

# ARITHMETIC PROPERTIES FOR UNRESTRICTED SINGULAR OVERPARTITIONS

YOSUA FERI WIJAYA<sup>1</sup>, UHA ISNAINI<sup>2</sup>, YENI SUSANTI<sup>3</sup> AND PEE CHOON TOH<sup>4</sup>

Department of Mathematics, Universitas Gadjah Mada, Indonesia<sup>1,2,3</sup>,  
Mathematics & Mathematics Education, National Institute of Education,  
Nanyang Technological University, Singapore<sup>4</sup>,

<sup>1</sup> yosua.feri.wijaya@mail.ugm.ac.id

<sup>2</sup> isnainiuha@ugm.ac.id

<sup>3</sup> yeni\_math@ugm.ac.id

<sup>4</sup> peechoon.toh@nie.edu.sg

**Abstract.** In this paper, we study a class of partitions called *unrestricted singular overpartitions* with parameters  $k$  and  $i$ , which is an overpartition such that no part is divisible by  $k$ , and any single occurrence of a part congruent to  $\pm i \pmod{k}$  may be overlined. The number of all such overpartitions of  $n$  is denoted by  $\overline{Cu}_{k,i}(n)$ . Previous studies have established several congruence properties for  $\overline{Cu}_{3,1}(n)$  modulo 2 and 4, and  $\overline{Cu}_{4,1}(n)$  modulo 2, 4, and 8. We extend these results by deriving new congruences for  $\overline{Cu}_{6,1}(n)$  modulo 36, 48, and 64. In addition, we establish connections between unrestricted singular overpartitions and other partition families, including  $(k, \ell)$ -regular bipartitions and overpartitions with  $\ell$ -regular nonoverlined parts.

*Keywords:* Partitions, Overpartitions, Congruences, Bijections

*Mathematics Subject Classification:* 05A17, 05A19, 11P83

## 1. INTRODUCTION

Overpartitions are one of the generalizations of classical integer partitions, in which the first occurrence of each distinct part may be overlined [5]. In this framework, parts may repeat as in ordinary partitions, while the overlining serves as an additional structural feature that enriches both combinatorial and arithmetic properties.

In 2015, Andrews [2] defined a specific type of overpartition called Andrews' singular overpartition with parameters  $k$  and  $i$ , which is an overpartition where no part is divisible by  $k$  and the first occurrence of any number that is congruent to  $\pm i \pmod{k}$  may be overlined. For any natural number  $n$ , the number of all such partitions is denoted by  $\overline{C}_{k,i}(n)$ . For example, the Andrews' singular overpartitions

---

2020 *Mathematics Subject Classification:*

Received: dd-mm-yyyy, accepted: dd-mm-yyyy.

of 5 with parameters  $k = 4$  and  $i = 1$  are as follows :

$$\begin{aligned} &5, \quad 5', \quad 3+2, \quad 3'+2, \quad 3+1+1, \quad 3'+1+1, \quad 3+1'+1, \quad 3'+1'+1, \\ &2+2+1, \quad 2+2+1', \quad 2+1+1+1, \quad 2+1'+1+1, \\ &1+1+1+1+1, \quad 1'+1+1+1+1. \end{aligned}$$

From this example, we have  $\overline{C}_{4,1}(5) = 14$ .

Inspired by Andrews' singular overpartition and the idea to remove the restriction that only the first occurrence of a part may be overlined, the concept of *unrestricted singular overpartition* with parameters  $k$  and  $i$  was introduced and extensively studied in 2025 by Wijaya et al. [15]. This type of partition is defined as one in which no part is divisible by  $k$ , and any single occurrence of a number congruent to  $\pm i \pmod{k}$  can be marked. The number of such partitions of  $n$  is denoted by  $\overline{Cu}_{k,i}(n)$ , while the set of all such partitions of  $n$  is denoted by  $\overline{Cu}_{k,i}[n]$ . For example, the elements of  $\overline{Cu}_{4,1}[4]$  are as follows:

$$\begin{aligned} &3+1, \quad 3'+1, \quad 3+1', \quad 3'+1', \quad 2+2, \quad 2+1+1, \quad 2+1'+1, \quad 2+1+1', \\ &1+1+1+1, \quad 1'+1+1+1, \quad 1+1'+1+1, \quad 1+1+1'+1, \quad 1+1+1+1'. \end{aligned}$$

Thus,  $\overline{Cu}_{4,1}(4) = 13$ .

Furthermore, Wijaya et al. [15] derived the generating function for  $\overline{Cu}_{k,i}(n)$  under certain restrictions on the parameters  $k$  and  $i$ . However, within these constraints, the generating function for  $\overline{Cu}_{2i,i}(n)$  cannot be obtained. Motivated by this limitation, the present paper establishes a more general and complete form of the generating function, which is stated as follows:

**Theorem 1.1.** *For any integers  $k \geq 2$  and  $1 \leq i \leq \frac{k}{2}$ , the generating function of  $\overline{Cu}_{k,i}(n)$  is given by*

$$\sum_{n=0}^{\infty} \overline{Cu}_{k,i}(n) q^n = \begin{cases} \frac{f_{2i}^2}{f_1 f_i}, & \text{if } i = \frac{k}{2}, \\ \frac{f_k}{f_1(q^i; q^k)_{\infty} (q^{k-i}; q^k)_{\infty}}, & \text{if } i \neq \frac{k}{2}, \end{cases} \quad (1)$$

where  $(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^k)$  and  $f_k = (q^k; q^k)_{\infty}$ .

Next, we would like to mention other kinds of partitions that can be connected to unrestricted singular overpartitions. We begin with  $(k, \ell)$ -regular bipartitions, which are defined as bipartitions where the first partition contains no multiples of  $k$  and the second partition contains no multiples of  $\ell$ .

Let  $B_{k,\ell}(n)$  denote the number of  $(k, \ell)$ -regular bipartitions of  $n$ , while the set of all such partitions is denoted by  $B_{k,\ell}[n]$ . For example, the elements of  $B_{2,3}[4]$  are as follows:

$$\begin{aligned} &(-, 4), \quad (-, 2+2), \quad (-, 2+1+1), \quad (-, 1+1+1+1), \quad (1, 2+1), \quad (1, 1+1+1), \\ &(1+1, 2), \quad (1+1, 1+1), \quad (3, 1), \quad (1+1+1, 1), \quad (3+1, -), \quad (1+1+1+1, -). \end{aligned}$$

Thus,  $B_{2,3}(4) = 12$ .

