Technical Report M2012-1
April 2012
Mathematics and Mathematics Education
National Institute of Education
Singapore

Exponential function and its derivative revisited

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February 3, 2012

Abstract

Most of the available proofs for $\frac{d}{dx}(e^x) = e^x$ rely on results involving either power series, uniform convergence or a round-about definition of the natural logarithm function $\ln(x)$ by the definite integral $\int_1^x \frac{1}{t}\,\mathrm{d}t$, and are thus not readily accessible by high school teachers and students. Even instructors of calculus courses avoid showing the complete proof to their undergraduate students because a direct and elementary approach is missing. This short paper fills in this gap by supplying a simple proof of the aforementioned basic calculus fact.

1 Introduction

The definition of the exponential function

$$e^x := \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

originated from Leonhard Euler ([1, p.363]). Apart from Euler himself, several authors based their proofs of $\frac{d}{dx}(e^x) = e^x$ on this definition; the unfortunate state of affairs being that most of these are spawned with gaps. It is impossible to be encyclopedic in documenting all such incomplete (but nonetheless published) proofs. But for illustration's sake, we quickly supply some evidence of such inadequacy. For instance, in [3], apart from the incomplete proof of the monotonicity of $(1+\frac{x}{n})^n$ when $x \geqslant 0$, the justification for $e^{x+y} = e^x \cdot e^y$, crucial in the proof of the result $\frac{d}{dx}(e^x) = e^x$, is missing. In another work [5], the result $e^x = \lim_{n\to\infty}(1+\frac{x}{n})^n$ was proven only for $0 \leq x \leq 1$. However, it does not appear to work in general due to the limitation of the functional equation (10) on [5, p.843]. Furthermore, a careful reading of the works [5] and [2] reveals that the density of $\mathbb{Q}$ in $\mathbb{R}$ has been exploited – this approach, in our view, cannot be considered as elementary.

Our paper uses a simple proof of $\frac{d}{dx}(e^x) = e^x$, free of the aforementioned deficiencies, that can be demonstrated even to (and easily understood by) a freshman calculus audience. To the best of the authors' knowledge, this proof is new. Such a demonstration, we believe, is beneficial to a calculus student as it weaves together many previously acquired notions and results into one meaningful mathematical fabric. In the ensuing development, the only two pre-requisites we assume of the reader are the Squeeze Theorem and the Archimedean property of real numbers.

2 Some crucial lemmata

In a typical calculus or real analysis course, the Bernoulli's and Arithmetic-Mean Geometric-Mean inequalities (AM-GM inequality, for short) are often included in foundational materials. For self-containment, we record them below:

Proposition 2.1 (Bernoulli's inequality)**.** *For any integer $n \geq 0$ and any real $x \geq -1$, it holds that $(1+x)^n \geq 1+nx$.*

Proposition 2.2 (Arithmetic-Geometric inequality)**.** *For any non-negative $x_1, \cdots, x_n \in \mathbb{R}$, it holds that $\frac{x_1+x_2+\cdots+x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots \cdot x_n}$.*

These famous inequalities collaborate to justify the following crucial lemmata:

Lemma 2.3. *If $N \in \mathbb{N}$ then*

$$\left(1+\frac{1}{N}\right)^N \leq 4.$$

Proof. The above inequality follows from the observation that for $N \geq 3$,

$$\sqrt[N]{\frac{1}{4}} = \sqrt[N]{\left(\frac{1}{2} \cdot \frac{1}{2} \cdot 1^N\right)} \leq \frac{1}{N}\left(\frac{1}{2}+\frac{1}{2}+(N-2)\cdot 1\right) \leq \frac{N}{N+1},$$

owing to AM-GM inequality, and that the first two terms are less than 4. □

Lemma 2.4. *If $x \geq 0$ and $n \in \mathbb{N}$, then $(1+\frac{x}{n})^n \leq 4^{1+\lfloor x \rfloor}$.*

Proof. By the Archimedean property there exists $N \in \mathbb{N}$ so large that if $n \geq N$ then

$$\begin{aligned}
\left(1+\frac{x}{n}\right)^n &\leq (1+\frac{1+\lfloor x \rfloor}{n})^n \\
&\leq (1+\frac{1+\lfloor x \rfloor}{N(1+\lfloor x \rfloor)})^{N(1+\lfloor x \rfloor)} \\
&\leq \left((1+\frac{1}{N})^N\right)^{1+\lfloor x \rfloor} \\
&= 4^{1+\lfloor x \rfloor} \qquad \text{(by Lemma 2.3).}
\end{aligned}$$

The proof is complete. □

Lemma 2.5. *If $c \in \mathbb{R}$, then* $\lim_{n\to\infty} \left(1 + \dfrac{c}{n^2}\right)^n = 1$.

