Some recent works on Baire class one functions and problem posing strategies involved

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Continuous functions

$f: \mathbb{R} \longrightarrow \mathbb{R}$ is cont. iff

$\forall \varepsilon > 0, \exists \delta : \mathbb{R} \longrightarrow \mathbb{R}^+$ such that $\forall x, y \in \mathbb{R}$,

$$|x-y| < \max\{\delta(x), \delta(y)\} \Longrightarrow |f(x) - f(y)| < \varepsilon. \quad (1)$$

Question: what functions are the f if in (1), $\min\{\delta(x), \delta(y)\}$ is repalaced by $\min\{\delta(x), \delta(y)\}$?

H-continuous functions

$f: \mathbb{R} \longrightarrow \mathbb{R}$ is H-cont. if

$\forall \varepsilon > 0, \exists \delta : \mathbb{R} \longrightarrow \mathbb{R}^+$ s.t. $\forall x, y \in \mathbb{R}$,

$$|x-y| < \min\{\delta(x), \delta(y)\} \Longrightarrow |f(x) - f(y)| < \varepsilon \quad (2)$$

Question: what are the properties of H-cont. functions.

- If f, g are H-cont, then so are $f+g$, fg
- If $\{f_n\}$ is a sequence of cont. functions and $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x \in \mathbb{R}$, then f is H-continuous.

Thus every Baire class one function is H-cont.

Question: Is the converse of above also true?

Theorem. (Lee, Tang, Zhao 2007)

Let $f: X \longrightarrow Y$ be a mapping between complete metric spaces. Then TFAE:

(1) $\forall \varepsilon > 0$, $\exists \delta : X \longrightarrow \mathbb{R}^+$,

$d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \min\{\delta(x), \delta(y)\}$.

(2) f is of the first Baire class.

Question: what if $\min\{\delta(x), \delta(y)\}$ is replaced by other two varable functions?

Jachymski & Lindner (2005)

For any $M : (0,\infty)\times(0,\infty) \to (0,\infty)$, let

$\mathcal{A}_M(X,Y)$: $f : X \to Y$,

$\forall \varepsilon > 0$, $\exists\ \delta : X \to (0,\infty)$ s.t. $\forall x_1, x_2 \in X$

$$d_X(x_1,x_2) < M(\delta(x_1),\delta(x_2)) \Longrightarrow d_Y(f(x_1), f(x_2)) < \varepsilon.$$

Proposition. $\forall M : (0,\infty)\times(0,\infty) \to (0,\infty)$, $\exists$ symmetric $\hat{M} : (0,\infty)\times(0,\infty) \to (0,\infty)$ s.t.

$$\mathcal{A}_M(X,Y) = \mathcal{A}_{\hat{M}}(X,Y).$$

Theorem. Let $M : (0,\infty)\times(0,\infty) \to (0,\infty)$ be a symmetric and non-decreasing. Then

$\mathcal{A}_M(\mathbb{R},\mathbb{R}) = \mathcal{B}_1(\mathbb{R},\mathbb{R})$

or $= \mathcal{C}(\mathbb{R},\mathbb{R})$.

E.g. if $M(u,v) = \frac{1}{2}(u,v)$, $\mathcal{A}_M(\mathbb{R},\mathbb{R}) = \mathcal{C}(\mathbb{R},\mathbb{R})$

$= \sqrt{uv}$, $\mathcal{A}_M(\mathbb{R},\mathbb{R}) = \mathcal{B}_1(\mathbb{R},\mathbb{R})$.

Question what if the $\varepsilon > 0$ in (1) is replaced by a function $\varepsilon : \mathbb{R} \to \mathbb{R}^+$?

K-Continuous functions

$f : \mathbb{R} \to \mathbb{R}$ is K-continuous if $\forall \varepsilon : \mathbb{R} \to \mathbb{R}^+$, $\exists\ \delta : \mathbb{R} \to \mathbb{R}^+$ such that $\forall x_1, x_2 \in \mathbb{R}$,

$$|x_1 - x_2| < \min\{\delta(x_1), \delta(x_2)\} \Longrightarrow |f(x_1) - f(x_2)| < \min\{\varepsilon(f(x_1)), \varepsilon(f(x_2))\}$$

Properties (Zhao D 2007)

(1) If $f, g : \mathbb{R} \to \mathbb{R}$ are K-Cont, then so is $g\circ f$.

(2) If f, g are inj. K-Cont, then $f+g$, fg are K-Cont.

(3) If $\forall$ B1 $h : \mathbb{R} \to \mathbb{R}$, $h\circ f$ is B1.

A function f satisfies Condition (3) is called a right B1 Compositor (Zhao D 2007)

Question: Is every right B1 Compositor K-continuous?

Theorem (Fenecios and Emmanuel A. Cabral 2012)

(1) A function $f : \mathbb{R} \to \mathbb{R}$ is a right B_1 compositor iff it is K-continuous.

(2) The sum/product of two K-cont. functions is K-continuous.

(3) Every B1 function is the uniform limit of a sequence of K-cont. functions.

Question: what about left B1 compositors?

Theorem (Fenecios and Cabral 2013)

A function $f : \mathbb{R} \to \mathbb{R}$ is a left B1 compositor iff it is continuous.