Chapter 1

Digraphs and Tournaments

1.1 Definition of directed graphs

A **directed graph** can be obtained from a graph by giving every edge a direction.

Example 1.1.1 *A directed graph is shown below.*

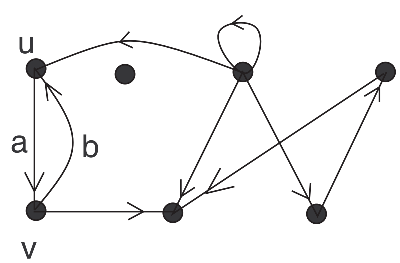


Figure 1.1

A directed graph is usually called a **digraph**.

A digraph consists of **vertices** and **arcs**.

An **arc** is an ordered pair of vertices. For example, in the digraph above,

- the arc a is denoted by (u, v), called "a joins u to v",
- the arc b is denoted by (v, u), called "b joins v to u".

Hence in a digraph, the arc (u, v) is different from (v, u).

Definition 1.1.1 *(1) A digraph $D = (V, A)$ consists of two finite sets V and D, where*

(1.1) V is the set of vertices in D, and

(1.2) A is the set of arcs in D, where each arc is an ordered pair of vertices.

(2) If $a = (u, v)$ is an arc of D, then

(2.1) a is said to join u to v,

(2.2) u is called the **initial vertex** *or the* **tail** *of a,*

(2.3) v is called the **terminal vertex** *or the* **head** *of a.*

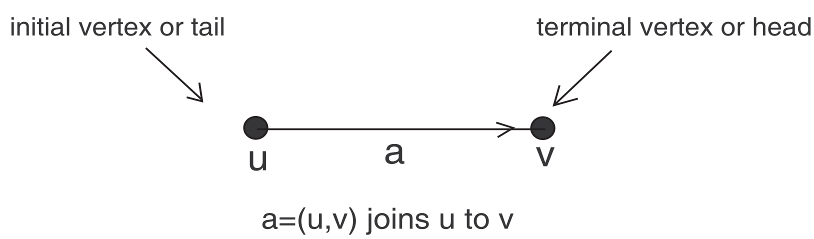


Figure 1.2

An arc is also called a **directed edge**.

The underlying graph of a digraph

Given a digraph D, we can obtain a graph G from D by "removing all the arrows" from the arcs. G is called the **underlying graph** of D.

Exercise 1.1.1 *Draw the underlying graph of the following digraph.*

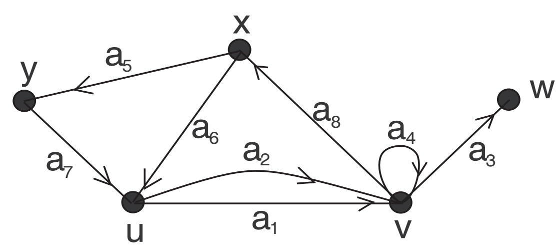


Figure 1.3

Orientations of a graph

Given a graph G, we can obtain a digraph from G by "assigning each edge in G a direction". Such a digraph D is called an **orientation** of G. Each non-empty graph has more than one orientation.

Exercise 1.1.2 *Find all orientations of* K_3.

1.2 Strongly Connected digraphs

Let D be a digraph. Then a **directed walk** in D is a finite sequence

$$W = v_0 a_1 v_1 a_2 v_2 \cdots a_k v_k,$$

where $a_1, a_2, \cdots, a_k$ are arcs and $v_0, v_1, v_2, \cdots, v_k$ are vertices of D such that v_{i-1} and v_i are respectively the tail and the head of a_i for all $i = 1, 2, \cdots, k$.

Exercise 1.2.1 *List three different directed walks from* v *to* u *in the following digraph.*

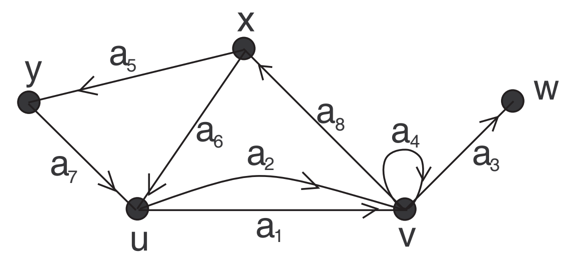


Figure 1.4

The **length** of a directed walk is the number of arcs in this walk.

A **directed trail** is a directed walk which does not repeat any arc.

A **directed path** is a directed walk which does not repeat any vertex except possibly the two ends.

A **directed cycle** is a directed path in which the two ends are the same vertex.

Exercise 1.2.2 *In the digraph of* Figure 1.4,

(1) find all directed trails from v to u;

(2) find all directed paths from v to u;

(3) find all directed cycles containing v.

Exercise 1.2.3 *Let D be a digraph and u, v be vertices in D. Show that if there is a directed walk from u to v, then there is a directed path from u to v.*

Unilaterally connected digraphs

A digraph D is said to be **weakly connected** if its underlying graph is connected.

For vertices u and v in a digraph D, v is said to be **reachable from** u if there is a directed path in D from u to v.

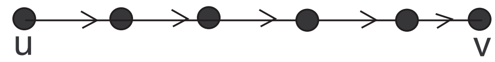


Figure 1.5

A digraph D is said to be **unilaterally connected** if for every pair of vertices u and v in D, either u is reachable from v or v is reachable from u.

Exercise 1.2.4 *Are the two digraphs in Figure 1.6 unilaterally connected?*

Exercise 1.2.5 *Draw a digraph of five vertices which is weakly connected but not unilaterally connected.*

Theorem 1.2.1 *Let D be a digraph. Then D is unilaterally connected if and only if D has a directed walk including all vertices in D.*

Exercise 1.2.6 *Prove Theorem 1.2.1.*

Strongly connected digraphs

A digraph D is said to be **strongly connected** if for every pair of vertices u and v in D, there exist a directed path from u to v and a directed path from v to u, i.e., each is reachable from the other.