Furthermore, the generating function of  $B_{k,\ell}(n)$  is as follows :

**Theorem 1.2.** [14] *For any integers  $k, \ell \geq 2$ , the generating function of  $B_{k,\ell}(n)$  is given by*

$$\sum_{n=0}^{\infty} B_{k,\ell}(n) q^n = \frac{f_k f_\ell}{f_1^2}. \quad (2)$$

There has been substantial recent interest in the study of congruences satisfied by  $B_{k,\ell}(n)$ . For examples where  $k = \ell$ , we can refer to Cui and Gu [6], and Lin [11, 12]. For examples where  $k$  and  $\ell$  are distinct odd primes, we can refer to Wang [14], Dou [7] and Kathiravan [10].

Recently, Alanazi et al. [1] studied a class of overpartitions in which the nonoverlined parts are required to be  $\ell$ -regular, meaning that they are not divisible by a fixed integer  $\ell$ . The number of such overpartitions of  $n$  is denoted by  $\overline{R}_\ell^*(n)$ , and the set of all such overpartitions is written as  $\overline{R}_\ell^*[n]$ .

For instance, the set  $\overline{R}_3^*[4]$  consists of the following overpartitions:

$$\begin{aligned} &4, \overline{4}, \overline{3} + 1, \overline{3} + \overline{1}, 2 + 2, \overline{2} + 2, 2 + 1 + 1, \\ &\overline{2} + 1 + 1, 2 + \overline{1} + 1, \overline{2} + \overline{1} + 1, 1 + 1 + 1 + 1, \overline{1} + 1 + 1 + 1. \end{aligned}$$

Hence,  $\overline{R}_3^*(4) = 12$ .

The generating function of  $\overline{R}_\ell^*(n)$  can then be expressed as follows.

**Theorem 1.3.** [1] *For any integers  $\ell \geq 2$ , the generating function of  $\overline{R}_\ell^*(n)$  is given by*

$$\sum_{n=0}^{\infty} \overline{R}_\ell^*(n) q^n = \frac{f_2 f_\ell}{f_1^2}. \quad (3)$$

Sellers [13] extended the earlier results by employing elementary generating function techniques to derive infinite families of congruences for such partitions. His work provides deeper insight into the arithmetic structure of overpartition functions with  $\ell$ -regular nonoverlined parts.

By comparing the generating functions in (1), (2), and (3), we obtain explicit connections between unrestricted singular overpartitions  $\overline{Cu}_{k,i}(n)$ ,  $(k, \ell)$ -regular bipartitions  $B_{k,\ell}(n)$ , and overpartitions with  $\ell$ -regular nonoverlined parts  $\overline{R}_\ell^*(n)$ .

**Theorem 1.4.** *For every integer  $n \geq 0$ ,*

$$\overline{Cu}_{2,1}(n) = B_{2,2}(n) = \overline{R}_2^*(n), \quad (4)$$

$$\overline{Cu}_{3,1}(n) = B_{3,3}(n), \quad (5)$$

$$\overline{Cu}_{4,1}(n) = B_{2,4}(n) = \overline{R}_4^*(n), \quad (6)$$

$$\overline{Cu}_{6,1}(n) = B_{2,3}(n) = \overline{R}_3^*(n). \quad (7)$$

Although Theorem 1.4 can be established directly using generating function manipulations, we will present alternative proofs based on explicit bijections between the corresponding partition families. These bijective arguments provide natural combinatorial interpretations of the identities.

A central objective of this paper is to investigate congruence properties of  $\overline{Cu}_{6,1}(n)$ . Several congruences have already been obtained in [1], [13], and [14]. In particular, the following results are relevant to our study.

**Theorem 1.5.** [1] *For every integer  $n \geq 0$ ,*

$$\overline{Cu}_{6,1}(9n+r) = \overline{R}_3^*(9n+r) \equiv 0 \pmod{3}, r = 4, 7.$$

**Theorem 1.6.** [13] *For every integer  $n \geq 0$ ,*

$$\overline{Cu}_{6,1}(3n+2) = \overline{R}_3^*(3n+2) \equiv 0 \pmod{4};$$

$$\overline{Cu}_{6,1}(9n+r) = \overline{R}_3^*(9n+r) \equiv 0 \pmod{4}, r = 4, 7.$$

**Theorem 1.7.** [14, Theorem 1.1(1)] *For every integer  $m \geq 1$ , we have*

$$\sum_{n=0}^{\infty} B_{3,m}(n) q^{24n+m+1} \equiv \sum_{i,j=-\infty}^{\infty} (-1)^{i+j} (3i+1) q^{(6i+1)^2+m(6j+1)^2} \pmod{9}.$$

In this paper, we further strengthen these results by establishing new congruences for  $\overline{Cu}_{6,1}(n)$ .

**Theorem 1.8.** *For every integer  $n \geq 0$ ,*

$$\overline{Cu}_{6,1}(9n+7) \equiv 0 \pmod{48}, \tag{8}$$

$$\overline{Cu}_{6,1}(27n+19) \equiv 0 \pmod{64}. \tag{9}$$

Furthermore, we can prove the following family of congruences for  $\overline{Cu}_{6,1}(n)$ .

**Theorem 1.9.** *Let  $p \geq 5$  be a prime such that  $\left(\frac{-2}{p}\right) = -1$ . Let  $0 \leq n_p \leq p-1$  and  $0 \leq r_p \leq p-1$  such that  $p \mid 8n_p+1$  and  $r_p \equiv \frac{n_p(8n_p+1)}{p} \pmod{p}$ . Then,*

$$\overline{Cu}_{6,1}(p^2n+pr+n_p) \equiv 0 \pmod{36}, \tag{10}$$

*for all  $0 \leq r \leq p-1$ , except possibly when  $r = r_p$ .*

As a result of Theorem 1.9, we can establish the following congruences for  $\overline{Cu}_{6,1}(n)$ .

