One-step closure, weak one-step closure and meet continuity

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The Scott topology

One-step closure

weak one-step closure

The relationship between one-step closure and weak one-step closure

The Scott topology

A *Scott open* subset U of a poset P iff the following two conditions are satisfied:

- $U = \uparrow U$;
- $\sup D \in U$ implies $D \cap U \neq \emptyset$ for all directed subsets $D \subseteq P$.

The complements of Scott open sets are called *Scott closed sets*.

Definition(Zou, Li and Ho)

A poset P is said to have *one-step closure* if $cl(A) = A^{'}$ holds for any $A \subseteq P$, where $A^{'} = \{x \in P \mid \exists D \subseteq^{\uparrow} \downarrow A, x = \sup D\}$.

Theorem(Zou, Li and Ho)

Every continuous poset has one-step closure.

Z. Zou, Q. Li, W. Ho, **Domains via approximation operators,** *Logical Methods in Computer Science*, 14 (2018): 1-17.

Open problem

Problem(Zou, Li and Ho)

Is a poset with one-step closure continuous?

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Answer: **No**.

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Xu and Yang's work

Sorgenfrey line $\mathbb{R}_l$: $(\mathbb{R}, \tau)$ (τ taking $\{[x, y) \mid x < y, x, y \in \mathbb{R}\}$ as a base).

- **Sorgenfrey line** $\mathbb{R}_l$ is well-filtered and first countable;
- K is countable for any $K \in Q(\mathbb{R}_l)$ (the Smyth power space).

By the above two conclusions, Xu and Yang showed that:

- $Q(\mathbb{R}_l)$ is **well-filtered** and **first countable**;
- the upper Vietoris topology coincides with the Scott topology of $Q(\mathbb{R}_l)$.

X. Xu, Z. Yang, **Coincidence of the upper Vietoris topology and the Scott topology,** *Topology and its Applications*, 288 (2021): 107480.

Xu and Yang's work

It follows that:

Lemma(Xu and Yang)

$Q(\mathbb{R}_l)$ is **continuous** iff $\mathbb{R}_l$ is **core-compact**.

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Open problem

By again the fact that $Q(\mathbb{R}_l)$ is **well-filtered** and **first countable**, we have the following result:

Lemma

$Q(\mathbb{R}_l)$ *has one-step closure.*

Theorem

$Q(\mathbb{R}_l)$ *has one-step closure, but it is not continuous.*

This implies that "one-step closure" is only a necessary condition for the continuity.

Problem

Characterize continuous posets by the properties together with one-step closure.

Definition(Mashburn)

Let x, y be elements of a poset P. We say that x is *weakly way-below* y, denoted by $x \ll_w y$, if for any directed subset D of P for which $\sup D$ exists, $y = \sup D$ implies $D \cap \uparrow x \neq \emptyset$. A poset P is called *exact* if for any $x \in P$, $\twoheaddownarrow_w x = \{y \in P \mid y \ll_w x\}$ is directed and $\sup \twoheaddownarrow_w x = x$.

J. Mashburn. **A comparison of three topologies on ordered sets.** *Topology and its applications.* 129 (2) (2003): 177-186.

We have the following characterization for continuity by one-step closure.

Theorem(Miao, Li and Zhao)

A poset L is continuous iff L has one-step closure and is exact.

Theorem(Zou, Li and Ho)

All posets having one-step closure are meet-continuous.

Based on the above theorem, we can observe that a **quasicontinuous poset** may not have one-step closure.

Z. Zou, Q. Li, W. Ho, **Domains via approximation operators,** *Logical Methods in Computer Science*, 14 (2018): 1-17.

Definition(Zhao)

A poset P is said to have *weak one-step closure* if for any $A \subseteq P$, it holds that $cl(A) = A^{''}$, where $A^{''} = \{x \in P \mid \exists D \subseteq^{\uparrow} \downarrow A, \ x \leq \sup D\}$.

The property is introduced by Zhao in his doctoral thesis in order to obtain some characterizations of Z-continuous posets.

D. Zhao, **Generalizations of continuous lattices and frames.** *Ph.D. Dissertation, The University of Cambridge.*

We have the following theorem:

Theorem(Miao, Li and Zhao)

Every quasicontinuous dcpo has weak one-step closure.