Exercise 1.2.7 *Two digraphs are shown in* Figure 1.6. *Is any one of them strongly connected?*

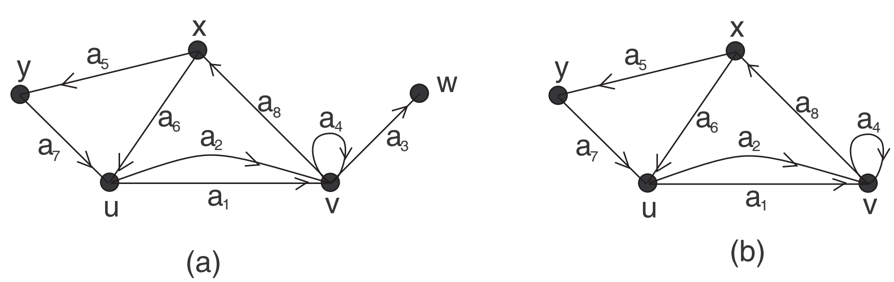


Figure 1.6

Exercise 1.2.8 *Draw a digraph of five vertices which is unilaterally connected but not strongly connected.*

Observations:

Let D be a digraph and u and v be two vertices in D.

(i) If u and v are on a directed closed walk of D, then v is reachable from u and u is reachable from v.

(ii) If v is reachable from u and u is reachable from v, then u and v are on a directed closed walk.

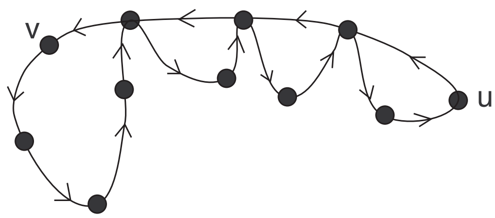


Figure 1.7

Theorem 1.2.2 *A digraph D is strongly connected if and only if it has a spanning closed walk, i.e., a closed walk passing through all vertices in D.*

A **strong component** S of a digraph D is a subdigraph of D which is strongly connected and is not a proper subdigraph of any other strongly connected subdigraph of D.

Observation: For any vertices u and v in D, if u is reachable from v and v is reachable from u, then u and v are in the same strong component.

Exercise 1.2.9 *Find all strong components of the following digraph.*

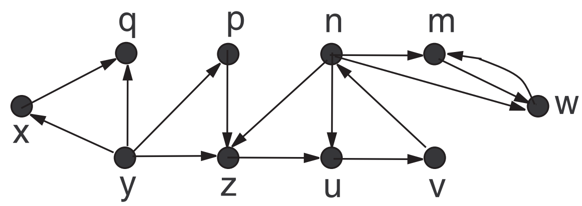


Figure 1.8

1.3 Matrix representations

Definition 1.3.1 *Let u and v be vertices in a digraph. If u and v are joined by an arc a, then u and v are said to be* **adjacent***. If the arc a is directed from u to v, then the arc a is said to be* **incident from** *u and* **incident to** *v.*

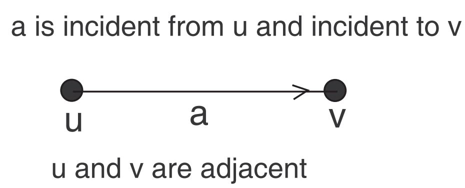


Figure 1.9

Definition 1.3.2 *Let D be a digraph and u be a vertex in D.*

(i) The **out-degree** *of u is the number of arcs incident from u, and is denoted by $od(u)$.*

(ii) The **in-degree** *of u is the number of arcs incident to u, and is denoted by $id(u)$.*

Example 1.3.1 *The out-degrees of the following digraph are shown below.*

$$od(y) = 3, \quad od(z) = 1, \quad od(u) = 1, \quad od(w) = 0, \quad od(p) = 1, \quad od(n) = 2.$$

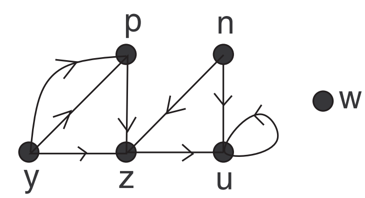


Figure 1.10

Exercise 1.3.1 *Find the in-degrees of the above digraph.*

Lemma 1.3.1 *In any digraph, the following three numbers are equal:*

(i) the sum of all the out-degrees;

(ii) the sum of all in-degrees; and

(iii) the number of arcs.

Proof. Since each arc has two ends, it must contribute exactly 1 to the sum of the out-degrees and exactly 1 to the sum of in-degrees. The result follows immediately. □

The Adjacency Matrix

Let D be a digraph with n vertices labeled $v_1, v_2, \cdots, v_n$. The **adjacency matrix**, denoted by $A(D)$, is the $n \times n$ matrix in which the entry in row i and column j is the number of arcs from vertex v_i to vertex v_j.

Example 1.3.2 *A digraph D and its adjacent matrix $A(D)$ are shown below.*

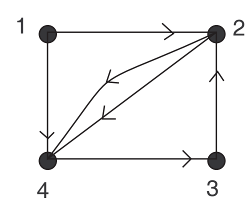


Figure 1.11

$$
\begin{array}{r}
\\ \\ \\
\text{row } 1 \rightarrow \\
\text{row } 2 \rightarrow \\
\text{row } 3 \rightarrow \\
\text{row } 4 \rightarrow
\end{array}
\begin{array}{c}
\begin{array}{cccc}
\text{col.} & \text{col.} & \text{col.} & \text{col.} \\
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow
\end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

Exercise 1.3.2 *Find the adjacency matrix $A(D)$ of the following digraph D.*

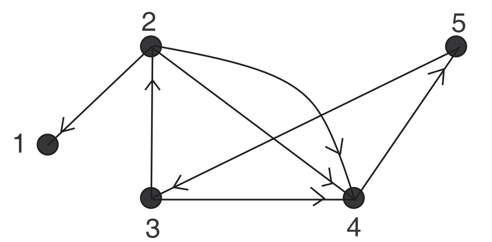


Figure 1.12

Exercise 1.3.3 *Find a digraph D with vertex set $\{1, 2, 3, 4\}$ such that its adjacency matrix is the following matrix.*

$$
\begin{pmatrix}
1 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1
\end{pmatrix}
$$

Theorem 1.3.1 *Let D be a digraph with vertex set $\{v_1, v_2, \cdots, v_n\}$, and $A(D)$ be the adjacency matrix of D. Assume that*

$$(A(D))^k = \left(a_{i,j}^{(k)}\right)_{n \times n}.$$

Then $a_{i,j}^{(k)}$ is the number of walks of length k from v_i to v_j. □

Exercise 1.3.4 *Prove Theorem 1.3.1 by induction.*

The incidence matrix

While the adjacency matrix of a digraph involves the adjacency of vertices, the incidence matrix involves the incidence of vertices and arcs.

