On a certain vector crank modulo 7

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Abstract
We define a vector crank to provide a combinatorial interpretation for a certain Ramanujan type congruence modulo 7.

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1 Introduction
In [7], one of the authors established several new Ramanujan type identities and congruences modulo 3, 5 and 7 for certain types of partition functions. For example, define \( Q_{p_o,p}(n) \) as the number of partitions of \( n \) into two colors, where the red colored parts form a partition into odd parts and the blue

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colored parts form an overpartition. Using the standard notation

\[(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j),\]

\[(a; q)_\infty = \lim_{n \to \infty} (a; q)_n,\]

\[(a_1, \ldots, a_m; q)_\infty = (a_1; q)_\infty \cdots (a_m; q)_\infty,\]

for \(|q| < 1\) and \(a, a_1, \ldots, a_m \neq 0\), we can write the generating function of \(Q_{po,p}(n)\) as

\[\sum_{n=0}^{\infty} Q_{po,p}(n)q^n = \frac{1}{(q; q^2)_\infty} \times \frac{(-q; q)_\infty}{(q; q)_\infty} = \frac{(-q, -q; q)_\infty}{(q; q)_\infty}.\]

Toh [7] proved that

\[\sum_{n=0}^{\infty} Q_{po,p}(7n + 2)q^n \equiv 0 \pmod{7}. \quad (1)\]

Zhou [9] subsequently provided alternative proofs of all of the congruences in [7] with the exception of (1). She re-interpreted these partition functions as partitions into multi-colors, introduced what she termed as *multiranks* – which are essentially vector cranks as defined by Garvan [4] – and proved that these vector cranks divided the partitions into equinumerous parts. The aim of this article is to define a vector crank that will explain (1) combinatorially.

## 2 A vector crank

If \(\lambda\) is a partition, we define \(\sigma(\lambda)\) and \(n(\lambda)\) as the sum of the parts and the number of parts of \(\lambda\) respectively. Let \(D, O, P\) denote the sets of partitions into distinct parts, partitions into odd parts, and unrestricted partitions respectively. Define the cartesian product

\[V = D \times D \times O \times O \times P \times P.\]

For a vector partition \(\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) \in V\) define a sum of parts \(s\), a weight \(w\) and a crank \(r\) by

\[s(\vec{\lambda}) = 2\sigma(\lambda_1) + \sigma(\lambda_2) + \sigma(\lambda_3) + \sigma(\lambda_4) + 2\sigma(\lambda_5) + 2\sigma(\lambda_6), \quad (2a)\]

\[w(\vec{\lambda}) = (-1)^{n(\lambda_1)}, \quad (2b)\]

\[r(\vec{\lambda}) = 2n(\lambda_3) - 2n(\lambda_4) + n(\lambda_5) - n(\lambda_6). \quad (2c)\]
The weighted count of vector partitions of $n$ with crank $m$, denoted by $N_V(m, n)$, is given by

$$N_V(m, n) = \sum_{\lambda \in V, s(\lambda) = n, r(\lambda) = m} w(\lambda). \quad (3)$$

We also define the weighted count of vector partitions of $n$ with crank congruent to $k$ modulo $t$ by

$$N_V(k, t, n) = \sum_{m = -\infty}^{\infty} N_V(mt + k, n) = \sum_{\lambda \in V, s(\lambda) = n, r(\lambda) \equiv k \pmod{t}} w(\lambda). \quad (4)$$

Finally, we have the following generating function for $N_V(m, n)$,

$$\sum_{m = -\infty}^{\infty} \sum_{n = 0}^{\infty} N_V(m, n) z^m q^n = \frac{(q^2; q^2)_{\infty} (-q; q)_{\infty}}{(z^2 q; q^2)_{\infty} (z^{-2} q; q^2)_{\infty} (z q^2; q^2)_{\infty} (z^{-1} q^2; q^2)_{\infty}}. \quad (5)$$

**Theorem 1.** The following equation holds for all nonnegative integers $n$.

$$N_V(0, 7, 7n + 2) = N_V(1, 7, 7n + 2) = \cdots = N_V(6, 7, 7n + 2) = \frac{Q_{p_0, p}(7n + 2)}{7}. \quad (7)$$