**Corollary 1.10.** *For every integer  $n \geq 0$ ,*

$$\overline{Cu}_{6,1}(25n+r) \equiv 0 \pmod{36}, r = 8, 13, 18, 23; \tag{11}$$

$$\overline{Cu}_{6,1}(49n+r) \equiv 0 \pmod{36}, r = 13, 20, 27, 34, 41, 48. \tag{12}$$

In the second section we will touch upon some lemmas that will be used in the subsequent sections, in which we establish the generating function in Theorem 1.1, prove the new congruences stated in Theorem 1.8, prove the family of congruences

stated in Theorem 1.9, and develop explicit bijections between selected pairs of  $\overline{Cu_{k,i}}(n)$  and  $B_{k,\ell}(n)$ .

## 2. PRELIMINARY RESULTS

In this section, we review some properties that will serve as tools for proving the theorems in the next section. We begin with the following 3-dissection which can be found in Berndt [4, p. 49, Corollary of Entry 31, (ii)].

**Lemma 2.1.** *We have the following equality:*

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9}. \quad (13)$$

Next, the following 3-dissection has been proven by Hirschhorn and Sellers [9].

**Lemma 2.2.** *We have the following equality:*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}. \quad (14)$$

Additionally, we also have the following 2-dissection which can be found in Hirschhorn [8, p. 196, (22.7.5)].

**Lemma 2.3.** *We have the following equality:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4}. \quad (15)$$

By using the binomial theorem, we have the following property.

**Lemma 2.4.** *For any natural number  $k$  and any prime number  $p$ , we have*

$$f_k^p \equiv f_{pk} \pmod{p}. \quad (16)$$

Next, we have the classical theta function  $\varphi(q)$  which satisfy the following identity [3, p.11].

**Lemma 2.5.** *The theta function  $\varphi(q)$  can be represented as an infinite product*

$$\varphi(q) = (-q; q^2)_\infty^2 (q^2; q^2)_\infty,$$

*in other words*

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \frac{f_2^5}{f_1^2 f_4^2}. \quad (17)$$

Lastly, we have the identity known as Euler's pentagonal number theorem [3, p.12].

**Lemma 2.6.** *The infinite product  $f_1$  can be represented as an infinite sum*

$$f_1 = \prod_{k=1}^{\infty} (1 - q^k) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}}. \quad (18)$$

## 3. PROOF OF THEOREM 1.1

First, consider the case where  $i \neq \frac{k}{2}$ . We observe that for any integer  $k \geq 2$  and  $1 \leq i \leq \frac{k}{2}$ , then  $i \neq \frac{k}{2} \Leftrightarrow k \geq 3$  and  $1 \leq i < \lceil \frac{k}{2} \rceil$ . Thus, from [15]<sup>1</sup>, we already know that

$$\sum_{n=0}^{\infty} \overline{Cu}_{k,i}(n)q^n = \frac{f_k}{f_1(q^i; q^k)_{\infty}(q^{k-i}; q^k)_{\infty}}, \quad \text{if } i \neq \frac{k}{2}. \quad (19)$$

Next, consider the case where  $i = \frac{k}{2}$ , or equivalently,  $k = 2i$ . In this case,  $\overline{Cu}_{k,i}(n) = \overline{Cu}_{2i,i}(n)$ , which counts the partitions of  $n$  where no part is divisible by  $2i$ , and any occurrence of a number congruent to  $\pm i \pmod{2i}$  may be marked. Therefore, if  $p \equiv i \pmod{k}$ , a partition can contain  $a$  occurrences of  $p$ , where we have  $a + 1$  choices: either to mark none or to mark one from the  $a$  occurrences of  $p$ . Thus, the generating function is given by:

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{Cu}_{2i,i}(n)q^n &= \prod_{p \not\equiv 0, i \pmod{2i}} \left( \sum_{r=0}^{\infty} q^{pr} \right) \prod_{p \equiv i \pmod{2i}} \left( \sum_{r=0}^{\infty} (1+r)q^{pr} \right) \\ &= \prod_{p \not\equiv 0, i \pmod{2i}} \left( \sum_{r=0}^{\infty} q^{pr} \right) \prod_{p \equiv i \pmod{2i}} \left( \sum_{r=0}^{\infty} q^{pr} \right)^2 \\ &= \prod_{p=1}^{\infty} \frac{(\sum_{r=0}^{\infty} q^{pr}) (\sum_{r=0}^{\infty} q^{(2pi-i)r})}{\sum_{r=0}^{\infty} q^{2pir}} \\ &= \prod_{p=1}^{\infty} \frac{(1 - q^{2pi})^2}{(1 - q^p)(1 - q^{pi})} \\ &= \frac{f_{2i}^2}{f_1 f_i}. \end{aligned}$$

In other words,

$$\sum_{n=0}^{\infty} \overline{Cu}_{k,i}(n)q^n = \frac{f_{2i}^2}{f_1 f_i}, \quad \text{if } i = \frac{k}{2}. \quad (20)$$

By combining Equations (19) and (20), we obtain Equation (1).  $\square$

---

<sup>1</sup>In this paper there is a typographical error in Theorem 1.1, it was written that  $1 \leq i < \lfloor \frac{k}{2} \rfloor$ , where it should have been  $1 \leq i < \lceil \frac{k}{2} \rceil$ .

## 4. PROOF OF THEOREM 1.8

By substituting  $(k, i) := (6, 1)$  into Equation (1), we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} \overline{Cu}_{6,1}(n) q^n &= \frac{f_6}{f_1(q; q^6)_{\infty} (q^5; q^6)_{\infty}} \\
 &= \frac{f_6}{f_1} (q^3; q^6)_{\infty} \frac{1}{(q; q^2)_{\infty}} \\
 &= \frac{f_6}{f_1} \frac{f_3}{f_6} \frac{f_2}{f_1} \\
 &= \frac{f_2 f_3}{f_1^2}.
 \end{aligned} \tag{21}$$

Next, applying Equation (14) into Equation (21) yields

$$\begin{aligned}
 \sum_{n=0}^{\infty} \overline{Cu}_{6,1}(n) q^n &= f_3 \frac{f_2}{f_1^2} \\
 &= f_3 \left( \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6} \right) \\
 &= \frac{f_6^4 f_9^6}{f_3^7 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^6} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^5}.
 \end{aligned}$$

By separating  $q$  based on the residue modulo 3 of its power and simplifying the results, we get

