A note on the category of c-spaces

Zhenchao Lyu, Xiaolin Xie, Hui Kou

Sichuan University, Chengdu
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The motivation of directed spaces

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In order to give semantics for the programming languages, people require the extended spaces have the following properties.

- Cartesian closedness of the category of the space with continuous maps;
- Approximation on the space.

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Is there a larger category which contains **Poset** and is cartesian closed?
Yes.

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Directed space

Let X be a T_0 space. There is a natural ordering (**specialization order**) $\leq$ on X, $x \leq y$ iff x is included in the closure of $\{y\}$.

Definition

(Kou,Yu 2014) Let X be a T_0 space. For every directed set D of X, D can be naturally viewed as a monotone net $(d)_{d\in D}$.
We say that U is a **directed open** set if for any $x \in U$ and any directed set D of X such that the net $(d)_{d\in D}$ converges to x in X, D intersects U.

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Remark: Erné introduced the concept of monotone determined spaces in 2009. Directed spaces are equivalent to monotone determined T_0 spaces.

Some examples

Example

- Any poset with Scott topology. In particular, every Scott space is a directed space.
- Any locally finitary compact space. Hence every c-space is a directed space. In particular every Alexandroff space is a directed space.
- A non-directed space. The set of natural numbers with cofinite topology.

Actually, for any T_0 space X, denote $d(X)$ the set of all directed open subsets of X, then $\mathcal{D}(X) = (X, d(X))$ is a directed space. We say that $\mathcal{D}(X)$ is the directed space generated by X.

c-space and locally finitary compact space

Definition

(Erné) Let X be a T_0 space. We say that X is a c-space iff for any open set U and any point $x \in U$, there is some point $y \in U$ such that $\uparrow y$ is a neighborhood of x.

Every continuous dcpo with Scott topology is a c-space.

Definition

Let X be a T_0 space. We say that X is a locally finitary compact space iff for any open set U and any point $x \in U$, there is a finite set $F \subseteq U$ such that $\uparrow F$ is a neighborhood of x. Then we call X a locally finitary compact space.

Every quasicontinuous dcpo with Scott topology is locally finitary compact.

The product and exponential object in DTop

Theorem

*(Kou, Yu 2014) The category of directed spaces and continuous maps (**DTop** for short) is cartesian closed.*

*For any two directed spaces X and Y, the **categorical product** $X \otimes Y$ is isomorphic to $\mathcal{D}(X \times Y)$ where $X \times Y$ is the space of the product topology of X and Y.*

*For any two directed spaces X and Y, the **exponential object** Y^X is isomorphic to $\mathcal{D}([X \to Y]_p)$ where $[X \to Y]_p$ is the set of all continuous functions from X to Y with the pointwise topology.*

If we view **DCPO** as a subcategory of **DTop**, the embedding functor preserves the product and exponential object.

Cartesian closed full subcategories of DTop

Firstly, we recall a similar result about **DCPO**.

Theorem

(Jung 1988) Let $\mathcal{C}$ be a cartesian closed full subcategory of **DCPO**. *For any two objects D, E in $\mathcal{C}$. The categorical product of D and E is isomorphic to $D \times E$ in* **DCPO**.
The exponential object E^D is isomorphic to $[D \to E]$ in **DCPO**.

Can this result be extended to **DTop**?

Theorem

Let $\mathcal{C}$ be a cartesian closed full subcategory of **DTop**. *For any two objects D, E in $\mathcal{C}$. The categorical product of D and E is isomorphic to $D \otimes E$ in* **DTop**.

We can not generalize the case of exponential object to **DTop**.

A counterexample

The category (**Alex** for short) of Alexandroff spaces and continuous maps is a cartesian closed full subcategory of **DTop**. Let $\mathbb{N}$ equipped with the Scott topology under the usual order.

Theorem

The exponential object $\mathbb{N}^{\mathbb{N}}$ *in* **Alex** *and* **DTop** *are different.*

Proof.