The main ingredient in the proof of the theorem is Winquist’s identity [8], which is a variant of the $B_2$ case of the Macdonald identities [5]. We state the identity in the following symmetric form [6, Eq. (3.1)]. If we define

$$F_1(x) = \sum_{j = -\infty}^{\infty} (-1)^j q^{3j^2} (x^{3j} + x^{-3j}), \quad (6a)$$

$$F_2(x) = \sum_{k = -\infty}^{\infty} (-1)^k q^{3k^2 + 2k} (x^{3k+1} + x^{-3k-1}), \quad (6b)$$

we have

$$F_1(x)F_2(y) - F_1(y)F_2(x) = -\frac{2}{x} \left( xq, \frac{q}{x}, yq, \frac{q}{y}, xy, \frac{q^2}{xy}, \frac{x}{y}, \frac{yq^2}{x}, q^2, q^2, q^2 \right)_{\infty}. \quad (6c)$$
Proof of Theorem 1. If we set $\zeta = \exp(2\pi i/7)$ in (5), we obtain
\[
\sum_{t=0}^{6} \zeta^t \sum_{n=0}^{\infty} N_V(t, 7, n) q^n = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_V(m, n) \zeta^m q^n = (q^2; q^2)_\infty = (q, \zeta^2 q, q/\zeta^2, \zeta q^2, q^2/\zeta, \zeta^2 q^2, q^2/\zeta^2, q^2, \zeta^3 q^2, q^2/\zeta^3; q^2)_\infty = \frac{F_1(\zeta^3) F_2(\zeta) - F_1(\zeta) F_2(\zeta^3)}{2\zeta(1 - \zeta^2)(1 - \zeta^3)(q^2; q^2)_\infty},
\]
where we used (6) with $x = \zeta^3$ and $y = \zeta$.

Since $3j^2 \equiv 0, 3, 5, 6 \pmod{7}$ and $3k^2 + 2k \equiv 0, 1, 2, 5 \pmod{7}$, the power of $q$ in $q^{3j^2+3k^2+2k}$ is congruent to 2 modulo 7 exactly when $j \equiv 0 \pmod{7}$ and $k \equiv 2 \pmod{7}$. This means that the coefficient of $q^{7n+2}$ in
\[
F_1(\zeta^3) F_2(\zeta) - F_1(\zeta) F_2(\zeta^3)
\]
is zero since
\[
(-1)^{j+k}(\zeta^{9j} + \zeta^{-9j})(\zeta^{3k+1} + \zeta^{-3k-1}) - (-1)^{j+k}(\zeta^{-3j} + \zeta^{9j-3})(\zeta^{9k+3} + \zeta^{-9k-3}) = 0
\]
when $j \equiv 0 \pmod{7}$ and $k \equiv 2 \pmod{7}$. Thus
\[
\sum_{t=0}^{6} N_V(t, 7, 7n + 2) \zeta^t = 0. \tag{7}
\]
Since the minimal polynomial for $\zeta$ over the rational numbers is
\[
p(x) = 1 + x + x^2 + \cdots + x^6,
\]
we conclude that
\[
N_V(0, 7, 7n + 2) = N_V(1, 7, 7n + 2) = \cdots = N_V(6, 7, 7n + 2). \quad \square
\]
We end by indicating how one may prove (1) directly as the details were omitted in [7]. This can be done by observing that
\[ \sum_{n=0}^{\infty} Q_{p_0,p}(n)q^n = \frac{(q^2; q^2)_\infty^2}{(q; q^3)_\infty^2} \equiv \frac{(q^2; q^2)_\infty^9}{(q; q^3)_\infty^3} \times \frac{1}{(q^{14}; q^{14})_\infty} \pmod{7}. \] (8)
Thus (1) is equivalent to proving the coefficients of $q^{7n+2}$ in
\[ \frac{(q^2; q^2)_\infty^9}{(q; q^3)_\infty^3} \]
are all divisible by 7. We offer three alternative ways of doing this. The easiest way is to appeal directly to [3, Th. 2]. Alternatively, we can use one of the Macdonald identities associated with the $C_2^\vee$ root system [5, p. 137] or [6, Eq. 3.12], to express
\[ \frac{(q^2; q^2)_\infty^9}{(q; q^3)_\infty^3} = \sum_{\substack{\alpha \equiv 1 \pmod{8} \\ \beta \equiv 3 \pmod{8}}} \frac{1}{8} (\beta^2 - \alpha^2)q^{\alpha^2 + \beta^2 - 10}. \]
If the exponent of $q$ is congruent to 2 modulo 7, we have
\[ \alpha^2 + \beta^2 \equiv 16(2) + 10 \equiv 0 \pmod{7}. \]
Since $-1$ is a quadratic nonresidue modulo 7, 7 must divide both $\alpha$ and $\beta$. The third way is to apply the Hecke operator $T_7$ to $\eta(16\tau)^9/\eta(8\tau)^3$, a weight 3 cusp form of level 128. One can refer to [1] for examples of how this may be done.

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**Proposition 2.** If $|q|, |t| < 1$ then
\[ \frac{(at; q)_\infty}{(a; q)_\infty(t; q)_\infty} = \frac{1}{(a; q)_\infty} + \sum_{n=1}^{\infty} \frac{f^n}{(aq^n; q)_\infty(q; q)_n}. \]
References