Let D be a digraph without loops, with n vertices labeled as $v_1, v_2, \cdots, v_n$ and m arcs labeled as $a_1, a_2, \cdots, a_m$. The **incidence matrix** $I(D)$ is the $n \times m$ matrix in which the entry in row i and column j is

$$\begin{cases} 1, & \text{if arc } a_j \text{ is incident from } v_i; \\ -1, & \text{if arc } a_j \text{ is incident to } v_i; \\ 0, & \text{otherwise.} \end{cases}$$

Example 1.3.3 *The incidence matrix $I(D)$ of the digraph D in Figure 1.13 is shown below.*

	col. 1 ↓	col. 2 ↓	col. 3 ↓	col. 4 ↓	col. 5 ↓	col. 6 ↓
row 1 →	1	0	0	1	0	0
row 2 →	−1	−1	0	0	1	1
row 3 →	0	1	−1	0	0	0
row 4 →	0	0	1	−1	−1	−1

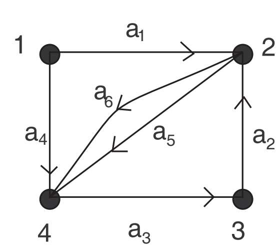


Figure 1.13

Exercise 1.3.5 *Find the incidence matrix $I(D)$ of the following digraph D.*

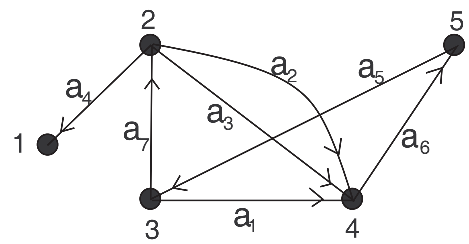


Figure 1.14

Exercise 1.3.6 *Find a digraph D such that its incidence matrix is the following matrix.*

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Exercise 1.3.7 *Let $F = (f_{i,j})$ be an $n \times m$ matrix. Find a characterization on the values of $f_{i,j}$'s such that F is the incidence matrix of some digraph D.*

1.4 Acyclic digraphs

A digraph D is called an **acyclic digraph** if D contains no directed cycles.

Theorem 1.4.1 *An acyclic digraph D has at least one vertex of out-degree zero.*

Proof. Let $P(x, y)$ be a longest directed path in D, where x and y are the first and the last vertex of this path respectively.

We shall show that $od(y) = 0$. Assume that the out-degree of y is not zero. Then D has an arc (y, z). Since D is acyclic, z should not be on the path $P(x, y)$. Thus the path $P(x, y)$ can be extended to z to get a longer path in D, a contradiction. Hence the out-degree of y is zero. □

By the same way, we can prove the following theorem.

Theorem 1.4.2 *An acyclic digraph D has at least one vertex of in-degree zero.*

Can you get some observation from the proof of Theorem 1.4.1?

We now give some characterizations for acyclic digraphs.

Theorem 1.4.3 *Let D be any digraph without loops. Then the following statements are equivalent:*

(i) D is acyclic;

(ii) every walk of D is a path;

(iii) there is an ordering of its vertices of D such that the adjacency matrix $A(D)$ with respect to this ordering is upper triangular.

Example 1.4.1 *The following digraph is acyclic. Find a suitable labeling of its vertices with $v_1, v_2, \cdots, v_5$ such that its adjacency matrix is upper triangular.*

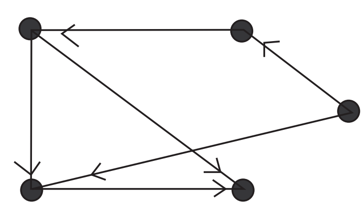


Figure 1.15

1.5 Tournaments

A **tournament** is an orientation of a complete graph. The following digraphs are tournaments with two vertices and three vertices.

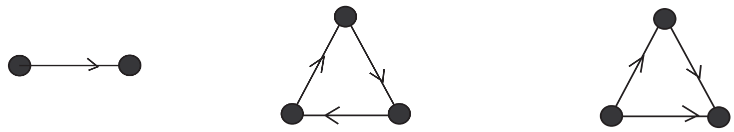


Figure 1.16

In some games, such as tennis and table tennis, each pair of players encounter each other and one beats the other. If the plays are represented by vertices, and for each pair of vertices, an arc is drawn from the winner to the loser, then a tournament is obtained.

Exercise 1.5.1 *How many different tournaments are there on n vertices?*

We present some results on tournaments.

Theorem 1.5.1 *Let D be a tournament and x be a vertex in D such that x has the maximum out-degree in D. Then for each $y \in V(D)$, there is a directed path from x to y of length at most 2.*

The proof of Theorem 1.5.1 is not difficult and left to the reader.

Exercise 1.5.2 *Prove Theorem 1.5.1.*

Theorem 1.5.2 (Rédei, 1934) *A tournament has a spanning directed path, i.e., a directed path passing through all vertices.*

Proof. Let D be a tournament of order n. Suppose that D has no spanning directed path. Let $P = x_1x_2\cdots x_k$ be a longest directed path in D, where $k < n$. Let u be a vertex of D such that u is not on P. If $(u, x_1) \in A(D)$, then $ux_1x_2\cdots x_k$ is a directed path longer than P, a contradiction. Hence $(x_1, u) \in A(D)$. Similarly, $(u, x_k) \in A(D)$, as shown in Figure 1.17.

