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# THE RIEMANN INTEGRAL USING ORDERED OPEN COVERINGS

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ABSTRACT. We define the Riemann integral for bounded functions defined on a general topological measure space. When the space is a compact metric space the integral is equivalent to the R-integral defined by Edalat using domain theory.

**1. Introduction.** Edalat [**1**] defined a Riemann type integral on a compact metric space, called the R-integral, using domain theory. The integral so defined has applications in various fields such as dynamic systems and chaos, and the work in [**1**] has also inspired other interesting researches, see [**2, 3, 5**]. The main properties of this new integral among others are: (1) If the space is $[a, b]$, then this integral coincides with the ordinary Riemann integral;(2) a bounded function $f$ is R-integrable if and only if it is continuous almost everywhere; (3) if $f$ is R-integrable then it is also Lebesgue integrable and the value of the R-integral equals that of the Lebesgue integral of $f$. However, as the definition of the R-integral and most of the proofs in [**1**] rely heavily on very technical details of domain theory, this integral is hardly accessible to those who know little about domain theory. Furthermore, unlike the Riemann sum over a partition, the Riemann sum over a simple valuation, the key structure in defining R-integral, lacks a clear geometric interpretation. In this paper we define a Riemann type integral with a domain-free approach. To make it easier to compare this integral with other known integrals we first introduce the more general $\mathcal{M}$-integral for a given collection $\mathcal{M}$ of some measurable subsets satisfying certain conditions. The integral introduced here is defined for bounded real valued functions on an arbitrary topological measure space $X$ which need not be a compact metric space as required in [**1**]; it is a generalization of the Riemann integral on intervals; a function $f$ is integrable if and only if it is continuous almost everywhere when the space is

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compact; when $f$ is integrable it is Lebesgue integrable and the values for these two integrals equal. All these then imply that this integral is equivalent to the R-integral defined by Edalat when the space $X$ is a compact metric space.

**1. Ordered coverings.** Let $X$ be a nonempty set and $\mathcal{M}$ be a collection of subsets of $X$ satisfying

(M1) $X$ and $\varnothing$ are in $\mathcal{M}$;

(M2) $A, B \in \mathcal{M}$ imply $A \cap B \in \mathcal{M}$.

*Definition* 1.1. An ordered $\mathcal{M}$-covering of X is an ordered tuple

$$\mathcal{A} = \langle A_1, A_2, \cdots, A_N \rangle$$

of sets $A_i$ in $\mathcal{M}$ such that $\cup_{i=1}^{N} A_i = X$. We also use the ordered chain

$$A_1 < A_2 < \cdots < A_N$$

to denote the above ordered covering. Here $N$ could be any positive integer.

Put $\Delta_{\mathcal{M}} = \{\mathcal{A} : \mathcal{A}$ is an ordered $\mathcal{M}$- covering of X$\}$.

*Remark* 1.2. (1) The set $A_i$ in an ordered covering could be empty.

(2) For any $X$ and any $\mathcal{M}$, $\langle X \rangle$ is an ordered $\mathcal{M}$- covering.

*Example* 1.3. (1) If $X$ is a topological space and $\mathcal{M}$ is the collection of all open sets then $\mathcal{M}$ satisfies (M1) and (M2). Such ordered $\mathcal{M}$-coverings will be called ordered open coverings of $X$. Ordered open coverings are used by Edalat in [**2**] to construct a sequence of simple valuations that approaches a given measure.

(2) If $X$ is a measure space and $\mathcal{M}$ is the set of all measurable sets of $X$, then $\mathcal{M}$ satisfies (M1) and (M2). Such ordered $\mathcal{M}$-coverings are called ordered measurable coverings.

(3) Let $X$ be a topological space and $\mathcal{M}$ the collection of all closed subsets of $X$. Then $\mathcal{M}$ satisfies (M1) and (M2).