Consider a set $\mathcal{U} = \{f \in [\mathbb{N} \to \mathbb{N}] : id \leq f\}$. $\mathcal{U}$ is an open set in $\mathbb{N}^{\mathbb{N}}$ in **Alex**, but it is not open in $\mathbb{N}^{\mathbb{N}}$ in **DTop**. □

The above example is due to Yuxu Chen.

c-spaces are introduced by Erné. Jung has shown that the category of continuous domains with Scott continuous maps is not cartesian closed by considering the function space. How about the category of c-spaces with continuous maps (**CS** for short)?

c-space

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The difference between these two categories is that the different behavior of the exponential object in its cartesian closed full subcategory. Hence we can not conclude that the category of c-spaces is not cartesian closed by just verifying that the space $\mathcal{D}([X \to Y]_p)$ is not a c-space for some c-spaces X and Y.

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Main result

Theorem

CS *is not cartesian closed.*

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CS *is not cartesian closed.*

We need a result about separate continuity and joint continuity.

Theorem

(Lawson 1985)Let E be a T_0 space. The following conditions are equivalent:

1. *E is locally finitary compact.*
2. *For all T_0 space X, if a map from $X \times E$ is separately continuous, then it is jointly continuous.*

Corollary

Let X be a c-space and Y a T_0 space. For any T_0 space Z, a map $f : X \times Y \to Z$ is continuous (i.e. jointly continuous) iff it is separately continuous.

A simple proof

Proof.

By contradiction. We consider the exponential object Y^Y in **CS** with a unique topology, where Y is the set of negative integers $(\mathbb{Z}^-)$ equipped with the Alexandroff topology under the usual order.
Given a continuous map $f: X \times \mathbb{Z}^- \to \mathbb{Z}^-$, we have a continuous map $\bar{f}$ from X to Y^Y. We can conclude that Y^Y is a d-space, hence a Scott space. Contradiction. □

Corollary

The category of locally finitary compact T_0 space with continuous maps is not cartesian closed.

Further direction

Classify the maximal cartesian closed full subcategories of **CS**.
Probably we should consider the L-domain like or FS-domain like directed spaces.

Related work

In order to model

- higher-order types (using cccs);
- computability (using ω-continuous domains);
- general computational effects such as nondeterminism (as free algebras for inequational theories).

However, Plotkin observed that it is not possible to combine all three features in traditional domain theory because none of the cartesian closed subcategories of ω-continuous dcpos are closed under the formation of free algebras. Hence there is a natural question: Find a category of domains that does support all three features in combination.

After some time, **qcb spaces** are introduced by Battenfeld, Schröder and Simpson (topological domain theory). The category of qcb spaces with continuous maps is cartesian closed. For studying cccs in Top instead of DCPO, we refer to the paper by Escardó, Lawson and Simpson.

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Property $\mathcal{M}$ and coherence for c-space

We extend the property $\mathcal{M}$ for domains to c-spaces.

Definition

Let X be a c-space. We say that X has **property** $\mathcal{M}$ if for any $x_1, y_1, x_2, y_2 \in X$ with $x_1 \in \mathrm{int}(\uparrow x_2), y_1 \in \mathrm{int}(\uparrow y_2)$, then there is a finite subset F of X such that

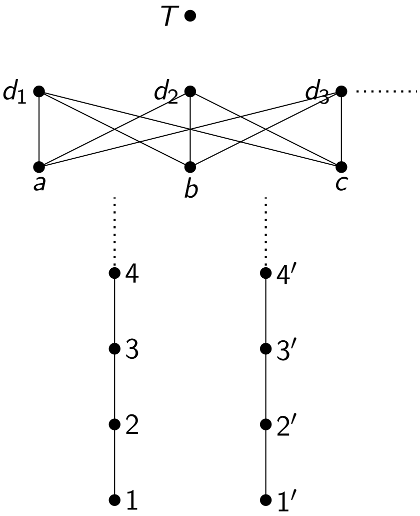
$$\uparrow x_1 \cap \uparrow y_1 \subseteq \uparrow F \subseteq \uparrow x_2 \cap \uparrow y_2.$$

Theorem

Let X be a c-space. If X is coherent, i.e. the intersection of every two compact saturated subsets is still compact, then X has **property** $\mathcal{M}$.

But unlike the case of domains, the converse direction is not true.

A counterexample



A counterexample

Let P be the poset. We consider a topology τ on P generated by $\uparrow n, \uparrow n', \uparrow d_i$ where $n \in \mathbb{N}$, $n' \in \mathbb{N}$, $i \in \mathbb{N}$. It is easy to verify that (P, τ) is a c-space.

Example

(P, τ) has property $\mathcal{M}$ but it is not coherent because $\uparrow a \cap \uparrow c$ is not compact.

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Thank You!