Z(A)$. Observe that $\Gamma_Z(A)$ is closed under arbitrary intersection. It is intended that the notion of $Z$-closed sets generalises that of Scott closed sets; just ask $Z$ to select all the directed subsets. We denote the $Z$-closure of a subset $X$ of a $Z$-complete semilattice $A$ by $\mathrm{cl}_Z(X) := \bigcap\{C \in \Gamma_Z(A) \mid X \subseteq C\}$. It can be shown that for a semilattice $A$, $A$ is a $Z$-frame if and only if $\Gamma_Z(A)$ is a frame. More importantly, **ZFrm** and **Frm** are connected by adjoint functors given below:

Theorem 2.6. *(*[**22**]*)*
*Suppose* $Z$ *is a subset system on* **SLat**. *Let* $F^Z : \mathbf{ZFrm} \longrightarrow \mathbf{Frm}$ *be the functor defined by* $F^Z(A) = (\Gamma_Z(A), \subseteq)$ *and*

$$F^Z(f : A \longrightarrow B) = (F^Z(f) : F^Z(A) \longrightarrow F^Z(B),\ X \mapsto \mathrm{cl}_Z(f(X)))$$

*and* $G^Z : \mathbf{Frm} \longrightarrow \mathbf{ZFrm}$ *the forgetful functor. Then* $F^Z$ *is left adjoint to* $G^Z$ *with unit given by*

$$\eta_A : A \longrightarrow \Gamma_Z(A),\ x \mapsto \downarrow x \quad (A \in \mathrm{obj}(\mathbf{ZFrm}))$$

*and co-unit given by*

$$\varepsilon_A : \Gamma_Z(A) \longrightarrow A,\ X \mapsto \bigvee X \quad (A \in \mathrm{obj}(\mathbf{ZFrm})).$$

**2.2. Comonads.** A *comonad* in a category **D** consists of a functor $U : D \longrightarrow D$ together with two natural transformations $\varepsilon : U \longrightarrow \mathrm{id}_{\mathbf{D}}$ (the *counit*) and $\nu : U \longrightarrow U^2$ (the *comultiplication*), subject to the following conditions:

(1) (Associativity) $U\nu_X \circ \nu_X = \nu_{UX} \circ \nu_X$, and
(2) (Unit laws) $\varepsilon_{UX} \circ \nu_X = U\varepsilon_X \circ \nu_X = id_{UX}$ for any object $X$ of **D**.

The naturality conditions mean that for all $f : X \longrightarrow Y$ , the following

$$\begin{array}{ccc} UX & \xrightarrow{Uf} & UY \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon_Y} \\ X & \xrightarrow[f]{} & Y \end{array} \quad \text{and} \quad \begin{array}{ccc} UX & \xrightarrow{Tf} & UY \\ {\scriptstyle \nu_X}\downarrow & & \downarrow{\scriptstyle \nu_Y} \\ U^2X & \xrightarrow[T^2 f]{} & U^2Y \end{array}$$

commute.

Let $U = (U, \varepsilon, \nu)$ be a comonad. A *$U$-coalgebra* is an object $A$ (the *underlying object*) together with an arrow $\beta : A \longrightarrow UA$ (the *co-structure map*) subject to the following conditions:

(1) (Associativity) $\nu_A \circ \beta = U\beta \circ \beta$, and
(2) (Unit law) $\epsilon_A \circ \beta = \mathrm{id}_A$

Sometimes, we refer to the underlying object of a co-algebra as the co-algebra.

Proposition 2.7. *For any object* $X$*,* $UX$ *is a coalgebra (called the* free $U$-coalgebra*) with the co-structure map* $\nu_X : UX \longrightarrow U^2X$*.*

Let $A$ and $B$ be $U$-coalgebras with co-structure maps $\alpha : A \longrightarrow UA$ and $\beta : B \longrightarrow UB$. A *$U$-coalgebra homomorphism* from $(A, \alpha)$ to $(B, \beta)$ is an arrow $h : A \longrightarrow B$ such that $Uh \circ \alpha = \beta \circ h$.