(4) Let $X$ be the set $\mathcal{R}$ of all real numbers. A subset $A$ of $X$ is said to be of density 1 at a point $c$ if

$$\lim_{h \to 0^+} \frac{\mu(A \cap (c-h, c+h))}{2h} = 1.$$

Let $\mathcal{M}$ be the collection of all subsets $A$ of $X$ such that $A$ is of density 1 at each point $c \in A$. Then $\mathcal{M}$ satisfies (M1) and (M2).

*Definition* 1.4. Let $\mathcal{A} = \langle A_1, A_2, \cdots, A_N \rangle$ and $\mathcal{B} = \langle B_1, B_2, \cdots, B_M \rangle$ be two ordered $\mathcal{M}$-coverings of $X$. Define $\mathcal{A} * \mathcal{B}$ to be the ordered covering in which the $\mathcal{M}$- sets are $A_i \cap B_j$, $1 \leq i \leq N$, $1 \leq j \leq M$, and $A_i \cap B_j < A_{i'} \cap B_{j'}$ if and only if either $i < i'$ or $i = i'$ and $j < j'$.

**2. The Riemann sums over ordered coverings.** We now define the lower and upper Riemann sums of a bounded function defined on a measure space and then use these to define the Riemann integral.

In the following we assume that $(X, \mathcal{H}, \mu)$ is a measure space with $\mu(X) = 1$, and $\mathcal{M}$ is a collection of measurable subsets satisfying (M1) and (M2). Let $f : X \to \mathcal{R}$ be a bounded real valued function on $X$ and $A \subseteq X$. Define

$$\inf\, f(A) = \inf\{f(x) : x \in A\} \quad \text{and} \quad \sup\, f(A) = \sup\{f(x) : x \in A\}.$$

We assume that $\inf\, f(\varnothing) = 0$ and $\sup\, f(\varnothing) = 0$.

*Definition* 2.1. Let $f : X \to \mathcal{R}$ be a bounded real valued function. For each $\mathcal{A} = \langle A_1, A_2, \cdots, A_N \rangle \in \Delta_{\mathcal{M}}$, define

$$S^l(f, \mathcal{A}) = \sum_{i=1}^{N} \mu(A_i^*) \inf\, f(A_i) \quad \text{and} \quad S^u(f, \mathcal{A}) = \sum_{i=1}^{N} \mu(A_i^*) \sup\, f(A_i),$$

where $A_1^* = A_1$ and $A_i^* = A_i - \cup_{j<i} A_j$, $i = 2, 3, \ldots, N$.

We call $S^l(f, \mathcal{A})$ and $S^u(f, \mathcal{A})$ the lower and upper Riemann sums of $f$ over $\mathcal{A}$, respectively.

**Lemma 2.2.** *Let $\mathcal{A}, \mathcal{B} \in \Delta_{\mathcal{M}}$. Then, for any bounded function $f : X \to \mathcal{R}$ we have*

$$S^l(f, \mathcal{A}) \leq S^l(f, \mathcal{A} * \mathcal{B}) \leq S^u(f, \mathcal{A} * \mathcal{B}) \leq S^u(f, \mathcal{A}),$$

*and*

$$S^l(f, \mathcal{B}) \leq S^l(f, \mathcal{A} * \mathcal{B}) \leq S^u(f, \mathcal{A} * \mathcal{B}) \leq S^u(f, \mathcal{B}).$$