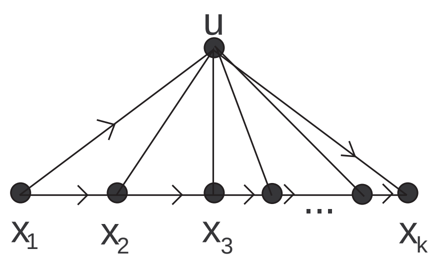


Figure 1.17

Let s be the maximum integer in $\{1, 2, \cdots, k\}$ such that $(x_s, u) \in A(D)$. Then $1 \leq s < k$ and $(u, x_{s+1}) \in A(D)$. Now we obtain a directed path in D:

$$x_1x_2\cdots x_sux_{s+1}\cdots x_k,$$

which is longer than the directed path P, a contradiction. □

Theorem 1.5.3 (Camion, 1959) *Every strongly connected tournament with n vertices has a directed cycle of length k for all $k = 3, 4, \cdots, n$.*

As a corollary, we have:

Corollary 1.5.1 *A tournament is strongly connected if and only if it has a spanning directed cycle.*

1.6 Orientations

Suppose the traffic in a section of a city has increased sufficiently that the city has decided to convert the two-way system in this area to a one-way system. After this conversion, one must, of course, be able to drive legally from any location in the section to any other. This leads to the following question:

Under what conditions can a traffic system have only one-way streets, yet allow one to travel from any location to any other location?

A connected graph G is called **strongly orientable** if it is possible to assign exactly one direction to each edge of G (thus changing the edges to arcs) to produce a strongly connected digraph D. The resulting digraph is called a **strong orientation**.

The traffic problem can now be rephrased in graphical terms: Let G be a graph that models the two-way street system. Let the vertices of G correspond to the street intersections. Two vertices of G are joined by an edge if it is possible to travel between the corresponding street intersections without passing through a third intersection.

A **bridge** is an edge of a connected graph G whose removal disconnects G. Note that an edge is a bridge if and only if it does not lie on a cycle.

First we have the following result.

Lemma 1.6.1 *Let G be a connected graph. If G contains a bridge, then G is not strongly orientable.*

Lemma 1.6.2 *Let D be a strongly connected digraph. Then adding any new directed path P to D with its two ends in D produces a new strongly connected digraph.*

The proof is left to the reader.

Lemma 1.6.3 *Let G be a connected graph which contains no bridges and G_1 be a subgraph of G. If G_1 has a strong orientation D_1, then D_1 can be extended to a strong orientation D of G.*

Proof. If $V(G_1) = V(G)$, then assigning any direction to each edge of G which is not in G_1 will produce a strong orientation D of G.

Now assume that $V(G_1)$ is proper subset of $V(G)$. Let u be a vertex in $V(G)\backslash V(G_1)$. We may choose such a vertex u so that it is adjacent to some vertex x in G_1. As G contains no bridges, the edge xu is contained in some cycle C of G. Thus the edge xu of G can be extended to a path $xuu_1 \cdots x_k y$ of G such that

$$\{x, u, u_1, \cdots, u_k, y\} \cap V(G_1) = \{x, y\}.$$

Now we assign directions to edges $xu, uu_1, \cdots, u_{k-1}u_k, u_k y$ of G to get arcs

$$(x, u), (u, u_1), \cdots, (u_{k-1}, u_k).$$

By Lemma 1.6.2, the digraph D' obtained from D_1 by adding new vertices $u, u_1, \cdots, u_k$ and the above new arcs is strongly connected.

Repeating the above process will get a strong orientation of G. □

Theorem 1.6.1 (Robbins (1939)) *A connected graph G is strongly orientable if and only if G contains no bridges.*

Proof. Suppose first that G is strongly orientable, and assume, to the contrary, that G contains a bridge $e = uv$. Let the digraph D be a strong orientation of G. Hence, D contains both a $u - v$ path and a $v - u$ path. We may assume that $uv \in A(D)$. Then uv is a $u - v$ path in D. We claim that D cannot contain a $v - u$ path. If $vv_1v_2 \cdots v_n u$ is a $v - u$ path in D, then $n \geq 1$ and $uv_n v_{n-1} \cdots v_1 v$ is a $u - v$ path in G that does not contain the edge uv. This contradicts our assumption that uv is a bridge.

Conversely, assume that G is a connected graph without bridges. We will produce a strong orientation D of G as follows. Since G contains no bridges, every edge of G lies on a cycle. Let $C_0 : v_1v_2\cdots v_nv_1$ be a cycle of G. We first assign directions to edges of C_0 such that $v_1v_2\cdots v_nv_1$ is a directed cycle.

Then, by Lemma 1.6.3, G is also strongly orientable. □

Exercise 1.6.1 *Let G be any connected graph. Write an algorithm to test whether G is strongly orientable. This algorithm should output a strong orientation if G is strongly orientable or a message that G is not strongly orientable otherwise.*

1.6.1 Acyclic orientations

For any graph $G = (V, E)$, an orientation D of G is called an *acyclic orientation* of G if D is acyclic, i.e., D contains no any cycles.

Let $\alpha(G)$ denote the number of acyclic orientations of G. By convention, let $\alpha(G) = 1$ if G is an empty graph. Observe that

(i) $\alpha(G) = 0$ if G has loops;

(ii) $\alpha(G) = 2$ if G is a graph with two vertices u, v and k parallel edges joining u and v, where $k \geq 1$;

(iii) $\alpha(K_3) = 6$;

(iv) $\alpha(C_k) = 2^k - 2$ for any $k \geq 2$.