Every adjunction between categories gives rise to a comonad. More precisely, given that $F : \mathbf{C} \longrightarrow \mathbf{D}$ is left adjoint to $G : D \longrightarrow C$, define $U : D \longrightarrow D$, $U = FG$. Set the co-unit to be that of the adjunction $f \dashv G$, i.e., $\varepsilon$, and the comultiplication to be $\nu = F\eta_G$. Then it can be shown that $(U, \varepsilon, \nu)$ is a comonad in $\mathbf{D}$.

EXAMPLE 2.8. (1) Specializing the above result to the adjunction $F \dashv G$ between the categories **OrdMon** and **NSRng**, $U : \mathbf{NSRng} \longrightarrow \mathbf{NSRng}$ is defined by $U = FG$. The counit $\varepsilon$ is given by

$$\varepsilon_R : UR \longrightarrow R,\ {\downarrow}A \mapsto \sum_{a \in A} a = \bigvee A \quad (A \subseteq_{\text{fin}} R \in \text{obj}(\mathbf{NSRng})),$$

while the comultiplication $\nu = F\eta_G$ is explicitly given by

$$\nu_R : {\downarrow}A \mapsto \{{\downarrow}B \in FGR \mid \exists a \in A.\ {\downarrow}B \subseteq {\downarrow}a\}.$$

(2) Applying the above result to the adjunction $F_Z \dashv G_Z$ between the categories **SLat** and **ZFrm**, $U_Z : \mathbf{SLat} \longrightarrow \mathbf{ZFrm}$ is defined by $U_Z = F_Z G_Z$. The counit $\varepsilon$ is given by

$$\varepsilon_S : Z(S) \longrightarrow S,\ D \mapsto \bigvee D,$$

while the co-multiplication $\nu = F_Z(\eta_{G_Z})$ is explicitly defined by

$$\nu_A : D \mapsto \{E \in Z(A) \mid \exists d \in D.\ E \subseteq {\downarrow}d\}.$$

(3) For the adjunction $F^Z \dashv G^Z$ between the categories **ZFrm** and **Frm**, $U^Z : \mathbf{ZFrm} \longrightarrow \mathbf{Frm}$ is defined by $U^Z = F^Z G^Z$. The counit $\varepsilon$ is given by

$$\varepsilon_A : \Gamma_Z(A) \longrightarrow A,\ X \mapsto \bigvee X,$$

while the comultiplication $\nu = F^Z(\eta_{G^Z})$ is explicitly given by

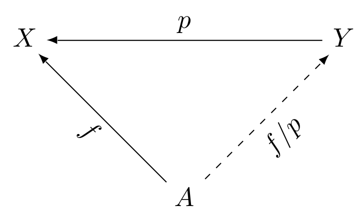
$$\nu_A : D \mapsto \{X \in \Gamma_Z(A) \mid \exists d \in D.\ X \subseteq {\downarrow}d\}.$$

**2.3. Kan lifts of morphisms in poset enriched categories.** In what follows, the categorical notion of Kan lift (which is the categorical dual of Kan extension, see [**15**, p.236] is applied to the special case of posets and monotone maps.

Recall that for posets $P$ and $Q$, a monotone map $f : P \longrightarrow Q$ is a function which preserves order, i.e., $x \leq_P y$ implies $f(x) \leq_Q f(y)$. If a pair of monotone maps $f : P \longrightarrow Q$ and $g : Q \longrightarrow P$ is such that $f(p) \leq q \iff p \leq g(q)$ for all $p \in P$ and $q \in Q$, then we say that $f$ is *left adjoint* to $g$, or equivalently $g$ is *right adjoint* to $f$. An adjunction pair as described above is denoted by $f \dashv g$. For monotone maps, $f \dashv g$ implies that $f \circ g \leq \text{id}_Q$ and $\text{id}_P \leq g \circ f$. If in addition $f \circ g = \text{id}_Q$, we say that $f \dashv g$ is *reflective*; and dually, *coreflective* if $g \circ f = \text{id}_P$.