Proof. In view of Lemma 2.4 and the Squeeze Theorem, it suffices to prove that

$$1 + \frac{c}{n} \leq \left(1 + \frac{c}{n^2}\right)^n \leqslant 1 - \frac{1}{n} + \frac{1}{n}\left(1 + \frac{c}{n^2}\right)^{n^2}$$

for all sufficiently large integers n. Indeed, if n is any positive integer satisfying $1 + \frac{c}{n^2} > 0$ (thanks to the Archimedean property), the Bernoulli's inequality immediately justifies the first inequality:

$$\left(1 + \frac{c}{n^2}\right)^n \geqslant 1 + n \cdot \frac{c}{n^2} = 1 + \frac{c}{n}.$$

The second inequality follows from the AM-GM inequality:

$$\left(1 + \frac{c}{n^2}\right)^n = \sqrt[n]{1^{n-1} \cdot \left(1 + \frac{c}{n^2}\right)^{n^2}} \leqslant \frac{1}{n} \cdot \left(\sum_{k=1}^{n-1} 1 + \left(1 + \frac{c}{n^2}\right)^{n^2}\right).$$

$\square$

As a consequence of Lemma 2.5, we obtain the following:

Corollary 2.6. *If $(x_n)_{n=1}^{\infty}$ is a bounded sequence of real numbers, then*

$$\lim_{n\to\infty} \left(1 + \frac{x_n}{n^2}\right)^n = 1.$$

3 Elementary properties of $(x \mapsto e^x)$

This section is devoted to some elementary properties of the exponential function $(x \mapsto e^x)$, and to do so we must first establish its functional status.

Proposition 3.1. *If $x \in \mathbb{R}$, the sequence*

$$\left(\left(1 + \frac{x}{n}\right)^n\right)_{n=1}^{\infty} \tag{1}$$

converges.

Proof. Based on the observation that for any $n \in \mathbb{N}$, $\lim_{n\to\infty} \left(1 - \frac{x^2}{n^2}\right)^n = 1$ by virtue of Lemma 2.5, once the convergence of the sequence (1) on $x \geq 0$ is proven it then follows that

$$\lim_{n\to\infty} \left(1 - \frac{x}{n}\right)^n = \lim_{n\to\infty} \frac{\left(1 - \frac{x^2}{n^2}\right)^n}{\left(1 + \frac{x}{n}\right)^n}$$

exists, thus accounting for the convergence of the sequence (1) on $x < 0$. It thus remains to prove that the sequence (1) converges on $x \geq 0$. Now for any $n \in \mathbb{N}$, we have

$$\sqrt[n+1]{\left(1+\frac{x}{n}\right)^n} = \sqrt[n+1]{\left(1+\frac{x}{n}\right)^n \cdot 1} \leq \frac{1}{n+1}\left(1+n\left(1+\frac{x}{n}\right)\right) = 1+\frac{x}{n+1},$$

where the second-to-last inequality is the AM-GM inequality. Thus, the sequence (1) is monotone. Because every bounded monotone sequence converges, the proof is complete. $\square$

Crucially, the preceding proposition justifies the existence of

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

which is denoted by e^x. Interested readers can find an alternative (but more complicated) proof of this convergence on [4, p.193] using power series.

We now establish the following for the *exponential* function ($x \mapsto e^x$):

Theorem 3.2. (i) $e^0 = 1$.