*Proof.* Let $\mathcal{A} = \langle A_1, A_2, \cdots, A_N\rangle$ and $\mathcal{B} = \langle B_1, B_2, \cdots, B_M\rangle$. Notice that $\mathcal{A} * \mathcal{B} = \{A_i \cap B_j\}$ in which $A_i \cap B_j < A_{i'} \cap B_{j'}$ if either $i < i'$, or $i = i'$ and $j < j'$. So we have

$$(A_k \cap B_l)^* = A_k \cap B_l - \bigcup_{(i,j)<(k,l)} A_i \cap B_j,$$

where $(i, j) < (k, l)$ if either $i < k$ or $i = k$ and $j < l$. Notice that $\cup_{1\le j\le M} B_j = X$, hence

$$\begin{aligned}\bigcup_{(i,j)<(k,l)} A_i \cap B_j &= \bigcup_{i<k}\left(\bigcup_{1\le j\le M}(A_i \cap B_j)\right) \cup \bigcup_{j<l}(A_k \cap B_j)\\ &= \bigcup_{i<k}\left(A_i \cap \left(\bigcup_{1\le j\le M} B_j\right)\right) \cup \left(A_k \cap \bigcup_{j<l} B_j\right)\\ &= \bigcup_{i<k} A_i \cup \left(A_k \cap \bigcup_{j<l} B_j\right).\end{aligned}$$

Hence

$$(A_k \cap B_l)^* = (A_k \cap B_l) - \left(\bigcup_{i<k} A_i \cup \left(A_k \cap \bigcup_{j<l} B_j\right)\right).$$

We first prove

$$S^l(f, \mathcal{A}) \le S^l(f, \mathcal{A} * \mathcal{B}) \le S^u(f, \mathcal{A} * \mathcal{B}) \le S^u(f, \mathcal{A}).$$

Now

$$
\begin{aligned}
&S^l(f, \mathcal{A} * \mathcal{B}) \\
&\quad = \sum_{1\le i\le N, 1\le j\le M} \mu((A_i \cap B_j)^*) \inf\ f(A_i \cap B_j) \\
&\quad = \mu(A_1 \cap B_1) \inf\ f(A_1 \cap B_1) + \mu(A_1 \cap B_2 - A_1 \cap\ B_1) \\
&\quad \times \inf\ f(A_1 \cap B_2) + \cdots \\
&\qquad + \mu\bigg(A_1 \cap\ B_M - A_1 \cap \bigcup_{j<M} B_j\bigg) \inf\ f(A_1 \cap\ B_M) + \cdots \\
&\qquad + \mu\bigg(A_k \cap B_1 - \bigcup_{i<k} A_i\bigg) \inf\ f(A_k \cap B_1) \\
&\qquad + \mu\bigg(A_k \cap B_2 - (\bigg(A_k \cap\ B_1) \cup \bigcup_{i<k} A_i\bigg)\bigg) \inf\ f(A_k \cap\ B_2) + \cdots \\
&\qquad + \mu\bigg(A_k \cap\ B_M - \bigg(\bigg(A_k \cap \bigcup_{j<M} B_j\bigg) \cup \bigcup_{i<k} A_i\bigg)\bigg) \\
&\quad \times \inf\ f(A_k \cap\ B_M) + \cdots \\
&\qquad + \mu\bigg(A_N \cap B_1 - \bigcup_{i<N} A_i\bigg) \inf\ f(A_N \cap B_1) \\
&\qquad + \mu\bigg(A_N \cap B_2 - \bigg((A_N \cap\ B_1) \cup \bigcup_{i<N} A_i\bigg)\bigg) \\
&\quad \times \inf\ f(A_N \cap\ B_2) + \cdots \\
&\qquad + \mu\bigg(A_N \cap\ B_M - \bigg(\bigg(A_N \cap \bigcup_{j<M} B_j\bigg) \cup \bigcup_{i<N} A_i\bigg)\bigg) \\
&\quad \times \inf\ f(A_N \cap\ B_M).
\end{aligned}
$$