Let $p : Y \leq X$ and $f : A \longrightarrow X$ be monotone maps between posets. A *left Kan lift* of $f$ along $p$ is a monotone map $f/p : A \longrightarrow Y$ such that

(K1) $p \circ (f/p) \leq f$
(K2) $p \circ g \leq f$ for $g : A \longrightarrow Y$ implies that $g \leq (f/p)$.




In other words, the left Kan lift of f along p, if it exists, is the greatest map $g : A \longrightarrow Y$ for which $p \circ g \leq f$. Since this amounts to the condition $p \circ g \leq f \iff g \leq (f/p)$ for all $g : A \longrightarrow Y$, the composition map

$$p \circ - : [A, Y] \longrightarrow [A, X]$$

is right adjoint to

$$-/p : [A, X] \longrightarrow [A, Y].$$

Here $[P, Q]$ denotes the set of all monotone maps from $P$ to $Q$.

Let **C** be a poset-enriched category and **E** a subclass of **C**-morphisms. An object $A$ of **C** is called a *left Kan object* over **E** if for every **E**-morphism $p : Y \longrightarrow X$ the composition map

$$p \circ - : [A, Y] \longrightarrow [A, X]$$

is right adjoint to

$$-/p : [A, X] \longrightarrow [A, Y].$$

If $A$ is a left Kan object over **E** and also the left Kan lift $f/p$ is an actual lift (i.e., $p \circ (f/p) = f$), then we call $A$ a *left projective object* over **E**.

## 3. KZ-comonadic machinery

We now develop the KZ-comonadic machinery, with which we characterize certain **E**-projectives in poset-enriched categories. Because what we are doing here is merely dualizing Escardó's monadic set-up into a comonadic one, we choose to omit all proofs. Experts in the KZ-monadic technology may choose to skip this entire section based on the duality principle. Readers who encounter this material for the first time should verify all the results reported here, and look up for hints (if one has trouble) from the original (monadic) version [**9**, p.31].

**3.1. Definitional matters.** Recall that a *poset-enriched category* is a category whose hom-sets are posets and whose composition operation is monotone. A *poset-functor* between poset-enriched categories is a functor which is monotone on hom-posets. A *poset functor* $U : C \longrightarrow D$ is *poset-faithful* if all $A$ and $B$ in **C** and all **C**-morphisms $f,\ g : A \longrightarrow B$,

$$f \leq g \iff Uf \leq Ug.$$

Example 3.1. Both **OrdMon** and **NSRng** are poset-enriched categories. Furthermore, the functors $F$ and $G$ are both poset-functors which are poset-faithful. Indeed, for any ordered monoid morphisms $f,\ g : M \longrightarrow N$ with $f \leq g$ pointwise, then for any finite set $A \subseteq M$, we have $Ff(\downarrow A) = \downarrow f(A) \subseteq \downarrow g(A) = Fg(\downarrow A)$. Conversely, if $Ff \leq Fg$ then for every $x \in A$, one has

$$Ff(\downarrow x) \subseteq Fg(\downarrow x) \Longrightarrow \downarrow f(x) \subseteq \downarrow g(x) \Longrightarrow f(x) \leq g(x),$$

and hence $f \leq g$.

EXAMPLE 3.2. The categories **SLat** and **ZFrm** (and of course, **Frm**) are poset-enriched. Furthermore, it is easy to check that $F_Z$, $F^Z$ and the forgetful functors are all poset-functors.

