Mathematical Research – Problem Posing, Problem Solving and Making Connections

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Outline of Presentation

- Problem Posing and Problem Solving
- Illustration with Example on Principal Elements and Principal Mappings
- Making Connections
- How to Enhance One's Ability to Make the Connections
- Concept of "Point-free" in Mathematics
- Conclusion
- Selected References

Some Important Aspects of Mathematical Research

- Problem Posing
- Problem Solving
- Making Connections

Problem Posing

- New problems can be posed when reading journal articles or attempting to generalise results.
- The statement of the problem must be phrased in such a way that it does not restrict the scope of the problem, but yet precise enough so as to avoid ambiguity.

Problem Solving

- Polya's problem solving framework is very useful and relevant.
- Some of the heuristics such as observing patterns and working with simpler cases could be used when testing if the conjecture to the problem posed is true.
- The importance of using certain tools and techniques when trying to prove certain results.

We defined principal mappings between lattices in our paper **"A generalization of Dilworth's Principal Elements"** in 2012.

From
$$(J \cap [K : I])I = JI \cap K,$$
$$[(K + JI) : I] = [K : I] + J,$$
which are satisfied by a principal ideal I of a commutative ring R **(hence a principal element of the lattice of all ideals of R)** and
$$F_I(J \cap G_I(K)) = F_I(J) \cap K,$$
$$G_I(K + F_I(J)) = G_I(K) + J,$$
where F_I is a mapping from the lattice of all ideals of R to itself given by $F_I(J) = IJ$,

we define a mapping $f : L \to M$ between two lattices to be a **principal mapping** if there exists a mapping $g : M \to L$ such that for all $x \in L, y \in M$,
$$f(x \wedge g(y)) = f(x) \wedge y, \quad g(y \vee f(x)) = g(y) \vee x.$$

From $f(x \wedge g(y)) = f(x) \wedge y$, $g(y \vee f(x)) = g(y) \vee x$,

we later generalized the concept of principal mappings between lattices to principal mappings between posets in our paper "**Principal Mappings between Posets**" in 2014.

A mapping $f : P \to Q$ between two posets P and Q is called a **principal mapping** if there is a mapping $g : Q \to P$ such that the following equations hold for all $x \in P$, $y \in Q$:
$$f(\downarrow x \cap \downarrow g(y)) = \downarrow f(x) \cap \downarrow y,$$
$$g(\uparrow y \cap \uparrow f(x)) = \uparrow g(y) \cap \uparrow x.$$

According to Anderson D.D. and Johnson E.W., an element a of a multiplicative lattice L is called a **weak principal element** of L if for all $x \in L$,
$$[x : a]a = a \wedge x \quad \text{and}$$
$$[xa : a] = [0_L : a] \vee x.$$

We generalized the concept of weak principal elements to weak principal mappings in 2014.

A mapping $f : P \to Q$ between two posets P (with top element 1_P) and Q (with bottom element 0_Q), is called a **weak principal mapping** if there is a mapping $g : Q \to P$ such that for all $x \in P$, $y \in Q$
$$f(g(y)) = f(1_P) \wedge y \quad \text{and} \quad g(f(x)) = g(0_Q) \vee x.$$

THEOREM (ANDERSON, D.D.) :

An element in a modular multiplicative lattice L is principal if and only if it is weak principal.

Theorem (Nai, Y.T. and Zhao, D.) :

Let $f : L \to M$ be a mapping between bounded modular lattices with an upper adjoint. Then f is a principal mapping if and only if f is a weak principal mapping.

Anderson D.D. raised the following question :

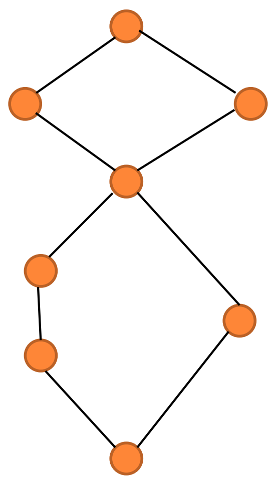
If every weak principal element of a bounded multiplicative lattice L is principal, must L be modular ?

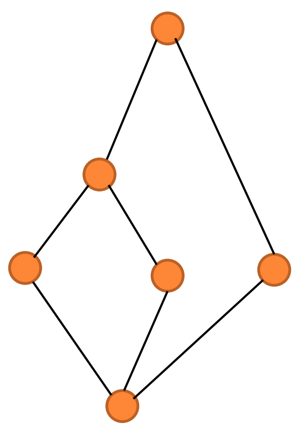
We posed the following problem :

If every weak principal mapping from a bounded lattice L to itself is principal, must L be modular ?

The answer to the second problem is negative.

Counter-Examples :





We posed another related problem :

Let L be a bounded lattice such that for any bounded lattice M, every weak principal mapping $f : L \to M$ is meet principal. Must L be modular ?

Making Connections

- There are interplay between the various areas of Mathematics; this is especially so for the areas of topology, algebra and order theory.
- It is necessary to make connections between the area of research with other areas of Mathematics so as to make the results more meaningful.

Making Connections between Multiplicative Lattice Theory (Order Theory) and Semi-ring Theory (Algebra)

We posed another problem :

Is it true that for any semiring S, if every weak principal mapping f : Idl(S) → Idl(S) is principal, then the lattice Idl(S) of all ideals of S is modular ?

How to enhance one's ability to make the connections

Extracted from the "Introduction : Stone's Theorem in Historical Perspective" by Peter T. Johnstone

"A second, related, point concerns the danger of adopting a narrowly specialist approach to Mathematics."

"Theorems and techniques which are commonplace in one field are laboriously and imperfectly rediscovered in adjacent ones."

"In contrast, Stone stands as an example of a man who, although his interests may lie in one particular area of Mathematics, has nonetheless a sufficient general perspective on the whole subject to recognize the significance of his work for other fields."

Concept of "Point-Free" in Mathematics

- Some definitions and results in ring theory are written in terms of ideals of a ring, rather than elements of a ring.

Example : Prime Ideal

An ideal $I \neq R$ is prime if whenever $xy \in I$, then $x \in I$ or $y \in I$.

An ideal $I \neq R$ is prime if for any ideals J, K of R such that $JK \subseteq I$, then $J \subseteq I$ or $K \subseteq I$.

- As such, we are able to derive results relating to multiplicative lattices which are the natural abstraction of the lattice of all ideals of a commutative ring R.

Example : Prime Element of a Multiplicative Lattice L

An element $a \neq 1_L$ of a multiplicative lattice L is prime if whenever $xy \leq a$, then $x \leq a$ or $y \leq a$.

Conclusion

- The journey of mathematical research is both challenging as well as fulfilling.
- To do our part in contributing to the knowledge in Mathematics.

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THANK YOU!