For each $1 \le k \le N$, we have

$$
\mu\bigg(A_k \cap B_1 - \bigcup_{i<k} A_i\bigg) \inf\ f(A_k \cap B_1)
$$

$$
\begin{aligned}
&+\mu\Big(A_k\cap B_2-\Big((A_k\cap B_1)\cup\bigcup_{i<k}A_i\Big)\Big)\inf\, f(A_k\cap B_2)+\cdots\\
&+\mu\Big(A_k\cap B_M-\Big(\Big(A_k\cap\bigcup_{j<M}B_j\Big)\cup\bigcup_{i<k}A_i)\Big)\inf\, f(A_k\cap B_M)\\
\geq&\Big[\mu\Big(A_k\cap B_1-\bigcup_{i<k}A_i\Big)+\mu\Big(A_k\cap B_2-\Big((A_k\cap B_1)\cup\bigcup_{i<k}A_i\Big)\Big)+\cdots\\
&+\mu\Big(A_k\cap B_M-\Big(\Big(A_k\cap\bigcup_{j<M}B_j\Big)\cup\bigcup_{i<k}A_i\Big)\Big)\Big]\inf\, f(A_k)\\
=&\mu\,\Big(\Big(A_k\cap B_1-\bigcup_{i<k}A_i)\cup\Big(A_k\cap B_2-\Big((A_k\cap B_1)\cup\bigcup_{i<k}A_i\Big)\Big)\cup\cdots\\
&\cup\Big(A_k\cap B_M-\Big(\Big(A_k\cap\bigcup_{j<M}B_j\Big)\cup\bigcup_{i<k}A_i\Big)\Big))\inf\, f(A_k)\\
=&\mu\Big(A_k-\bigcup_{i<k}A_i\Big)\inf\, f(A_k),
\end{aligned}
$$

where the last and the second to the last equation follow from the fact that the sets

$$
\begin{aligned}
A_k\cap B_1-\bigcup_{i<k}A_i, A_k\cap B_2-(A_k\cap B_1\cup\bigcup_{i<k}A_i),\cdots,&\\
A_k\cap B_M-\Big(A_k\cap\bigcup_{j<M}B_j\cup\bigcup_{i<k}A_i\Big)&
\end{aligned}
$$

are pairwise disjoint and their union is $A_k-\bigcup_{i<k}A_i$.

Since $A_k-\cup_{i<k}A_i=A_k^*$, it then follows that

$$S^l(f,\mathcal{A}*\mathcal{B})\geq S^l(f,\mathcal{A}).$$

Similarly we can prove

$$S^u(f,\mathcal{A}*\mathcal{B})\leq S^u(f,\mathcal{A}).$$

Now we prove

$$S^l(f,\mathcal{B})\leq S^l(f,\mathcal{A}*\mathcal{B}).$$

For each $1 \leq l \leq M$, the sum of the terms in $S^l(f, \mathcal{A} * \mathcal{B})$ involving $B_l$ is

$$\begin{aligned} &\mu((A_1 \cap B_l)^*) \inf f(A_1 \cap B_l) + \mu((A_2 \cap B_l)^*) \inf f(A_2 \cap B_l) + \cdots \\ &\quad + \mu((A_N \cap B_l)^*) \inf f(A_N \cap B_l) \\ &\geq [\mu((A_1 \cap B_l)^*) + \mu((A_2 \cap B_l)^*) + \cdots + \mu((A_N \cap B_l)^*)] \inf f(B_l). \end{aligned}$$

In addition, $\mu((A_1 \cap B_l)^*) + \mu((A_2 \cap B_l)^*) + \cdots + \mu((A_N \cap B_l)^*)) = \mu((A_1 \cap B_l)^* \cup (A_2 \cap B_l)^* \cup \cdots \cup (A_N \cap B_l)^*)$ because $(A_1 \cap B_l)^*, (A_2 \cap B_l)^*, \cdots, (A_N \cap B_l)^*$ are pairwise disjoint.