For the poset-enriched category **NSRng**, the following property holds:

PROPOSITION 3.3. *For any normal semi-ring $R$, it holds that*

$$\varepsilon_{UR} \leq U\varepsilon_R.$$

PROOF. We perform the following calculations for a finite subset $\mathcal{A} \subseteq FGR$: $\varepsilon_{UR}(\downarrow\mathcal{A}) = \downarrow\bigcup\{A \mid \downarrow A \in \mathcal{A}\}$ and $U\varepsilon_R(\downarrow\mathcal{A}) = \downarrow\{\bigvee A \mid \downarrow A \in \mathcal{A}\}$. Clearly, $\varepsilon_{UR}(\downarrow\mathcal{A}) \subseteq U\varepsilon_R(\downarrow\mathcal{A})$. $\square$

**3.2. Main results.** Comonads which satisfy the inequality in the preceding proposition are of a special kind, namely the KZ-comonads. The definition of KZ-comonad, given below, is based on a result of Anders Kock [**14**], specialized to poset-enriched categories.

LEMMA 3.4. *Let $(U, \varepsilon\nu)$ be a comonad in a poset-enriched category $\mathbf{D}$, and assume that $U$ is a poset-functor. Then the following conditions are equivalent:*

(KZ$_0$) $\varepsilon_{UX} \leq U\varepsilon_X$ *for all* $X \in \mathbf{D}$.
(KZ$_1$) *For all $X \in \mathbf{D}$, an arrow $\beta : X \longrightarrow UX$ is a co-structure map if and only if $\beta \dashv \varepsilon_X$ is a coreflective adjunction (i.e., $\varepsilon_X \circ \beta = \mathrm{id}_X$).*
(KZ$_2$) $\nu_X \dashv \varepsilon_{UX}$ *for all* $X \in \mathbf{D}$.
(KZ$_3$) $U\varepsilon_X \dashv \nu_X$ *for all* $X \in \mathbf{D}$.

DEFINITION 3.5. Let $\mathbf{D}$ be a poset-enriched category. A *left KZ-comonad* in $\mathbf{D}$ is a comonad $(U, \varepsilon, \nu)$ in $\mathbf{D}$ with $U$ a poset functor, subject to the equivalent conditions of Lemma 3.4. Poset-dually, one defines *right KZ-comonads*. Whenever there is no confusion, we just write 'KZ-comonad' for 'left KZ-comonad'. Here "KZ" abbreviates "Kock-Zöberlein".

EXAMPLE 3.6. By Proposition 3.3, the comonad induced by the adjunction between **OrdMon** and **NSRng** is a left KZ-comonad.

EXAMPLE 3.7. It is quite straightforward to check that the comonads $U_Z = F_Z G_Z$ (induced by the adjunction $F_Z \dashv G_Z$) and $U^Z = F^Z G^Z$ (induced by the adjunction $F^Z \dashv G^Z$) are KZ-comonads, following the respective definitions of the functors and the co-multiplications explicitly given earlier in Example 2.8(2) and (3).

REMARK 3.8. By the condition (KZ1), each object $A$ has at most one co-structure map. We denote the unique co-structure map of a $U$-coalgebra $A$ of a KZ-comonad $U = (U, \varepsilon, \nu)$ by $m_A$.

DEFINITION 3.9. Let $F : D \longrightarrow D$ be a poset-functor on a poset-enriched category $\mathbf{D}$. A *left $F$-arrow* is a morphism $f : X \longrightarrow Y$ in $\mathbf{D}$ such that $Ff : FX \longrightarrow FY$ has a right adjoint denoted by $\hat{f} : FY \longrightarrow FX$. If the adjunction is reflective (i.e., $Ff \dashv \hat{f}$ and $Ff \circ \hat{f} = \mathrm{id}_{FY}$), we say that $f$ is a *left $F$-quotient*.

PROPOSITION 3.10. *Let $(U, \varepsilon, \nu)$ be a left KZ-comonad in a poset-enriched category $\mathbf{D}$. Then $\varepsilon_X : UX \longrightarrow X$ is a left $U$-quotient with $\hat{\varepsilon_X} = \nu_X$.*

PROOF. The condition (KZ3) states that $U\varepsilon_X \dashv \nu_X$. By the unit law $\varepsilon_{UX} \circ \nu_X = \mathrm{id}_{UX}$, it follows that this adjunction is reflective. $\square$

We arrive at the main theorem of this section.