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1}(3n) q^n = \frac{f_2^4 f_3^6}{f_1^7 f_6^3}, \tag{22}$$

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1}(3n+1) q^n = 2 \frac{f_2^3 f_3^3}{f_1^6}, \tag{23}$$

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1}(3n+2) q^n = 4 \frac{f_2^2 f_6^3}{f_1^5}. \tag{24}$$

From Equations (23) and (24), we conclude

$$\begin{aligned}
 \overline{Cu}_{6,1}(3n+1) &\equiv 0 \pmod{2}, \\
 \overline{Cu}_{6,1}(3n+2) &\equiv 0 \pmod{4}.
 \end{aligned} \tag{25}$$

Furthermore, applying Equation (14) into Equation (23) yields

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (3n+1) q^n &= 2 \frac{f_2^3 f_3^3}{f_1^6} \\
&= 2 f_3^3 \left( \frac{f_2}{f_1^2} \right)^3 \\
&= 2 f_3^3 \left( \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6} \right)^3 \\
&= 2 f_3^3 \left( \frac{f_6^{12} f_9^{18}}{f_3^{24} f_{18}^9} + 6q \frac{f_6^{11} f_9^{15}}{f_3^{23} f_{18}^6} + 24q^2 \frac{f_6^{10} f_9^{12}}{f_3^{22} f_{18}^3} + 56q^3 \frac{f_6^9 f_9^9}{f_3^{21}} \right. \\
&\quad \left. + 96q^4 \frac{f_6^8 f_9^6 f_{18}^3}{f_3^{20}} + 96q^5 \frac{f_6^7 f_9^3 f_{18}^6}{f_3^{19}} + 64q^6 \frac{f_6^6 f_{18}^9}{f_3^{18}} \right),
\end{aligned}$$

in other words,

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (3n+1) q^n &= 2 \left( \frac{f_6^{12} f_9^{18}}{f_3^{21} f_{18}^9} + 56q^3 \frac{f_6^9 f_9^9}{f_3^{18}} + 64q^6 \frac{f_6^6 f_{18}^9}{f_3^{15}} \right) \\
&\quad + 12q \left( \frac{f_6^{11} f_9^{15}}{f_3^{20} f_{18}^6} + 16q^3 \frac{f_6^8 f_9^6 f_{18}^3}{f_3^{17}} \right) \\
&\quad + 48q^2 \left( \frac{f_6^{10} f_9^{12}}{f_3^{19} f_{18}^3} + 4q^3 \frac{f_6^7 f_9^3 f_{18}^6}{f_3^{16}} \right).
\end{aligned}$$

By separating  $q$  based on the residue modulo 3 of its power and simplifying the results, we get

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (9n+1) q^n = 2 \left( \frac{f_2^{12} f_3^{18}}{f_1^{21} f_6^9} + 56q \frac{f_2^9 f_3^9}{f_1^{18}} + 64q^2 \frac{f_2^6 f_6^9}{f_1^{15}} \right), \quad (26)$$

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (9n+4) q^n = 12 \left( \frac{f_2^{11} f_3^{15}}{f_1^{20} f_6^6} + 16q \frac{f_2^8 f_3^6 f_6^3}{f_1^{17}} \right), \quad (27)$$

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (9n+7) q^n = 48 \left( \frac{f_2^{10} f_3^{12}}{f_1^{19} f_6^3} + 4q \frac{f_2^7 f_3^3 f_6^6}{f_1^{16}} \right). \quad (28)$$

From Equations (27) and (28), we conclude

$$\begin{aligned}
\overline{Cu}_{6,1} (9n+4) &\equiv 0 \pmod{12}, \\
\overline{Cu}_{6,1} (9n+7) &\equiv 0 \pmod{48}.
\end{aligned} \quad (29)$$

This proves Equation (8).

Putting Equation (26) under modulo 64, we will have

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (9n+1) q^n \equiv 2 \frac{f_2^{12} f_3^{18}}{f_1^{21} f_6^9} - 16q \frac{f_2^9 f_3^9}{f_1^{18}} \pmod{64}. \quad (30)$$

By applying Equations (14) and (13) into Equation (30), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{Cu}_{6,1} (9n+1) q^n &\equiv 2 \left( \frac{f_2}{f_1^2} \right)^{10} \frac{f_2^2 f_3^{18}}{f_1 f_6^9} - 16q \left( \frac{f_2}{f_1^2} \right)^9 f_3^9 \\ &\equiv 2 \left( \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6} \right)^{10} \left( \frac{f_6 f_9^2}{f_3 f_{18}} + q \frac{f_{18}^2}{f_9} \right) \frac{f_3^{18}}{f_6^9} \\ &\quad - 16q \left( \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6} \right)^9 f_3^9 \pmod{64}. \end{aligned}$$

By calculating the above equation and simplifying the result, we get

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{Cu}_{6,1} (9n+1) q^n &\equiv 2 \left( \frac{f_6^{32} f_9^{62}}{f_3^{63} f_{18}^{31}} + 12q^3 \frac{f_6^{29} f_9^{53}}{f_3^{60} f_{18}^{22}} \right) \\ &\quad + 6q \left( 7 \frac{f_6^{31} f_9^{59}}{f_3^{62} f_{18}^{28}} + 8 \frac{f_6^{36} f_9^{54}}{f_3^{63} f_{18}^{27}} \right) \\ &\quad + 32q^2 \left( \frac{f_6^{30} f_9^{56}}{f_3^{61} f_{18}^{25}} + \frac{f_6^{35} f_9^{51}}{f_3^{62} f_{18}^{24}} + q^3 \frac{f_6^{27} f_9^{47}}{f_3^{58} f_{18}^{16}} \right) \pmod{64}. \end{aligned}$$

By separating  $q$  based on the residue modulo 3 of its power and simplifying the results, we get

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (27n+1) q^n \equiv 2 \left( \frac{f_2^{32} f_3^{62}}{f_1^{63} f_6^{31}} + 12q \frac{f_2^{29} f_3^{53}}{f_1^{60} f_6^{22}} \right) \pmod{64}, \quad (31)$$

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (27n+10) q^n \equiv 6 \left( 7 \frac{f_2^{31} f_3^{59}}{f_1^{62} f_6^{28}} + 8 \frac{f_2^{36} f_3^{54}}{f_1^{63} f_6^{27}} \right) \pmod{64}, \quad (32)$$