Notice that for any four sets $A, B, C$ and $D$ we have the equation $(A - B) \cap (C - D) = A \cap C - ((B \cap C) \cup D)$. Then for each $m \leq N$,

$$\begin{aligned} (A_m \cap B_l)^* &= (A_m \cap B_l) - \left(\left(A_m \cap \bigcup_{j<l} B_j\right) \cup \bigcup_{i<m} A_i\right) \\ &= \left(B_l - \bigcup_{j<l} B_j\right) \cap \left(A_m - \bigcup_{i<m} A_i\right) \\ &= {B_l}^* \cap \left(A_m - \bigcup_{i<m} A_i\right). \end{aligned}$$

Hence

$$\begin{aligned} &(A_1 \cap B_l)^* \cup (A_2 \cap B_l)^* \cup \cdots \cup (A_N \cap B_l)^* \\ &\qquad = {B_l}^* \cap \bigcup_{i=1}^{N} \left(A_i - \bigcup_{j<i} A_j\right) \\ &\qquad = {B_l}^* \cap \bigcup_{i=1}^{N} A_i = {B_l}^* \cap X = {B_l}^*. \end{aligned}$$

Therefore, $S^l(f, \mathcal{A} * \mathcal{B}) \geq \sum_{j \leq M} \mu(B_j^*) inf\, f(B_j) = S^l(f, \mathcal{B})$. Similarly, we can show $S^u(f, \mathcal{A} * \mathcal{B}) \leq S^u(f, \mathcal{B})$. The proof is complete.

**Corollary 2.3.** *For any* $\mathcal{A}, \mathcal{B} \in \Delta_{\mathcal{M}}$, $S^l(f, \mathcal{A}) \leq S^u(f, \mathcal{B})$.

*Proof.* This follows from

$$S^l(f, \mathcal{A}) \leq S^l(f, \mathcal{A} * \mathcal{B}) \leq S^u(f, \mathcal{A} * \mathcal{B}) \leq S^u(f, \mathcal{B}).$$

*Definition* 2.4. Let $\mathcal{M}$ be a collection of measurable sets of $X$ satisfying (M1) and (M2). For any bounded function $f : X \to \mathcal{R}$ define

$$(\mathcal{M}) \underline{\int} f d\mu = \text{Sup}\{S^l(f, \mathcal{A}) : \mathcal{A} \in \mathcal{M}\},$$

$$(\mathcal{M}) \overline{\int} f d\mu = \text{Inf}\{S^u(f, \mathcal{A}) : \mathcal{A} \in \mathcal{M}\}.$$

*Remark* 2.5. (1) From Corollary 2.3 it follows immediately that

$$(\mathcal{M}) \underline{\int} f d\mu \leq (\mathcal{M}) \overline{\int} f d\mu.$$

(2) If $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then obviously

$$(\mathcal{M}_1) \underline{\int} f d\mu \leq (\mathcal{M}_2) \underline{\int} f d\mu \leq (\mathcal{M}_2) \overline{\int} f d\mu \leq (\mathcal{M}_1) \overline{\int} f d\mu.$$

(3) If in an ordered covering $\mathcal{A} = \langle A_1, A_2, \cdots, A_N \rangle$, $A_i$ is contained in the union of those $A_j$ with $j < i$, then we can remove $A_i$ from $\mathcal{A}$ without effecting the values of the Riemann sums. In particular we can always remove the empty set from $\mathcal{A}$.

**3. The $\mathcal{M}$-integral.** Now we can define a Riemann type integral for each $\mathcal{M}$ satisfying the conditions (M1) and (M2) which includes both the Riemann integral and the Lebesgue integral as special cases when the functions considered are bounded.