Example 1.6.1 *Determine $\alpha(K_4 - e)$, where $K_4 - e$ is the graph obtained from K_4 by deleting one edge.*

Lemma 1.6.4 *Let G be any graph. If G is disconnected with components $G_1, G_2, \cdots, G_k$, or G is connected with blocks $G_1, G_2, \cdots, G_k$, then*

$$\alpha(G) = \alpha(G_1)\alpha(G_2)\cdots\alpha(G_k).$$

Lemma 1.6.5 *Let G be any graph and e be an edge in G.*

(i) *If e is a loop, then $\alpha(G) = 0$;*

(ii) *If e is a bridge, then $\alpha(G) = 2\alpha(G/e)$;*

(iii) *If e is not a loop nor a bridge, then*

$$\alpha(G) = \alpha(G - e) + \alpha(G/e).$$

Example 1.6.2 *Apply Lemmas 1.6.4 and 1.6.5 to find $\alpha(K_4)$ and $\alpha(K_{2,3})$.*

1.6.2 Totally cyclic orientations

For any graph $G = (V, E)$, an orientation D of G is called a *totally cyclic orientation* of G if every arc of D is contained in some cycles of D.

Let $\alpha^*(G)$ denote the number of totally cyclic orientations of G. By convention, let $\alpha^*(G) = 1$ if G is an empty graph. Observe that

(i) $\alpha^*(G) = 0$ if G has a bridge;

(ii) $\alpha^*(C_k) = 2$, where $k \geq 2$;

(iii) $\alpha^*(G) = 2^k - 2$ if G is a graph with two vertices and k parallel edges joining these two vertices, where $k \geq 1$.

Example 1.6.3 *Determine $\alpha^*(K_4 - e)$, where $K_4 - e$ is the graph obtained from K_4 by deleting one edge.*

Lemma 1.6.6 *Let G be any graph. If G is disconnected with components $G_1, G_2, \cdots, G_k$, or G is connected with blocks $G_1, G_2, \cdots, G_k$, then*

$$\alpha^*(G) = \alpha^*(G_1)\alpha^*(G_2)\cdots\alpha^*(G_k).$$

Lemma 1.6.7 *Let G be any graph and e be an edge in G.*

(i) *If e is a bridge, then $\alpha^*(G) = 0$;*

(ii) *If e is a loop, then $\alpha^*(G) = 2\alpha^*(G-e)$;*

(iii) *If e is not a loop nor a bridge, then*

$$\alpha^*(G) = \alpha^*(G-e) + \alpha^*(G/e).$$

Example 1.6.4 *Apply Lemmas 1.6.6 and 1.6.7 to find $\alpha^*(K_4)$ and $\alpha^*(K_{2,3})$.*

Remark: For any graph G, let $\tau(G)$ denote the number of spanning trees of G. Merino and Welsh made the following conjecture in the paper in the footnote[1] relating these three graphical parameters $\tau(G)$, $\alpha(G)$ and $\alpha^*(G)$: For any bridgeless and loopless graph G,

$$max\{\alpha(G), \alpha^*(G)\} \geq \tau(G).$$

Up to now, it has been proved for a few cases.

1.7 Acyclic ordering, strong decomposition, etc

1.7.1 Transitive

A tournament T is said to be **transitive** if, whenever uv and vw are arcs of T, then uw is also an arc of T.

[1] C. Merino, D.J.A. Welsh, Forests, colorings and acyclic orientations of the square lattice, *Ann. Comb.* **3** (24) (1999) 417-429. http://dx.doi.org/10.1007/BF01608795. On combinatorics and statistical mechanics.

Exercise 1.7.1 *Can you draw a tournament T of order 4 such that T is transitive?*

For vertices u and v in T, $u \to v$ means that u is adjacent to v.

Theorem 1.7.1 *A tournament is transitive if and only if it is acyclic.*

Proof. Let T be an acyclic tournament. Suppose uv and vw are arcs of T. To avoid the cycle $uvwu$, wv should not be an arc of T and thus uw is also an arc of T.

For the converse, suppose that T is a transitive tournament. Now assume to the contrary that T is not acyclic. Let $C : v_1v_2 \cdots v_nv_1$ be a cycle of minimum length n in T. Clearly, $n > 3$ since T is transitive. However, $v_1 \to v_2 \to v_3$ implies that $v_1 \to v_3$ since T is transitive. Thus, $v_1v_3 \cdots v_nv_1$ is a cycle of length $n-1$ in T which is a contradiction. □

Theorem 1.7.2 *A nondecreasing sequence s of p (≥ 1) nonnegative integers is a sequence of out-degrees of a transitive tournament if and only if s is the sequence $(0, 1, \cdots, p-1)$.*

Proof. We show first that $s = (0, 1, \cdots, p-1)$ is a sequence of outdegrees of a transitive tournament. Let T be a tournament defined by

$$V(T) = \{v_1, v_2, \cdots, v_p\}$$

and

$$A(T) = \{(v_i, v_j) | 1 \leq j < i \leq p\}.$$

Then $d^+(v_i) = i - 1$ for $i = 1, 2, \cdots, p$; so s is a sequence of outdegrees of a tournament. Suppose $v_i \to v_j \to v_k$. Then $i > j > k$ and so $v_i \to v_k$. Thus, T is transitive.

Conversely, assume that T is a transitive tournament. By Theorem 1.7.1, T is a acyclic, and so there exists a Hamiltonian path $v_1v_2 \cdots v_p$ in T. Since T is transitive and $v_1 \to v_2 \to \cdots \to v_p$, we have $v_i \to v_j$ for all i, j with $1 \le i < j \le p$. Thus $d^+(v_i) = p - i$ for $i = 1, 2, \cdots, p$ and so $s = (0, 1, \cdots, p-1)$ is the out-degrees sequence of T, in non-decreasing order. □

Corollary 1.7.1 *A nondecreasing sequence s of p (≥ 1) nonnegative integers is a sequence of indegrees of a transitive tournament if and only if s is the sequence $(0, 1, \cdots, p-1)$.*

Corollary 1.7.2 *For every positive integer p, there is exactly one transitive tournament of order p.*

1.7.2 Acyclic ordering

Let D be a digraph. Recall that D is **acyclic** if it has no directed cycle.

