An Identity via Arbitrary Polynomials

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Paper

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Started from two conjectures

The following conjectures were posed by Thomas Dence in a paper in CMJ in 2007.

Conjecture

Let n and k be odd positive integers with $k \leq n$. Then

$$\sum_{j=0}^{(n-1)/2} \binom{n}{j} (-1)^j (n-2j)^k = \begin{cases} 0, & \text{if } k < n; \\ 2^{n-1} n!, & \text{if } k = n. \end{cases}$$

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Conjecture

Let n and k be even positive integers with $k \leq n$. Then

$$\sum_{j=0}^{(n-2)/2} \binom{n}{j} (-1)^j (n-2j)^k = \begin{cases} 0, & \text{if } k < n; \\ 2^{n-1} n!, & \text{if } k = n. \end{cases}$$

The conjectures were proven in 2009

Hidefumi Katsuura proved these conjecture by establishing the following result:

Theorem

For complex numbers x and y and any positive integer n,

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j (xj+y)^k = \begin{cases} 0, & \text{if } 0 \le k < n \\ (-x)^n \times n!, & \text{if } k = n. \end{cases}$$

Our Contribution

- We observe that the previous theorem is actually a special case of a much more general result.
- We will prove this general result and the Theorem above and the conjectures above follow directly.

Combinatorial Number $\binom{n}{k}$

- $\binom{n}{k}$ is the number of k-element subsets of a set S with $|S| = n$.
- Special values:

$$\binom{n}{k} = \begin{cases} 1, & \text{if } k = 0 \text{ or } n; \\ n, & \text{if } k = 1 \text{ or } n-1; \\ 0, & \text{if } k < 0 \text{ or } k > n. \end{cases}$$

- In general, if $0 \le k \le n$,

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$

Examples

$$\binom{1}{0} - \binom{1}{1} = 0.$$

$$\binom{2}{0} - \binom{2}{1} + \binom{2}{2} = 0.$$

$$\binom{3}{0} - \binom{3}{1} + \binom{3}{2} - \binom{3}{3} = 0.$$

For any $n \ge 1$,

$$\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^n \binom{n}{n} = 0.$$

Examples

$$0 \times \binom{2}{0} - 1 \times \binom{2}{1} + 2 \times \binom{2}{2} = 0.$$

$$0 \times \binom{3}{0} - 1 \times \binom{3}{1} + 2 \times \binom{3}{2} - 3 \times \binom{3}{3} = 0.$$

For any $n \geq 2$,

$$0 \times \binom{n}{0} - 1 \times \binom{n}{1} + 2 \times \binom{n}{2} - \cdots + (-1)^n \times n \times \binom{n}{n}$$

$$= \sum_{k=0}^{n} (-1)^k \times k \times \binom{n}{k} = 0.$$

For any integer $0 \leq r < n$

- $$\sum_{k=0}^{n} (-1)^k \times k(k-1)(k-2)\cdots(k-r+1) \times \binom{n}{k} = 0.$$
- Let $(x)_r = x(x-1)(x-2)\cdots(x-r+1)$. Then

$$\sum_{k=0}^{n} (-1)^k \times (k)_r \times \binom{n}{k} = 0$$

holds for all $r = 0, 1, 2, \cdots, n-1$.

For an arbitrary polynomial $P(x)$

If the degree of $P(x)$ is less then n, then $P(x)$ has a unique linear combination of $(x)_0, (x)_1, \cdots, (x)_{n-1}$:

$$P(x) = a_0(x)_0 + a_1(x)_1 + \cdots + a_{n-1}(x)_{n-1} = \sum_{r=0}^{n-1} a_r(x)_r,$$

thus

$$\sum_{k=0}^{n} (-1)^k \times P(k) \times \binom{n}{k} = \sum_{r=0}^{n-1} a_r \sum_{k=0}^{n} (-1)^k (k)_r \binom{n}{k} = 0.$$

Try to extend

- The previous result: For any polynomial $P(x)$ of degree at most $n-1$,

$$\sum_{k=0}^{n} P(k)(-1)^k \binom{n}{k} = 0.$$

- Notice that $(1-z)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} z^k$.
- The result may be related to $(1-z)^n$ or $(z-1)^n$.

Extension

- Let $f(z) = c_0 + c_1 z + \cdots + c_m z^m$ be any polynomial over $\mathbb{C}$ such that $(z-1)^n$ divides $f(z)$.
- For any polynomial $P(x)$ of degree at most $n-1$,

$$\sum_{k=0}^{m} P(k)c_k = 0.$$

Extension

- Let $f(z) = c_0 + c_1 z + \cdots + c_m z^m$ be any polynomial over $\mathbb{C}$ such that $(z-1)^n$ divides $f(z)$.
- If $P(z)$ is a polynomial of degree n and leading coefficient c, then

$$\sum_{i=0}^{m} P(k)c_k = c\sum_{i=n}^{m} c_i (i)_n,$$

in particular,

$$\sum_{i=0}^{m} P(k)(-1)^k \binom{n}{k} = c(-1)^n n!.$$

Hidefumi Katsuura's result in 2009

Theorem

For complex numbers x and y and any positive integer n,

$$\sum_{j=0}^{n} \binom{n}{j} (-1)^j (xj+y)^r = \begin{cases} 0, & \text{if } 0 \le r < n \\ (-x)^n \times n!, & \text{if } r = n. \end{cases}$$

It follows from our result by choosing $f(z) = (xz+y)^r$, a polynomial of degree r with leading coefficient x^r.

Application

Our result implies some special identities:

$$\sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (k^2+ak+b)^r = \begin{cases} 0, & \text{if } 0 \le r < n \\ (2n)!, & \text{if } r = n. \end{cases}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)^r = \begin{cases} 0, & \text{if } 0 \le r < n \\ n!, & \text{if } r = n. \end{cases}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k}{r} = \begin{cases} 0, & \text{if } 0 \le r < n \\ (-1)^n, & \text{if } r = n. \end{cases}$$

Thanks

Thanks for your attendance!