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (27n+19) q^n \equiv 32 \left( \frac{f_2^{30} f_3^{56}}{f_1^{61} f_6^{25}} + \frac{f_2^{35} f_3^{51}}{f_1^{62} f_6^{24}} + q \frac{f_2^{27} f_3^{47}}{f_1^{58} f_6^{16}} \right) \pmod{64}. \quad (33)$$

By using Lemma 2.4 and Equation (15), we know that

$$\begin{aligned}
\frac{f_2^{30} f_3^{56}}{f_1^{61} f_6^{25}} + \frac{f_2^{35} f_3^{51}}{f_1^{62} f_6^{24}} + q \frac{f_2^{27} f_3^{47}}{f_1^{58} f_6^{16}} &\equiv \frac{f_3^6}{f_1} + f_2^4 f_3^3 + q \frac{f_3^3 f_6^6}{f_2^2} \\
&\equiv f_3^3 \left( \frac{f_3^3}{f_1} + f_2^4 + q \frac{f_6^6}{f_2^2} \right) \\
&\equiv f_3^3 \left( \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q \frac{f_{12}^3}{f_4} + f_2^4 + q \frac{f_6^6}{f_2^2} \right) \\
&\equiv f_3^3 \left( f_2^4 + q \frac{f_6^6}{f_2^2} + f_2^4 + q \frac{f_6^6}{f_2^2} \right) \\
&\equiv 2f_3^3 \left( f_2^4 + q \frac{f_6^6}{f_2^2} \right) \\
&\equiv 0 \pmod{2}.
\end{aligned} \tag{34}$$

By combining Equations (33) and (34), we have

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1} (27n + 19) q^n \equiv 0 \pmod{64}. \tag{35}$$

From Equation (35), we conclude

$$\overline{Cu}_{6,1} (27n + 19) \equiv 0 \pmod{64}.$$

This proves Equation (9), and thus the proof of Theorem 1.8 is complete.  $\square$

Additionally, due to the fact that  $\overline{Cu}_{6,1}(n) = \overline{R}_3^*(n)$ , the process of reaching Equations (25), (29), and (8) can be viewed as an alternative way to prove Theorem 1.5 and Theorem 1.6.

## 5. PROOF OF THEOREM 1.9

Suppose that we can represent the generating function of  $f(n)$  as

$$\sum_{n=0}^{\infty} f(n) q^n = \sum_{x,y=-\infty}^{\infty} a(x,y) q^{\frac{x^2+2y^2-3}{24}}, \tag{36}$$

where  $a(x,y)$  are coefficients.

The value of  $f(pm + n_p)$  is given by the sum of the coefficients  $a(x,y)$  that satisfies

$$\frac{x^2 + 2y^2 - 3}{24} = pm + n_p,$$

or in other words

$$x^2 + 2y^2 = 3p \left( 8m + \frac{8n_p + 1}{p} \right).$$

The above means that  $p \mid x^2 + 2y^2$ , thus  $x^2 y^{p-3} \equiv -2y^{p-1} \pmod{p}$ .

If  $p \nmid x$ , then  $p \nmid y$ . Using Fermat's little theorem we have  $(xy^{\frac{p-3}{2}})^2 \equiv -2 \pmod{p}$ , however this contradicts the fact that  $\left(\frac{-2}{p}\right) = -1$ .

Hence  $p \mid x$ , thus  $p \mid 2y^2$ , and since  $p \geq 5$  is prime, then  $p \mid y$ . Furthermore  $p^2 \mid x^2 + 2y^2$  and therefore

$$\begin{aligned} 3p \left( 8m + \frac{8n_p + 1}{p} \right) &\equiv 0 \pmod{p^2} \\ \Rightarrow 3 \left( 8m + \frac{8n_p + 1}{p} \right) &\equiv 0 \pmod{p} \\ \Rightarrow 8m + \frac{8n_p + 1}{p} &\equiv 0 \pmod{p}. \end{aligned}$$

If we set  $m = pk + r$  where  $0 \leq r \leq p - 1$  and using the fact that  $p \mid 8n_p + 1$ , we have

$$8m + \frac{8n_p + 1}{p} \equiv 8r - 8n_p \frac{8n_p + 1}{p} \equiv 8(r - r_p) \equiv 0 \pmod{p}.$$

Since  $p$  is an odd prime, then for all values of  $r$  except  $r = r_p$ , the above congruence does not hold and thus no such values of  $x$  and  $y$  exist. Hence, for all  $0 \leq r \leq p - 1$  except possibly when  $r = r_p$ , we have

$$f(pm + n_p) = f(p^2k + pr + n_p) = 0.$$

However, from Theorem 1.7 it is known that

$$\sum_{n=0}^{\infty} B_{3,2}(n) q^{24n+3} \equiv \sum_{i,j=0}^{\infty} (-1)^{i+j} (3i+1) q^{(6i+1)^2 + 2(6j+1)^2} \pmod{9},$$

or in other words

$$\sum_{n=0}^{\infty} \overline{Cu}_{6,1}(n) q^n \equiv \sum_{i,j=0}^{\infty} (-1)^{i+j} (3i+1) q^{\frac{(6i+1)^2 + 2(6j+1)^2 - 3}{24}} \pmod{9}$$

We can see that the generating function of  $\overline{Cu}_{6,1}(n)$  can be represented as in Equation (36), and thus for all  $0 \leq r \leq p - 1$  except possibly when  $r = r_p$ , we have

$$\overline{Cu}_{6,1}(p^2k + pr + n_p) \equiv 0 \pmod{9}. \quad (37)$$

The above result is essentially Theorem 1.2(2) of Wang [14].