Let $x_1, x_2, \cdots, x_n$ be an ordering of its vertices. We call this ordering an **acyclic ordering** if, for every arc (x_i, x_j) in D, we always have $i < j$. This definition is equivalent to that the adjacency matrix of D with the ordering $x_1, x_2, \cdots, x_n$ is upper-triangular.

Clearly, if D has an acyclic ordering, then D is an acyclic digraph.

On the other hand, the following holds:

Lemma 1.7.1 *Every acyclic digraph has an acyclic ordering of its vertices.*

Proof. We give a constructive proof by describing a procedure that generates an acyclic ordering of the vertices in an acyclic digraph D.

Recall that any acyclic digraph has a vertex of in-degree zero. At the first step, we choose a vertex v with in-degree zero. Set $x_1 = v$ and delete x_1 from D to get another acyclic digraph. At the i-th step, we find a vertex u of in-degree zero in the remaining acyclic digraph, set $x_i = u$ and delete x_i from the remaining acyclic digraph. Now we get an ordering $x_1, x_2, \cdots, x_n$, where n is the number of vertices in D by the above process.

Suppose that $x_i \to x_j$ in D, but $i \geq j$. As x_j was chosen before x_i, it means that x_j was not of in-degree zero at the j-th step of the procedure, a contradiction. □

For a digraph D and two vertices u and v, a $u - v$ *path* in D is a directed path in D with the initial vertex u and the terminal vertex v.

Lemma 1.7.2 *Let D be an acyclic digraph. Assume that x is the only vertex in D with $id(x) = 0$. Then, for every vertex $v \in V(D)$, there exists an $x - v$ path.*

Proof. Let $P : v_1 v_2 \cdots v_k$ be a longest path in D terminating at v, i.e., $v_k = v$. Since D is acyclic, there are no $i : 1 \leq i \leq k$ such that $v_i \to v_1$. Since P is a longest path in D terminating at v, there are no vertex $w \in V(D) \setminus \{v_1, v_2, \cdots, v_k\}$ such that $w \to v_1$. Thus the in-degree of v_1 is 0, i.e., $id(v_1) = 0$. Since x is the only vertex in D such that $id(x) = 0$, we have $v_1 = x$. Hence P is a $x - v$ path in D and the result is proved. □

Similarly, we have the following result.

Lemma 1.7.3 *Let D be an acyclic digraph with precisely one vertex y of out-degree zero in D. For every vertex $v \in V(D)$ there is a $v - y$ path.*

Question 1.7.1 *Prove Lemma 1.7.3.*

1.7.3 Strong decomposition

Recall that a **strong component** of a digraph D is a maximal induced subdigraph of D which is strongly connected.

If $D_1, \cdots, D_t$ are the strong components of D, then

$$V(D_1) \cup \cdots \cup V(D_t) = V(D)$$

and for every i, j with $i \neq j$,

$$V(D_i) \cap V(D_j) = \emptyset.$$

We call $V(D_1) \cup \cdots \cup V(D_t)$ the **strong decomposition** of D.

The **strong component digraph** $SC(D)$ of D is obtained by contracting strong components of D and merging all parallel arcs obtained in this process into one. In other words, if $D_1, \cdots, D_t$ are the strong components of D, then the vertex set of $SC(D)$ is

$$V(SC(D)) = \{d_1, \cdots, d_t\}.$$

and (d_i, d_j) is an arc in $SC(D)$ if and only if there exists an arc in D from some vertex in D_i to some vertex in D_j.

Exercise 1.7.2 *Let D be the following digraph. Find all strong components of D and draw the digraph $SC(D)$. Is it acyclic? Why?*

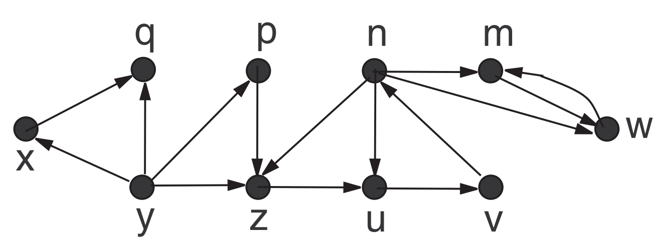


Figure 1.18

Lemma 1.7.4 *$SC(D)$ is acyclic for any digraph D.*

Proof. Let $D_1, D_2, \cdots, D_t$ be the strong components of D. Let $\{d_1, d_2, \cdots, d_t\}$ be the vertex set of $SC(D)$, where d_i corresponds to D_i.

Suppose $SC(D)$ is not acyclic. Then $SC(D)$ contains a directed cycle, say $d_1 \to d_2 \to \cdots \to d_k \to d_1$, where $2 \leq k \leq t$.

Note that $d_i \to d_{i+1}$ implies that there exists an arc e_i in D which adjacent from some vertex in D_i to some vertex in D_{i+1}.

Thus D has a cycle which passes through vertices in $D_1, D_2, \cdots, D_k$, implying that the subdigraph induced by $V(D_1) \cup V(D_2) \cup \cdots \cup V(D_k)$ is strongly connected. It further implies that each D_i, $1 \leq i \leq k$, is not a maximal strongly connected subdigraph of D, a contradiction. □

By Lemma 1.7.1, the vertices of SC(D) have an acyclic ordering. This implies that the strong components of D can be labelled $D_1, \cdots, D_t$ such that whenever $j > i$, D has no arcs from D_j to D_i. We call such an ordering an **acyclic ordering of the strong components** of D.

Strong components of D are divided into three groups:

(i) A strong component D_i of D is called an **initial strong components** of D if D_i corresponds to a vertex d_i in $SC(D)$ of in-degree zero.

(ii) Similarly, a strong component D_i of D is called a **terminal strong components** of D if D_i corresponds to a vertex d_i in $SC(D)$ of out-degree zero.