D. Zhao, **Generalizations of continuous lattices and frames.** *Ph.D. Dissertation, The University of Cambridge*.

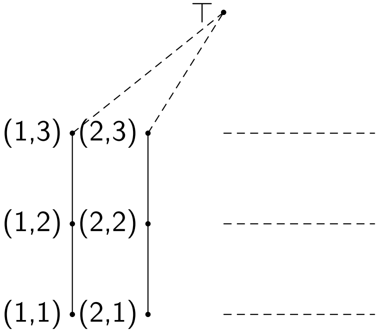

Figure: A non-quasicontinuous dcpo that has weak one-step closure.

So we have that “weak one-step closure” is only a necessary condition for the quasicontinuous dcpos.

Problem(Miao, Li and Zhao)

Characterize quasicontinuous dcpos by the properties together with weak one-step closure.

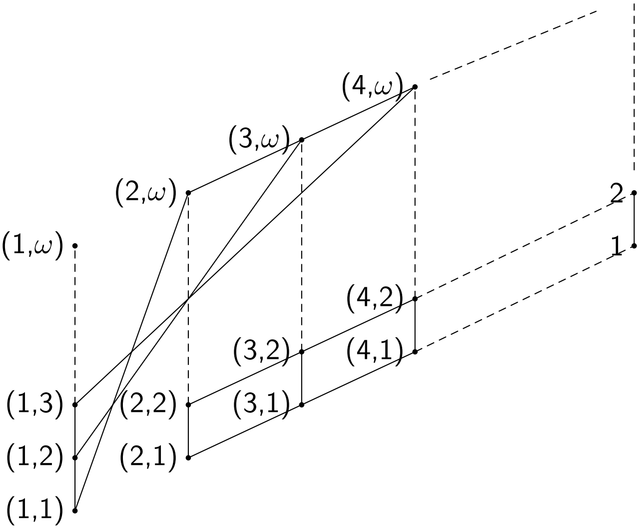

Figure: A quasicontinuous poset does not have weak one-step closure.

One-step closure $\underset{\not\longleftarrow}{\longrightarrow}$ Weak one-step closure

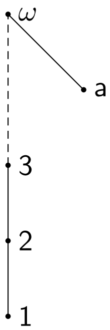

Figure: A poset with weak one-step closure does not have one-step closure.

One-step closure $\overset{\longrightarrow}{\not\longleftarrow}$ Meet continuity

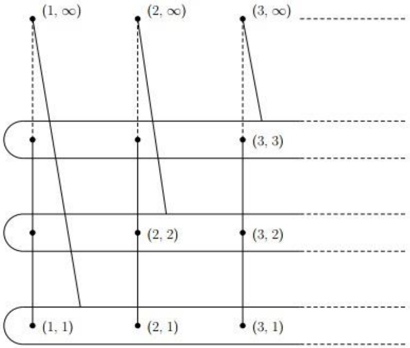

Figure: The Johnstone space $\mathbb{J}$

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Let $A = \{K_n^m \mid n \in \mathbb{N}, m \in \mathbb{N}\}$. Then

$$\sup_{m \in \mathbb{N}} \uparrow \{(1, m)\} = \uparrow \{(1, \infty)\} \in cl(A) \backslash A' \text{ (in } \mathbf{Q}(\mathbb{J})).$$

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One-step closure $\underset{\not\longleftarrow}{\longrightarrow}$ Meet continuity

Then $(Q(\mathbb{J}), \supseteq)$ (the set of all **compact saturated subsets** of $\mathbb{J}$) is meet continuous, but it does not have one-step closure.

One-step closure $\longleftrightarrow$ Weak one-step closure and Meet continuity

Theorem(Miao, Li and Zhao)

A poset has one-step closure if and only if it has weak one-step closure and is meet continuous.

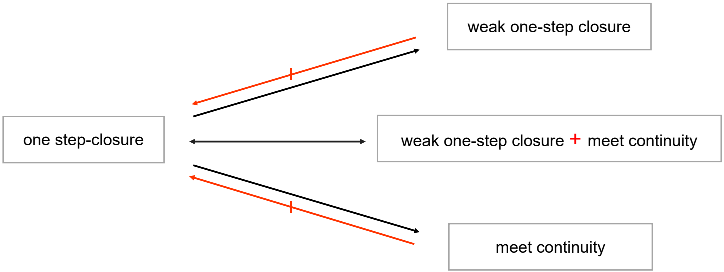

Figure: The relationship between one-step closure, weak one-step closure and meet continuity

Thank you !