Analogously, suppose that we can represent the generating function of  $g(n)$  as

$$\sum_{n=0}^{\infty} g(n) q^n = \sum_{x,y=-\infty}^{\infty} b(x,y) q^{\frac{x^2 + 2y^2 - 1}{8}}, \quad (38)$$

where  $b(x,y)$  are coefficients. Using a similar method as before, we can conclude that for all  $0 \leq r \leq p - 1$  except possibly when  $r = r_p$ , we have

$$g(pm + n_p) = g(p^2k + pr + n_p) = 0.$$

Since  $(1 - q^{2n})^4 \equiv (1 - q^{4n})^2 \pmod{4}$ , we have

$$f_2^4 \equiv f_4^2 \pmod{4}. \quad (39)$$

By substituting Equations (17), (18), and (39) on Equation (21), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \overline{Cu}_{6,1}(n) q^n &= f_3 \frac{f_2}{f_1^2} \\ &\equiv f_3 \frac{f_2^5}{f_1^2 f_4^2} \\ &= \left( \sum_{i=0}^{\infty} (-1)^i q^{\frac{3i(3i+1)}{2}} \right) \left( \sum_{j=0}^{\infty} q^{j^2} \right) \\ &= \sum_{i,j=0}^{\infty} (-1)^i q^{\frac{(6i+1)^2 + 2(2j)^2 - 1}{8}} \pmod{4}. \end{aligned}$$

We can see that the generating function of  $\overline{Cu}_{6,1}(n)$  can be represented as in Equation (38), and thus for all  $0 \leq r \leq p-1$  except possibly when  $r = r_p$ , we have

$$\overline{Cu}_{6,1}(p^2k + pr + n_p) \equiv 0 \pmod{4}. \quad (40)$$

Combining the congruences from Equation (37) and Equation (40) will complete our proof.  $\square$

## 6. BIJECTIVE PROOF OF THEOREM 1.4

We will first start by providing a bijection between  $\overline{Cu}_{6,1}[n]$  and  $B_{2,3}[n]$ . To help illustrate the idea, first we observe the following procedure.

- (1) For any overpartition in  $\overline{Cu}_{6,1}[n]$ , we first recall that no part is a multiple of 6. First color all the parts that is even in red, and the rest in black. Then, if a summand has one of its appearance marked, we color it and its subsequent appearance in red.
- (2) After the coloring is done, we remove all marks and separate the black colored parts and the red colored parts into two separate partitions with black in the front and red in the back.
- (3) Thus we get a bipartition. Additionally, since all even summands are red, then no even parts will appear in the first partition which means it is 2-regular. On the other hand, since there are no summands that are multiples of 6 and only summands that are equivalent to 1 or 5  $\pmod{6}$  may be marked, then each summand that is a multiple of 3 will always be black, thus no multiple of 3 will appear in the second partition which means it is 3-regular. Thus the resulting bipartition is in  $B_{2,3}[n]$ .

Example :

$$\begin{array}{c}
 9 + 8 + 7 + 7 + 5' + 5 + 4 + 4 + 3 + 3 + 2 + 2 + 1 + 1 + 1' \\
 \updownarrow \\
 (9 + 7 + 7 + 3 + 3 + 1 + 1, 8 + 5 + 5 + 4 + 4 + 2 + 2 + 1)
 \end{array}$$

It turns out that the above procedure is a bijection. We now proceed to prove it formally.

To simplify, we denote  $c^u = \overbrace{c + \dots + c}^{u \text{ times}}$  and  $c^{u,v} = \overbrace{c + \dots + c}^{u \text{ times}} + \overbrace{c' + \dots + c'}^{v \text{ times}}$ . For example, we may denote the unrestricted overpartition  $7' + 5 + 5' + 2 + 2 + 1 + 1' + 1 + 1$  as  $7^{0,1} + 5^{1,1} + 2^{2,0} + 1^{1,3}$  and the bipartition  $(5 + 2 + 2 + 1, 7 + 5 + 1 + 1 + 1)$  as  $(5^1 + 2^2 + 1^1, 7^1 + 5^1 + 1^3)$ .

Also, since  $c^0 = \underbrace{c + \dots + c}_{0 \text{ times}}$  and  $c^{0,0} = \underbrace{c + \dots + c}_{0 \text{ times}} + \underbrace{c' + \dots + c'}_{0 \text{ times}}$ , they denote nothing and can be erased.

We will now find a mapping  $\alpha : \overline{Cu}_{6,1}[n] \rightarrow B_{2,3}[n]$  such that  $\alpha$  is a bijection. First, we know that

$$c_1^{u_1, v_1} + c_2^{u_2, v_2} + \dots \in \overline{Cu}_{6,1}[n] \Rightarrow \begin{cases} c_i \not\equiv 0 \pmod{6}, \text{ and} \\ \text{If } c_i \equiv 2, 3, 4 \pmod{6}, \text{ then } v_i = 0. \end{cases}$$

Now, for any  $c_1^{u_1, v_1} + c_2^{u_2, v_2} + \dots \in \overline{Cu}_{6,1}[n]$  we define

$$\alpha(c_1^{u_1, v_1} + c_2^{u_2, v_2} + \dots) = (c_1^{a_1} + c_2^{a_2} + \dots, c_1^{b_1} + c_2^{b_2} + \dots),$$

where

$$(a_i, b_i) = \begin{cases} (u_i, v_i); & \text{if } c_i \equiv 1, 3, 5 \pmod{6}, \\ (v_i, u_i); & \text{if } c_i \equiv 2, 4 \pmod{6}. \end{cases}$$

Since  $c_i \not\equiv 0 \pmod{6}$  and if  $c_i \equiv 2, 4 \pmod{6} \Rightarrow c_i^{a_i} = c_i^{v_i} = c_i^0$ , then  $c_1^{a_1} + c_2^{a_2} + \dots$  will not contain any even summands, which means it is 2-regular.

Likewise, since  $c_i \not\equiv 0 \pmod{6}$  and if  $c_i \equiv 3 \pmod{6} \Rightarrow c_i^{b_i} = c_i^{v_i} = c_i^0$ , then  $c_1^{b_1} + c_2^{b_2} + \dots$  will not contain any multiple of 3, which means it is 3-regular.

Hence  $(c_1^{a_1} + c_2^{a_2} + \dots, c_1^{b_1} + c_2^{b_2} + \dots) \in B_{2,3}[n]$ , thus  $\alpha : \overline{Cu}_{6,1}[n] \rightarrow B_{2,3}[n]$  is a function.

Next we show that  $\alpha$  is invertible, let  $(c_1^{a_1} + c_2^{a_2} + \dots, d_1^{b_1} + d_2^{b_2} + \dots) \in B_{2,3}[n]$ , thus  $c_i \not\equiv 0 \pmod{2}$  and  $d_i \not\equiv 0 \pmod{3}$  for any  $i$ .