(iii) A strong component of D which is not initial nor terminal is called an **intermediate strong component** of D.

Exercise 1.7.3 *Let D be the digraph in* Figure 1.18. *Determine the initial and terminal strong components of D.*

Recall that a digraph is called unilaterally connected if for every pair of vertices x, y in D, there must be either a $x - y$ path or a $y - x$ path in D. A unilaterally connected digraph is also called a *unilateral digraph*.

Lemma 1.7.5 *If D is a unilateral digraph, then for every acyclic ordering of the strong components $D_1, \cdots, D_t$ of D, there exists an arc from some vertex in D_i to some vertex in D_{i+1} for all $i = 1, 2, \cdots, t-1$.*

Proof. Let $D_1, \cdots, D_t$ be any acyclic ordering of the strong components of D.

Let i be any integer with $1 \leq i < t$ and $x \in V(D_i)$ and $y \in V(D_{i+1})$.

By the definition of acyclic ordering of the strong components, x is not reachable from y.

By the definition of unilateral digraphs, y must be reachable from x. Let $x_0 \to x_1 \to \cdots \to x_k$, where $x_0 = x$ and $x_k = y$, be a path from x to y. For every arc $x_r \to x_{r+1}$ in this path, whenever $x_r \in V(D_l)$, then $x_{r+1} \in V(D_f)$ for some $f \geq l$. Thus, if there exists $j : 1 \leq j \leq k$ such that $x_j \in V(D_l)$ for some $l > i+1$, then **no vertex** in $\{x_j, x_{j+1}, \cdots, x_k\}$ is contained in $V(D_i)$, a contradiction. Thus all vertices of $\{x_1, x_2, \cdots, x_k\}$ is contained in $V(D_i)$ or $V(D_{i+1})$. Since $x_0 \in V(D_i)$ and $x_k \in V(D_{i+1})$, there exists minimum j with $1 \leq j \leq k$ such that $x_j \in V(D_{i+1})$. Hence $x_{j-1} \to x_j$ and the result thus holds. □

The following is a characterization of unilateral digraphs.

Theorem 1.7.3 *A digraph D is unilateral if and only if for any acyclic ordering of the strong components $D_1, \cdots, D_t$ of D, there exists an arc from some vertex in D_i to some vertex in D_{i+1} for all $i = 1, 2, \cdots, t-1$.*

Proof. The sufficiency is trivial. The necessity follows from Lemma 1.7.5 □

1.7.4 In-branching and out-branching

A subdigraph T of a digraph D is called a **spanning oriented tree** of D if the underlying graph of T is a spanning tree of the underlying graph of D.

For a spanning oriented tree T of D, T is called an **in-branching** of D if T has only one vertex of out-degree zero. The vertex is called a **root** of T.

Similarly, for a spanning oriented tree T of D, T is called an **out-branching** of D if T has only one vertex of in-degree zero. The vertex is also called a **root** of T.

Since each spanning directed tree R of a connected digraph is acyclic, R has at least one vertex of out-degree zero and at least one vertex of in-degree zero. Hence, the out-branchings and in-branchings capture the important cases of uniqueness of the corresponding vertices. The following is a characterisation of digraphs with in-branchings (out-branchings).

Lemma 1.7.6 *For a digraph D and a vertex w in D, if every vertex in D is reachable from w, then D has an out-branching T with root w.*

Proof. We construct a sequence of oriented trees $T_1, T_2, \cdots, T_n$ as follows, where $n = |V(D)|$, such that $w \in V(T_i)$ all $i = 1, 2, \cdots, n$ and w is the only vertex of in-degree zero in T_i for all $i = 2, 3, \cdots, n$.

Let T_1 consist of vertex w only. Assume that T_i has be obtained, where $i < n$. Since $w \in V(T_i)$ and every vertex of D is reachable from w, there exists $u \in V(T_i)$ and $v \in V(D) \setminus V(T_i)$ such that $u \to v$. Let T_{i+1} be obtained from T_i by adding the vertex v and the arc uv.

Note that T_n is an out-branching of D with root w. □

Similarly, we have:

Lemma 1.7.7 *For a digraph D and a vertex w in D, if w is reachable from every vertex in D, then D has an in-branching T with root w.* □

Lemma 1.7.8 *A connected digraph D contains an out-branching if and only if D has only one initial strong component.*

Proof. We need only prove the lemma for out-branchings since the in-branching case follows by considering the converse of D.

(Necessity). Assume that D contains an out-branching T and x is the root of T. Suppose that D contains at least two initial strong components, D_1 and D_2. Then either $x \notin V(D_1)$ or $x \notin V(D_2)$. Assume that $x \notin V(D_1)$. Since T is an out-branching of D, and x is the root of T, every vertex of D_1 is reachable from x, implying that D has an arc uv, where $u \in V(D) \setminus V(D_1)$ and $v \in V(D_1)$, contradicting the assumption that D_1 is an initial strong component. Hence the necessity holds.

(Sufficiency) Now we suppose that D contains only one initial strong component D_1 and w is an arbitrary vertex of D_1. We prove that D has an out-branching with root w.

In $SC(D)$, the vertex x corresponding to D_1 is the only vertex of in-degree zero and, hence, by Lemma 1.7.2, every vertex of $SC(D)$ is reachable from x. Thus, every vertex of D is reachable from w. By Lemma 1.7.6, D has an out-branching with root w. □

Similarly, we have:

Lemma 1.7.9 *A connected digraph contains an in-branching of and only if D has only one terminal strong component.*

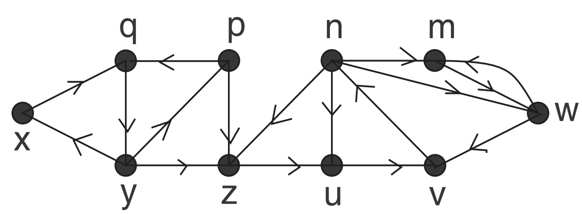
Questions for Chapter 1

(1.1) Let G be a graph with m edges. What is the total number of different orientations of G?