*Definition* 3.1. Given a collection $\mathcal{M}$ of measurable sets satisfying the conditions (M1) and (M2). A bounded real valued function $f : X \to \mathcal{R}$ is called $\mathcal{M}$-integrable if

$$(\mathcal{M}) \underline{\int} f d\mu = (\mathcal{M}) \overline{\int} f d\mu.$$

In this case we call $(\mathcal{M}) \underline{\int} f d\mu = (\mathcal{M}) \overline{\int} f d\mu$ the $\mathcal{M}$-integral of $f$ on X and denote it by

$$(\mathcal{M}) \int f d\mu.$$

**Corollary 3.2.** *If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ then by Remark 2.5, every $\mathcal{M}_1$-integrable function is also $\mathcal{M}_2$-integrable, and in this case*

$$(\mathcal{M}_1)\int f d\mu = (\mathcal{M}_2)\int f d\mu.$$

For any scalar $k$ and any two functions $f$ and $g$ we have

$$(\mathcal{M})\int_{-} k\, f d\mu = k(\mathcal{M})\int_{-} f d\mu, \quad (\mathcal{M})\int^{-} k\, f d\mu = k(\mathcal{M})\int^{-} f d\mu$$

and

$$\begin{aligned}(\mathcal{M})\int_{-} f d\mu + (\mathcal{M})\int_{-} g\, d\mu & \\ &\leq (\mathcal{M})\int_{-} (f+g)\, d\mu \leq (\mathcal{M})\int^{-} (f+g) d\mu \\ &\leq (\mathcal{M})\int^{-} f d\mu + (\mathcal{M})\int^{-} g\, d\mu.\end{aligned}$$

From these we obtain

**Corollary 3.3.** *If $f$ and $g$ are $\mathcal{M}$-integrable functions and $k$ is any scalar, then both $kf$ and $f+g$ are $\mathcal{M}$- integrable, and in these cases*

$$\begin{aligned}(\mathcal{M})\int (f+g)\, d\mu = (\mathcal{M})\int f d\mu + (\mathcal{M})\int g d\mu, (\mathcal{M})\int k f d\mu \\ = k(\mathcal{M})\int f d\mu.\end{aligned}$$

The following lemma can be verified directly.

**Lemma 3.4.** *Let $f : X \to \mathcal{R}$ be any bounded function. Then the following are equivalent*:

(1) *The function $f$ is $\mathcal{M}$-integrable.*

(2) *For any* $\varepsilon > 0$ *there exists* $\mathcal{A} \in \Delta_{\mathcal{M}}$ *such that*

$$S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) < \varepsilon.$$

(3) *There is a number* $b$ *such that for any* $\varepsilon > 0$ *there exists* $\mathcal{A} = \langle A_1, A_2, \cdots, A_N \rangle \in \Delta_{\mathcal{M}}$ *such that*

$$\left| \sum_{i=1}^{N} \mu(A_i^*) f(\xi_i) - b \right| < \varepsilon$$

*holds for arbitrary points* $\xi_i \in A_i$, $i = 1, 2, \ldots, N$.

(4) *For any* $\varepsilon > 0$ *there is* $\mathcal{A} = \langle A_1, A_2, \cdots, A_N \rangle \in \Delta_{\mathcal{M}}$ *such that*

$$\sum_{i=1}^{N} \mu(A_i^*) \omega(f, A_i) < \varepsilon,$$

*where* $\omega(f, A_i)$ *is the oscillation of* $f$ *on* $A_i$.

**4. The Lebesgue integral for bounded functions.** In this section we consider the $\mathcal{L}$-integral where $\mathcal{L}$ is the set of all measurable sets of $X$. It turns out with no surprise that this is exactly the Lebesgue integral.

The Lebesgue integral of a bounded real valued function can be defined in various equivalent ways. Here we adopt the following definition. For the case when $X = [a, b]$ see [**4**, Definition 3.6].