Let  $\{e_1, e_2, \dots\} = \{c_1, c_2, \dots\} \cup \{d_1, d_2, \dots\}$ , such that  $e_1 > e_2 > \dots$ . Thus we have  $e_i \not\equiv 0 \pmod{6}$ .

We will also have that

$$\left(c_1^{a_1} + c_2^{a_2} + \cdots, d_1^{b_1} + d_2^{b_2} + \cdots\right) = \left(e_1^{f_1} + e_2^{f_2} + \cdots, e_1^{g_1} + e_2^{g_2} + \cdots\right),$$

where

$$f_i = \begin{cases} a_j; & \text{if } e_i \in \{c_1, c_2, \dots\} \text{ and } e_i = c_j, \\ 0; & \text{if } e_i \notin \{c_1, c_2, \dots\}, \end{cases}$$

$$g_i = \begin{cases} b_j; & \text{if } e_i \in \{d_1, d_2, \dots\} \text{ and } e_i = d_j, \\ 0; & \text{if } e_i \notin \{d_1, d_2, \dots\}. \end{cases}$$

We define

$$\alpha' \left(c_1^{a_1} + c_2^{a_2} + \cdots, d_1^{b_1} + d_2^{b_2} + \cdots\right) = e_1^{u_1, v_1} + e_2^{u_2, v_2} + \cdots$$

where

$$(u_i, v_i) = \begin{cases} (f_i, g_i); & \text{if } e_i \equiv 1, 3, 5 \pmod{6}, \\ (g_i, f_i); & \text{if } e_i \equiv 2, 4 \pmod{6}. \end{cases}$$

Now we observe that if  $e_i \equiv 2, 4 \pmod{6}$ , then  $e_i \equiv 0 \pmod{2}$ , hence  $e_i \notin \{c_1, c_2, \dots\}$ , which implies  $f_i = 0$ , and since  $(u_i, v_i) = (g_i, f_i)$ , then  $v_i = 0$ .

On the other hand, if  $e_i \equiv 3 \pmod{6}$ , then  $e_i \equiv 0 \pmod{3}$ , hence  $e_i \notin \{d_1, d_2, \dots\}$ , which implies  $g_i = 0$ , and since  $(u_i, v_i) = (f_i, g_i)$ , then  $v_i = 0$ .

Thus, the overpartition  $e_1^{u_1, v_1} + e_2^{u_2, v_2} + \cdots$  satisfies that  $e_i \not\equiv 0 \pmod{6}$  for any  $i$ , and if  $e_i \not\equiv \pm 1 \pmod{6}$ , then  $v_i = 0$ . Therefore,  $e_1^{u_1, v_1} + e_2^{u_2, v_2} + \cdots \in \overline{Cu}_{6,1}[n]$ . Thus  $\alpha' : B_{2,3}[n] \rightarrow \overline{Cu}_{6,1}[n]$  is a function.

Moreover, we observe that

$$\begin{aligned} \alpha \left( \alpha' \left( c_1^{a_1} + c_2^{a_2} + \cdots, d_1^{b_1} + d_2^{b_2} + \cdots \right) \right) &= \alpha \left( e_1^{u_1, v_1} + e_2^{u_2, v_2} + \cdots \right) \\ &= \left( e_1^{a'_1} + e_2^{a'_2} + \cdots, e_1^{b'_1} + e_2^{b'_2} + \cdots \right), \end{aligned}$$

where

$$(a'_i, b'_i) = \begin{cases} (u_i, v_i); & \text{if } c_i \equiv 1, 3, 5 \pmod{6}, \\ (v_i, u_i); & \text{if } c_i \equiv 2, 4 \pmod{6}. \end{cases}$$

However, since

$$(u_i, v_i) = \begin{cases} (f_i, g_i); & \text{if } e_i \equiv 1, 3, 5 \pmod{6}, \\ (g_i, f_i); & \text{if } e_i \equiv 2, 4 \pmod{6}, \end{cases}$$

then

$$\alpha \left( \alpha' \left( c_1^{a_1} + c_2^{a_2} + \cdots, d_1^{b_1} + d_2^{b_2} + \cdots \right) \right) = \left( e_1^{f_1} + e_2^{f_2} + \cdots, e_1^{g_1} + e_2^{g_2} + \cdots \right).$$

And also, since

$$f_i = \begin{cases} a_j; & \text{if } e_i \in \{c_1, c_2, \dots\} \text{ and } e_i = c_j, \\ 0; & \text{if } e_i \notin \{c_1, c_2, \dots\}, \end{cases}$$

$$g_i = \begin{cases} b_j; & \text{if } e_i \in \{d_1, d_2, \dots\} \text{ and } e_i = d_j, \\ 0; & \text{if } e_i \notin \{d_1, d_2, \dots\}, \end{cases}$$

then

$$\alpha \left( \alpha' \left( c_1^{a_1} + c_2^{a_2} + \dots, d_1^{b_1} + d_2^{b_2} + \dots \right) \right) = \left( c_1^{a_1} + c_2^{a_2} + \dots, d_1^{b_1} + d_2^{b_2} + \dots \right).$$

Thus,  $\alpha$  is invertible with  $\alpha'$  as its inverse.

In conclusion,  $\alpha : \overline{Cu}_{6,1}[n] \rightarrow B_{2,3}[n]$  is a bijection.

□

To illustrate, below is an example of how the mapping works.

$$\begin{aligned} & \alpha(9 + 8 + 7 + 7 + 5' + 5 + 4 + 4 + 3 + 3 + 2 + 2 + 1 + 1 + 1') \\ &= \alpha(9^{1,0} + 8^{1,0} + 7^{2,0} + 5^{0,2} + 4^{2,0} + 3^{2,0} + 2^{2,0} + 1^{2,1}) \\ &= (9^1 + 8^0 + 7^2 + 5^0 + 4^0 + 3^2 + 2^0 + 1^2, 9^0 + 8^1 + 7^0 + 5^2 + 4^2 + 3^0 + 2^2 + 1^1) \\ &= (9 + 7 + 7 + 3 + 3 + 1 + 1, 8 + 5 + 5 + 4 + 4 + 2 + 2 + 1). \end{aligned}$$

We can see that the mapping  $\alpha$  is essentially the same as the coloring procedure we mentioned at the start of this section.