Theorem 3.11. *The following statements are equivalent for a left KZ-comonad* $(U, \varepsilon, \nu)$ *in a poset-enriched category* $\mathbf{D}$ *and any object* $A \in \mathbf{D}$*:*

(1) *$A$ is a left projective over left $U$-quotients.*
(2) *$A$ is projective over left $U$-quotients.*
(3) *$A$ is a $U$-coalgebra.*

*These conditions imply*

(4) *$A$ is a left Kan object over the left $U$-arrows.*

*Moreover, assuming that any one of the equivalent conditions (1) - (3) holds, if* $p : Y \longrightarrow X$ *is a left $U$-arrow and* $f : A \longrightarrow X$ *is any arrow in* $\mathbf{D}$*, then*

$$f/p = \varepsilon_Y \circ \hat{p} \circ Uf \circ m_A,$$

*where* $m_A : U \longrightarrow UA$ *is the co-structure map of the coalgebra* $A$*.*

## 4. Applications

**4.1. E-projective normal semi-rings.** A semi-ring morphism $f : R \longrightarrow S$ between normal semi-rings $R$ and $S$ is said to be *perfect* if $Uf$ has a reflective right adjoint, i.e., a semi-ring morphism $s : US \longrightarrow UR$ such that $Uf \dashv s$ and $Uf \circ s = \mathrm{id}_{US}$.

Remark 4.1. Every perfect semi-ring morphism has a section in NSRng.

Throughout this section, we denote by $\mathbf{E}$ the class of perfect semi-ring morphisms. For the remaining part of this section, we specialize the definitions of $F, G, U, \varepsilon$ and $\nu$ to be those of Example 2.8.

Remark 4.2. For any normal semi-ring $R$, the co-unit map

$$\varepsilon : UR \longrightarrow R, \ \downarrow A \mapsto \bigvee A$$

is such that $U\varepsilon_R : U^2R \longrightarrow UR$ has a reflective right adjoint $\nu_R : UR \longrightarrow U^2R$ given by

$$\downarrow A \mapsto \{\downarrow B \mid \exists a \in A. \ \downarrow B \subseteq \downarrow a\}.$$

This provides a natural example of a perfect semi-ring morphism.

Before arriving at the main result of this section, we need to introduce two definitions new to the theory of semi-rings.

Definition 4.3. Let $(R, +, \cdot)$ be a semi-ring. Define an auxiliary relation $\ll_F$ (read as finitely way-below) on $R$ as follows:

$$x \ll_F y \iff \forall A \subseteq_{\text{fin}} R. \left( \sum_{a \in A} a \geq y \Longrightarrow \exists a \in A. x \leq a \right).$$

Viewing a normal semi-ring as an ordered monoid, the sum $\sum_{a \in A} a$ may be seen as $\bigvee A$.

Definition 4.4. A normal semi-ring $(R, +, \cdot)$ is said to be $F$-continuous if for any $x \in R$,

(i) $\Downarrow x := \{r \in R \mid r \ll_F x\}$ is the lower closure (w.r.t. the induced partial order) of a finite subset of $R$, and
(ii) $x = \bigvee \Downarrow x$.

A semi-ring $R$ is said to be *stably $F$-continuous* if in addition to (i) and (ii) it satisfies the following condition:

(iii) $x \ll_F y \cdot z$ if and only if there exist $y'$ and $z'$ in $R$ such that $y' \ll_F y$, $z' \ll_F z$ and $x \leq y' \cdot z'$.