(ii) *If $a, b \in \mathbb{R}$, then $e^a \cdot e^b = e^{a+b}$. In particular, $e^x > 0$ for all $x \in \mathbb{R}$.*

(iii) *If $a, b \in \mathbb{R}$, then $e^a \div e^b = e^{a-b}$.*

(iv) *If $a, b \in \mathbb{R}$, then $e^a(b-a) \leqslant e^b - e^a \leqslant e^b(b-a)$.*

(v) *The function $x \mapsto e^x$ is strictly increasing; in particular, it is injective.*

(vi) *The function $x \mapsto e^x$ is continuous on $\mathbb{R}$.*

Proof. (i) is obvious. To prove (ii), we observe that if $n \in \mathbb{N}$ then

$$\left(1+\frac{a}{n}\right)^n \cdot \left(1+\frac{b}{n}\right)^n = \left(1+\frac{a+b}{n}\right)^n \cdot \left(1+\frac{ab}{n^2 \cdot (1+\frac{a+b}{n})}\right)^n.$$

Consequently, (ii) follows from Lemma 2.6 and (i). It is also clear that (iii) follows from (ii). We now establish (iv). Without loss of generality, we suppose that $b \geq a$. Then for any $n \in \mathbb{N}$ satisfying $1 + \frac{a}{n} > 0$, we have $1 + \frac{b}{n} \geq 1 + \frac{a}{n}$ and hence

$$\begin{aligned}
(b-a)\left(1+\frac{a}{n}\right)^{n-1} &\leqslant \frac{b-a}{n}\sum_{k=0}^{n-1}\left(1+\frac{b}{n}\right)^{n-1-k}\left(1+\frac{a}{n}\right)^k \\
&= \left(1+\frac{b}{n}\right)^n - \left(1+\frac{a}{n}\right)^n \\
&\leqslant \frac{b-a}{n}\sum_{k=0}^{n-1}\left(1+\frac{b}{n}\right)^{n-1} \\
&= (b-a)\cdot\left(1+\frac{b}{n}\right)^{n-1}.
\end{aligned}$$

Letting $n \to \infty$ completes the argument.

It is easy to check that (v) is an immediate consequence of (iv) and (ii). Also, (vi) follows from (iv). $\square$

4 An elementary proof of $\frac{d}{dx}(e^x) = e^x$

Theorem 4.1. *The exponential function* $(x \mapsto e^x)$ *is differentiable on* $\mathbb{R}$ *and*

$$\frac{d}{dx}(e^x) = e^x \quad (x \in \mathbb{R}).$$

Proof. Let $x \in \mathbb{R}$ be given. Then, for any real number $h \neq 0$, we use Theorem 3.2(iv) with $a = x$ and $b = x + h$ to get

$$e^x \cdot h \leq e^{x+h} - e^x \leq e^{x+h} \cdot h$$

or equivalently,

$$\left| \frac{e^{x+h} - e^x}{h} - e^x \right| \leq \left| e^{x+h} - e^x \right|.$$

Letting $h \to 0$ and invoking (vi) complete the argument. $\square$

5 Concluding remarks

In a (currently on-going) survey[1] conducted in 2011 among Singapore secondary school and junior college teachers, preliminary findings revealed that over 80% of the participants has either no or incomplete (i.e., poorly or wrongly justified) knowledge of the definition of the exponential function $(x \mapsto e^x)$. As a result, a number of classroom approaches based the proof of $\frac{d}{dx}(e^x) = e^x$ on the unjustified fact $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$. There were also a number of teachers who conveniently chose a sufficiently convincing but inadequately justified proof of

$$\frac{d}{dx}(e^x) = \frac{d}{dx} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{d}{dx} \left(\frac{x^k}{k!} \right) = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

The rest of the participants, forming the majority, supplied no proofs at all.

A direct disaster of this deficiency in the mathematical pedagogical content knowledge (*mpck*, for short) is the emergence of recent generations of mathematics students memorizing $\frac{d}{dx}(e^x) = e^x$ as a meaninglessly isolated fact. Repercussions abound; for instance, there is a widespread lack of understanding as to why e is called the natural base (especially when the decimal representation

[1]The results of this survey will be analyzed and reported in greater detail in an upcoming report.

$2.718281828\cdots$ is far from being natural) since the special role of $y = e^x$ as a fixed point of the *naturally* occurring differential equation $\frac{dy}{dx} = y$ is completely removed from the student's learning experience.

Responding to this gap in the teachers' *mpck*, our present paper delivers an elementary proof of that the derivative of exponential function is itself – starting from Euler's original definition of the exponential function ($x \mapsto e^x$). Because our simple proof does not rely on any results of power series and uniform convergence, it is easily accessible by both instructors and learners of calculus.

References

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