Let $s : X \to \mathcal{R}$ be a measurable function. The function $s$ is a simple function if it has a finite range, equivalently, if there are pairwise disjoint measurable sets $E_1, E_2, \cdots, E_n$ of $X$ which form a covering of $X$ and $s = \sum_{k=1}^{n} c_k \chi_{E_k}$, where $\chi_{E_k}$ is the characteristic function of $E_k$. The Lebesgue integral of the simple function $s = \sum_{k=1}^{n} c_k \chi_{E_k}$ is defined by

$$\int s \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k).$$

*Definition* 4.1. Let $f$ be a bounded measurable function on $X$. The lower and the upper Lebesgue integrals of $f$ are defined by

$$\int_{-} f = \sup\Big\{ \int \phi\, d\mu : \phi \le f \text{ is a simple function} \Big\},$$
$$\int^{-} f = \inf\Big\{ \int \psi\, d\mu : \psi \ge f \text{ is a simple function} \Big\}.$$

If these two integrals are equal, then $f$ is called Lebesgue integrable on $X$ and the common value is denoted by $(L)\int_X f\, d\mu$, or simply $\int f\, d\mu$.

**Lemma 4.2.** *A bounded function $f$ is Lebesgue integrable if and only if for any $\varepsilon > 0$ there is an ordered measurable covering $\mathcal{A} = \langle E_1, E_2, \cdots, E_n\rangle$ of $X$ such that*

$$S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) < \varepsilon.$$

*Proof.* Suppose the condition is satisfied. For any $\varepsilon > 0$, let $\mathcal{A} = \langle E_1, E_2, \cdots, E_n\rangle$ be an ordered measurable covering satisfying

$$S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) < \varepsilon.$$

If necessary we can replace $\mathcal{A}$ by the ordered measurable covering $\mathcal{B}^*$ obtained by removing the empty sets from the covering $\mathcal{A}^* = \langle E_1, E_2 - E_1, \ldots, E_k - \cup_{j<k} E_j, \ldots, E_n - \cup_{j<n} E_j\rangle$. This is possible because $S^l(f, \mathcal{A}) \le S^l(f, \mathcal{A}^*) \le S^u(f, \mathcal{A}^*) \le S^u(f, \mathcal{A})$, and $S^l(f, \mathcal{B}^*) = S^l(f, \mathcal{A}^*), S^u(f, \mathcal{B}^*) = S^u(f, \mathcal{A}^*)$. Thus we can assume the sets $E_i$ are pairwise disjoint and nonempty. Define two simple functions $\psi$ and $\phi$ as follows:

$$\psi = \sum_{i=1}^{n} s_i \chi_{E_i}, \qquad \phi = \sum_{i=1}^{n} l_i \chi_{E_i},$$

where $s_i = \sup f(E_i)$, $l_i = \inf f(E_i)$. Obviously $\phi \le f \le \psi$, and $\int \phi\, d\mu = S^l(f, \mathcal{A}), \int \psi\, d\mu = S^u(f, \mathcal{A})$. This then deduces that $\int^{-} f - \int_{-} f \le S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) < \varepsilon$. Thus $f$ is Lebesgue integrable.

Conversely if $f$ is Lebesgue integrable, then for any $\varepsilon > 0$ there are simple functions $\phi$ and $\psi$ such that

$$\phi = \sum c_k \chi_{E_k} \leq f \leq \psi = \sum s_i \chi_{B_i}$$

and

$$\int \psi \, d\mu - \int \phi \, d\mu < \varepsilon.$$

Let $\mathcal{A}$ be the ordered measurable covering formed by the pairwise disjoint sets $E_k \cap B_i$ in any fixed order. Then one easily verifies that $\int \phi \, d\mu \leq S^l(f, \mathcal{A}) \leq S^u(f, \mathcal{A}) \leq \int \psi \, d\mu$, hence

$$S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) \leq \int \psi \, d\mu - \int \phi \, d\mu < \varepsilon.$$

**Corollary 4.3.** *A bounded function $f$ is Lebesgue integrable if and only if it is $\mathcal{L}$-integrable. In this case the values for the two integrals are equal.*

Since $\mathcal{L}$ is the largest collection of measurable sets satisfying the conditions (M1) and (M2), by Corollary 3.2 we deduce the following.