LEMMA 4.5. *A normal semi-ring $(R, +, \cdot)$ is the underlying object of a $U$-coalgebra if and only if it is stably $F$-continuous.*

PROOF. By Lemma 3.4, $R$ is the underlying object of a $U$-coalgebra if and only if its co-unit $\varepsilon_R : UR \longrightarrow R$ has a coreflective right adjoint $\beta : R \longrightarrow UR$. This is equivalent to the condition that $\beta \circ \varepsilon_r \leq \mathrm{id}_{UR}$ and $\varepsilon \circ \beta = \mathrm{id}_R$. Since $\varepsilon_R(\downarrow A) = \sum_{a \in A} a = \bigvee A$, the adjoint situation forces the preceding inequalities to be equivalent to

$$\beta(r) = \bigcap \{\downarrow A \in FR \mid r \leq \bigvee A\}.$$

Thus, $s \in \beta(r)$ if and only if $s$ belongs to the lower closure of every finite set $A \subseteq R$ with $r \leq \bigvee A$, i.e., $s \ll_F r$. Hence $R$ is a continuous normal semi-ring. Since $\beta$ preserves the multiplication $\cdot$, it follows that for any $y$ and $z \in R$, $\beta(y \cdot z) = \beta(y) \otimes \beta(z)$. Thus, this second condition is equivalent to:

$$\begin{aligned} x \ll_F (y \cdot z) \quad &\iff x \in \beta(y) \otimes \beta(z) \\ &\iff (\exists y' \ll_F y) \wedge (\exists z' \ll_F z).\ (x \leq y' \cdot z'), \end{aligned}$$

which is just the condition that $R$ is stably $F$-continuous. □

EXAMPLE 4.6. By virtue of the Fundamental Theorem of Arithmetic, the semi-ring $R = (\mathbb{N}, \mathrm{lcm}, \mathrm{gcd})$ with lowest common multiple as addition and greatest common divisor as multiplication is normal. Viewed as an ordered monoid, the partial order $\leq$ on $R$ is just the divisibility relation, i.e., $a \leq b \iff a \mid b$. Clearly, $0 \ll_F 0$ and $1 \ll_F 1$ in $R$, and crucially, if $y \neq 0, 1$, then $x \ll_F y \iff x = pk$ for some prime $p$ and $x \mid y$. It follows immediately by Euclid's lemma that $R$ is a stably continuous normal semi-ring.

THEOREM 4.7. *The following are equivalent for any normal semi-ring $R$:*

(i) *$R$ is $\mathbf{E}$-projective.*
(ii) *$R$ is the underlying object of a $U$-coalgebra.*
(iii) *$R$ is stably $F$-continuous.*

PROOF. A direct consequence of Lemmas 3.4 and 4.5, and Theorem 3.11. □

REMARK 4.8. An independent work [**16**] of N. T. Nai has established results that are similar to ours but her class $\mathbf{E}$ of semi-ring morphisms $f$ are those for which $G(f)$ has a section. Not surprising, Nai has to separately prove that for a normal semi-ring $S$, the canonical structure $(D_0(S), \bigcup, \otimes)$ is stably finitely continuous. (see [**16**, Proposition 5.2.10, p.67]) since she relies only on Lemma 1.2.

**4.2. E-projective $Z$-frames and frames.** In this subsection, we overload some of the definitions used in the preceding subsection.

DEFINITION 4.9. A $Z$-frame homomorphism $f : A \longrightarrow B$ between $Z$-frames $A$ and $B$ is said to be *perfect* if $U_Z f$ has a reflective right adjoint, i.e., a $Z$-frame homomorphism $s : U_Z B \longrightarrow U_Z A$ such that $U_Z f \dashv s$ and $U_Z f \circ s = \mathrm{id}_{U_Z B}$.

A frame homomorphism $f : A \longrightarrow B$ between frames $A$ and $B$ is said to be *perfect* if $U^Z f$ has a reflective right adjoint, i.e., a frame homomorphism $s : U^Z B \longrightarrow U^Z A$ such that $U^Z f \dashv s$ and $U^Z f \circ s = \mathrm{id}_{U^Z B}$.