(1.2) If G is the complete graph with n vertices, then what is the total number of different orientations of G?

(1.3) Let D be a digraph and u and v be two vertices in D. Show that if there is a directed walk from u to v, then there is a direct path from u to v.

(1.4) Find all strong components of the following digraph.

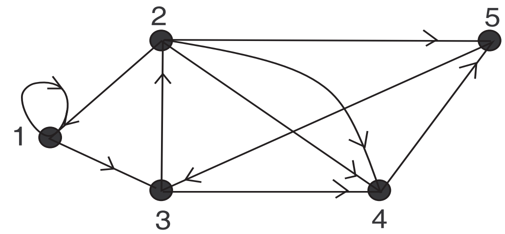


(1.5) Prove that a digraph D is strongly connected if and only if it has a closed directed walk going through all its vertices.

(1.6) Describe a method for finding all strong components of a digraph D.

(1.7) Let D be a digraph and $k = \min\{od(v) : v \in V(D)\}$ (i.e., k is the minimum out-degree of D). Assume that D has no multiple arcs, i.e., every two arcs are different ordered pairs of vertices. Show that D contains a directed path of length at least k. Is the result also true if $k = \min\{id(v) : v \in V(D)\}$?

(1.8) Find the adjacency matrix $A(D)$ of the following digraph D.

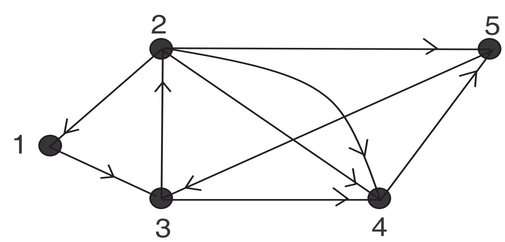


(1.9) Find a digraph D such that its adjacency matrix is the following matrix.

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

(1.10) Let D be a digraph with vertex set $\{1, 2, 3, \cdots, n\}$ and $A = (a_{i,j})$ be its adjacency matrix. Find a characterization on entries of A such that D is unilaterally connected.

(1.11) Find the incidence matrix $I(D)$ of the following digraph D.



(1.12) Find a digraph D such that its incidence matrix is the following matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

(1.13) Let D be a digraph with vertex set $\{v_1, v_2, \cdots, v_n\}$ and arc set $\{a_1, a_2, \cdots, a_m\}$. Let I be the incidence matrix of D with respect to this labeling of its vertice and arcs, i.e., I is an $n \times m$ matrix in which the entry at (i, j) is

$$\begin{cases} 1, & \text{if arc } a_j \text{ is incident from } v_i; \\ -1, & \text{if arc } a_j \text{ is incident to } v_i; \\ 0, & \text{otherwise.} \end{cases}$$

Let α_i be the i'th column vector of I for each i with $1 \leq i \leq m$. Show that D has a directed cycle consisting of arcs $a_1, a_2, \cdots, a_k$ if and only if $\alpha_1 + \alpha_2 + \cdots + \alpha_k$ is a zero vector but any proper subset of $\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$ is linearly independent.

(1.14) Let D be a digraph. Show that D is acyclic if and only if there exists an ordering of vertices in D such that the adjacency matrix of D is upper triangular.

(1.15) Show that every graph without loops always has an orientation D such that D is acyclic.

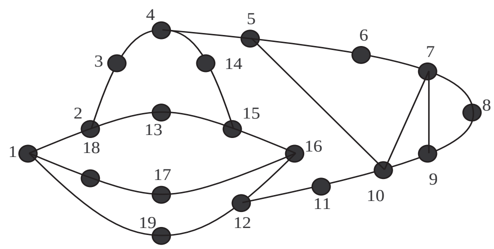
(1.16) Let n be any positive integer. Show that there exists a simple digraph D with n vertices $x_1, x_2, \cdots, x_n$ such that $od(x_k) = k-1$ and $id(v_k) = n-k$ for all $k = 1, 2, \cdots, n$.

(1.17) Show that there are exactly $n!$ tournaments on n vertices which are acyclic.

(1.18) Let s be a positive integer and D be a simple digraph such that $id(x) \geq s$ for every vertex x in D. Show that D has a directed cycle of length at least $s + 1$.

(1.19) A digraph D is called *irreducible* if it is impossible to split the vertices into two sets V_1 and V_2 such that whenever an arc in D has two ends in V_1 and V_2 respectively, then this arc is incident from a vertex in V_1 and incident to a vertex in V_2. Show that a digraph is irreducible if and only if it is strongly connected.

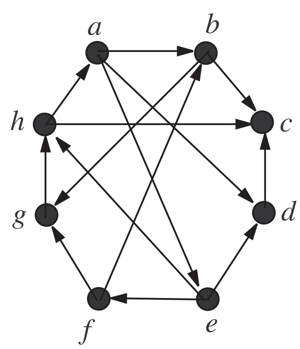
(1.20) Let G be the following graph. Clearly show the steps to find a strong orientation D on G.



(1.21) Find a strong orientation D on the Petersen graph.

(1.22) (*) Show that any connected graph without bridges can be obtained from a cycle by a series of the operations: *adding a new path, where all internal vertices are also new while the two ends are in the previous graph.*

(1.23) Let D be the following digraph.



(i) Find all strong components of D.

(ii) Draw the digraph $SC(D)$.

(iii) Find an out-branching with root a in D.

(iv) Find an in-branching with root c in D.

(1.24) Prove that a directed digraph D is strongly connected if and only if it has an out-branching rooted at v for every vertex v in D.

(1.25) Let D be a digraph. A **source** in D is a vertex in D which can reach all other vertices in D, and a **sink** in D is a vertex in D to which all other vertices in D can reach.

(i) Find all sources in the digraph of Question (1.23).

(ii) Find all sinks in the digraph of Question (1.23).

(1.26) Call a digraph D **resourceful** if every two vertices in D can be reached from a common vertex. Then D is resourceful if and only if it has a source.