For a further example, here is a table that shows the bijection between the elements of  $\overline{Cu}_{6,1}[5]$  and  $B_{2,3}[5]$ .

| $p \in \overline{Cu}_{6,1}(5)$ | $\alpha(p) \in B_{2,3}(5)$ | $p \in \overline{Cu}_{6,1}(5)$ | $\alpha(p) \in B_{2,3}(5)$ |
|--------------------------------|----------------------------|--------------------------------|----------------------------|
| 5                              | (5, -)                     | 5'                             | (-, 5)                     |
| 4 + 1                          | (1, 4)                     | 4 + 1'                         | (-, 4 + 1)                 |
| 3 + 2                          | (3, 2)                     | 3 + 1 + 1                      | (3 + 1 + 1, -)             |
| 3 + 1' + 1                     | (3, 1 + 1)                 | 3 + 1 + 1'                     | (3 + 1, 1)                 |
| 2 + 2 + 1                      | (1, 2 + 2)                 | 2 + 2 + 1'                     | (-, 2 + 2 + 1)             |
| 2 + 1 + 1 + 1                  | (1 + 1 + 1, 2)             | 2 + 1' + 1 + 1                 | (-, 2 + 1 + 1 + 1)         |
| 2 + 1 + 1' + 1                 | (1, 2 + 1 + 1)             | 2 + 1 + 1 + 1'                 | (1 + 1, 2 + 1)             |
| 1 + 1 + 1 + 1 + 1              | (1 + 1 + 1 + 1 + 1, -)     | 1' + 1 + 1 + 1 + 1             | (-, 1 + 1 + 1 + 1 + 1)     |
| 1 + 1' + 1 + 1 + 1             | (1, 1 + 1 + 1 + 1)         | 1 + 1 + 1' + 1 + 1             | (1 + 1, 1 + 1 + 1)         |
| 1 + 1 + 1 + 1' + 1             | (1 + 1 + 1, 1 + 1)         | 1 + 1 + 1 + 1 + 1'             | (1 + 1 + 1 + 1, 1)         |

Analogously, we can also provide bijections between the other three pairs of  $\overline{Cu}_{k,i}[n]$  and  $B_{k,\ell}[n]$  mentioned in Theorem 1.4. Those bijections are:

(1)  $\beta : \overline{Cu}_{2,1}[n] \rightarrow B_{2,2}[n]$ , where

$$\beta(c_1^{u_1, v_1} + c_2^{u_2, v_2} + \dots) = (c_1^{u_1} + c_2^{u_2} + \dots, c_1^{v_1} + c_2^{v_2} + \dots).$$

(2)  $\gamma : \overline{Cu}_{3,1}[n] \rightarrow B_{3,3}[n]$ , where

$$\gamma(c_1^{u_1, v_1} + c_2^{u_2, v_2} + \dots) = (c_1^{u_1} + c_2^{u_2} + \dots, c_1^{v_1} + c_2^{v_2} + \dots).$$

(3)  $\delta : \overline{Cu_{4,1}}[n] \rightarrow B_{2,4}[n]$ , where

$$\delta(c_1^{u_1, v_1} + c_2^{u_2, v_2} + \cdots) = (c_1^{a_1} + c_2^{a_2} + \cdots, c_1^{b_1} + c_2^{b_2} + \cdots),$$

$$(a_i, b_i) = \begin{cases} (u_i, v_i); & \text{if } c_i \equiv 1, 3 \pmod{4}, \\ (v_i, u_i); & \text{if } c_i \equiv 2 \pmod{4}. \end{cases}$$

**Acknowledgement** We thank the Indonesia Endowment Fund for Education Agency (LPDP) under the Ministry of Finance of the Republic of Indonesia for providing a doctoral scholarship to the first author.

## REFERENCES

- [1] Alanazi, A., Alenazi, B., Keith, W., Munagi, A.: Refining overpartitions by properties of non-overlined parts. *Contributions to Discrete Mathematics*. **17**(2), 96-111 (2022)
- [2] Andrews, G.E.: Singular overpartitions. *Int. J. Number Theory*. **11**(5), 1523-1533 (2015)
- [3] Berndt, B.C.: *Number theory in the spirit of ramanujan*, vol 34. American Mathematical Soc. (2006)
- [4] Berndt, B.C.: *Ramanujan's notebooks: Part iii*. Springer Science & Business Media (2012)
- [5] Corteel, S., Lovejoy, J.: Overpartition. *Transactions of the American Mathematical Society*. **356**(4), 1623-1635 (2004)
- [6] Cui, S.P., Gu, N.S.: Congruences for  $\ell$ -regular partitions and bipartitions. *Rocky Mountain J. Math.* **50**(2), 513-526 (2020)
- [7] Dou, D.Q.J.: Congruences for (3,11)-regular bipartitions modulo 11. *The Ramanujan Journal*. **40**(3), 535-540 (2016)
- [8] Hirschhorn, M.D.: *The power of q*. Springer (2017)
- [9] Hirschhorn, M.D., Sellers, J.A.: Arithmetic relations for overpartitions. *J. Combin. Math. Combin. Comput.* **53**(1), 65-73 (2005)
- [10] Kathiravan, T.: Ramanujan-type congruences modulo m for  $(\ell, m)$ -regular bipartitions. *Indian Journal of Pure and Applied Mathematics*. **53**(2), 375-391 (2022)
- [11] Lin, B.L.S.: Arithmetic of the 7-regular bipartition function modulo 3. *The Ramanujan Journal*. **37**(3), 469-478 (2015)
- [12] Lin, B.L.S.: An infinite family of congruences modulo 3 for 13-regular bipartitions. *The Ramanujan Journal*. **39**(1), 169-178 (2016)
- [13] Sellers, J.A.: Extending congruences for overpartitions with  $\ell$ -regular nonoverlined parts. *Bulletin of the Australian Mathematical Society*. **111**(3), 478-489 (2025)
- [14] Wang, L.: Arithmetic properties of  $(k, \ell)$ -regular bipartitions. *Bulletin of the Australian Mathematical Society*. **95**(3), 353-364 (2017)
- [15] Wijaya, Y.F., Isnaini, U., Susanti, Y.: On some congruences for unrestricted singular overpartitions. *Asian-European Journal of Mathematics*. **18**(3), 2450116 (2025)