In domain theory, the way-below relation, $\ll$, is the central relation that has been extensively studied [**11**, pp.49–78]. We modify it for the situation of $Z$-frame:

DEFINITION 4.10. Let $A$ be a $Z$-frame. Define an auxiliary relation $\ll_Z$ (read as $Z$-way-below) on $A$ as follows:

$$x \ll_Z y \iff \forall D \in Z(A).\ \left(\bigvee D \geq y \Longrightarrow \exists d \in D.\ x \leq d\right).$$

Corresponding to the adjunction between **ZFrm** and **Frm**, we have to define a new binary relation $\prec_Z$ for a frame:

DEFINITION 4.11. Let $A$ be a frame. Define an auxiliary relation $\prec_Z$ (read as $Z$-beneath) on $A$ as follows:

$$x \prec_Z y \iff \forall C \in \Gamma_Z(A).\ \left(\bigvee C \geq y \Longrightarrow x \in C\right).$$

REMARK 4.12. A special instance of $\prec_Z$ has appeared in both the work of Escardó([**10**]), and the joint work of the author and Zhao ([**12**]) independently. In their works, $Z$ selects all the directed subsets.

DEFINITION 4.13. A $Z$-frame $A$ is said to be *$Z$-continuous* if for any $x \in A$,

(i) $\Downarrow_Z x := \{a \in A \mid a \ll_Z x\} \in Z(A)$ and
(ii) $x = \bigvee \Downarrow_Z x$.

A $Z$-frame $A$ is said to be *stably $Z$-continuous* if in addition to (i) and (ii) it satisfies the following conditions:

(iii) $1 \ll_Z 1$, and
(iv) $x \ll_Z y \wedge z$ if and only if there exist $y'$ and $z'$ in $A$ such that $y' \ll_Z y$, $z' \ll_Z z$ and $x \leq y' \wedge z'$.

DEFINITION 4.14. A frame $A$ is said to be *$\Gamma_Z$-continuous* if for any $x \in A$, it holds that $x = \bigvee \gamma(x)$, where $\gamma(x) := \{a \in A \mid a \prec_Z x\}$. A frame $A$ is said to be *stably $\Gamma_Z$-continuous* if in addition it satisfies the following conditions:

(i) $1 \prec_Z 1$ and
(ii) $x \prec_Z y \wedge z$ if and only if there exist $y'$ and $z'$ in $A$ such that $y' \prec_Z y$, $z' \prec_Z z$ and $x \leq y' \wedge z'$.

REMARK 4.15. Notice that it always holds that $\gamma(x) \in \Gamma_Z(A)$ for any $x \in A$. This follows immediately from the definitions of $\prec_Z$ and $Z$-closed sets.

LEMMA 4.16. *Let $Z$ be a subset system on* **SLat**. *Then the following statements hold:*

(i) *A $Z$-frame $A$ is the underlying object of a $U_Z$-coalgebra if and only if $A$ is stably $Z$-continuous.*
(ii) *A frame $A$ is the underlying object of a $U^Z$-coalgebra if and only if $A$ is stably $\Gamma_Z$-continuous.*

PROOF. Similar to the proof of Lemma 4.5. □

THEOREM 4.17. (1) *Let* **E** *be the set of perfect $Z$-frame homomorphisms. The following are equivalent for any $Z$-frame $A$:*

(i) *$A$ is* **E**-*projective.*
(ii) *$A$ is the underlying object of a $U_Z$-coalgebra.*
(iii) *$A$ is stably $Z$-continuous.*

(2) *Let* **E** *be the set of perfect frame homomorphisms. The following are equivalent for any frame A:*
  (i) *A is* **E**-*projective.*
  (ii) *A is the underlying object of a* $U^Z$-*coalgebra.*
  (iii) *A is stably* $\Gamma_Z$-*continuous.*

Proof. A direct consequence of Lemmas 3.4 and 4.16, and Theorem 3.11. □

## 5. Conclusion

This paper makes a simple adaptation of Escardó's KZ-monadic machinery to deal with the characterisation of projectives. Instead of re-inventing the wheel, the main idea here is to turn the wheel in the *opposite* direction. Our Theorem 3.11, which is the category-dual of Theorem 1.1, allows us to characterise the underlying coalgebras of the comonads in consideration as precisely the **E**-projectives with **E** as the class of those left monad-quotients. Such a technique is applied to the several comonadic situations arising from the theory of ordered monoids and normal semirings, and from the theory of frames, and in each case characterises the **E**-projectives as precisely those stably continuous structures (each respect to $\ll_F, \ll_Z$ and $\prec_Z$). In summary, we demonstrated that it is easy to formulate and apply the dual version of Escardó's monadic machinery, and that his method is, in fact, just as powerful in characterising projectives as it is for injectives. One important piece of future work is to find explicit characterisations of the various perfect morphisms for each of the different categories in the running examples. Currently, we have not obtained such characterisations.

## References


[1] Abramsky, S. and Jung, A., "Domain Theory," Volume 3 of Handbook of Logic in Computer Science, Clarendon Press, Oxford, 1994.

[2] Balbes, R., *Projective and injective distributive lattices*, Pacific J. Math. **21** (1967), 405–420.

[3] Balbes, R., and Dwinger, P., "Distributive Lattices," University of Missouri Press, 1975.

[4] Banaschewski, B., and Niefeld, S. B., *Projective and supercoherent frames*, J. Pure Appl. Algebra **70** (1991), 45–51.

[5] Crown, G. D., *Projectives and Injectives in the Category of Complete Lattices with Residuated Mappings*, Math. Ann. **187** (1970), 295–299.

[6] Davey, B. A., and Priestley, H. A., "Introduction to Lattices and Order," Cambridge University Press, 1990.

[7] Erné, M., *Z-continuous posets and their topological manifestation*, Applied Categorical Structures **7** (1999), 31–70.

[8] Escardó. *Injective spaces via the filter monad*, Topology proceedings **22** (1997), 97–110.

[9] Escardó, M. H., *Properly injective spaces and function spaces*, Topology and its Applications **89(1-2)** (1998), 75–120.

[10] Escardó, M. H., *Injective locales over perfect embeddings and algebras of the upper powerlocale monad*, Applied General Topology **4(1)** (2003), 193–200.

[11] Gierz, G., Hofmann, K. H., Keimel, K., Lawson, J. D., Mislove, M.W., and Scott, D.S., "Continuous Lattices and Domains," Number 93 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2003.

[12] Ho, W. K., and Zhao, D., *Lattices of Scott-closed sets*, Comment. Math. Univ. Carolinae **50(2)** (2009), 297–314.

[13] Johnstone, P. T., "Stone Spaces," Number 3 in Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1982.

[14] Kock, A., *Monads for which structures are adjoint to units*, Journal of Pure Applied Algebra **104** (1995), 41–59.

[15] Mac Lane, S., "Categories for the Working Mathematician," 2nd Ed., Springer-Verlag, 1998.

[16] Nai, Y. T., "Order structures in algebra," Confirmation report for Ph.D., National Institute of Education, Nanyang Technological University, May 2012.
[17] Poigné, A., "Basic Category Theory," In *Handbook of Logic in Computer Science* **1** (1992), Oxford University Press, New York, 413–640.
[18] Scott, D.S., Continuous lattices. In *Toposes, Algebraic Geometry and Logic*, Lecture Notes in Mathematics (1972), Springer-Verlag, 97–136.
[19] Sikorski, R., *A theorem on extensions of homomorphisms,* Ann. Soc. Polon. Math. **21** (1948), 332–335.
[20] Wright, J.B., Wagner, E. G., and Thatcher, J. W., *A uniform approach to inductive posets and inductive closure*, Theoret. Comput. Sci. **7** (1978), 57–77.
[21] Zhao, B., and Wang., K. Y., *$Z$-quantales and its categorical properties*, Acta Mathematica SINICA, Chinese Series **53(5)** (2010), 997–1006.
[22] Zhao., D., Generalisation of Continuous Lattices and Frames. Ph.d., University of Cambridge, 1992.
[23] Zhao, D., *On Projective Frames*, Canad. Math. Bull. **40(1)** (1997), 39–46.