**Corollary 4.4.** *Let $\mathcal{M}$ be a collection of measurable sets satisfying the conditions* (M1) *and* (M2). *If a bounded function $f$ is $\mathcal{M}$-integrable it is also Lebesgue integrable, and in this case*

$$(\mathcal{M}) \int f \, d\mu = (L) \int f \, d\mu.$$

**5. The $R$-integral.** In this section we consider an integral for bounded real valued functions defined on a topological space $X$ equipped with a normed Borel measure $\mu$, that is $\mu(X) = 1$. Let $\mathcal{O}$ be the collection of all open sets of $X$. The $\mathcal{O}$-integrable functions will be called $R$-integrable functions. We shall prove that $R$-integral is a generalization of the Riemann integral on intervals.

An ordered $\mathcal{O}$-covering of $X$ is called an ordered open covering.

Let $f : X \to \mathcal{R}$ be a bounded function. Recall that, for each subset $A$ of $X$, the oscillation of $f$ on $A$ is defined by

$$\omega(f, A) = \sup\{f(x) : x \in A\} - \inf\{f(x) : x \in A\},$$

and for each point $a \in X$, the oscillation of $f$ at $a$ is defined by

$$\omega(f, a) = \inf\{\omega(f, U) : \text{U is an open neighborhood of } a\}.$$

It is well known that $f$ is continuous at $a$ if and only if $\omega(f, a) = 0$. For each $\varepsilon > 0$, the set $D(f; \varepsilon) = \{x : \omega(f, x) \geq \varepsilon\}$ is a closed subset of $X$, and the set of discontinuity points of $f$, denoted by $D(f)$, is

$$D(f) = \bigcup_{n=1}^{+\infty} D(f; 1/n).$$

A function $f$ is said to be continuous almost everywhere if $\mu(D(f)) = 0$.

**Lemma 5.1.** *If a function $f$ is R-integrable, then $f$ is continuous almost everywhere.*

*Proof.* Suppose, on the contrary, $\mu(D(f)) \neq 0$. Then $\mu(D(f; 1/n)) \neq 0$ for some $n$. Now for any ordered open covering $\mathcal{A} = \langle A_1, A_2, \cdots, A_N\rangle$ of $X$,

$$\begin{aligned} S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) &= \sum_{k=1}^{N} \mu(A_k^*)\omega(f, A_k) \\ &\geq \sum_{k=1}^{N} \mu(D(f; 1/n) \cap A_k^*)\omega(f, A_k). \end{aligned}$$

Notice that $D(f; 1/n) \cap A_k^* \subseteq D(f; 1/n) \cap A_k$. If $\mu(D(f; 1/n) \cap A_k^*) \neq 0$, then $D(f; 1/n) \cap A_k \neq \varnothing$. Since $A_k$ is open, it follows that $\omega(f, A_k) \geq 1/n$, thus $\mu(D(f; 1/n) \cap A_k^*)\omega(f, A_k) \geq \mu(D(f; 1/n) \cap A_k^*)1/n$. If $\mu(D(f; 1/n) \cap A_k^*) = 0$, then trivially $\mu(D(f; 1/n) \cap A_k^*)\omega(f, A_k) = \mu(D(f; 1/n) \cap A_k^*)1/n$.

Hence we have

$$\sum_{k=1}^{N}\mu(D(f;1/n)\cap A_k^*)\omega(f,A_k)\geq\sum_{k=1}^{N}\mu(D(f;1/n)\cap A_k^*)\frac{1}{n}$$
$$=\frac{1}{n}\sum_{k=1}^{N}\mu(D(f;1/n)\cap A_k^*)=\frac{1}{n}\mu(D(f;1/n)).$$

The last equation follows from the fact that the $A_k^*$'s are pairwise disjoint and their union is $X$. This contradicts the assumption that $f$ is $R$-integrable. Hence $\mu(D(f))=0$.

For the converse conclusion to be true we need the measure to have the following property:

For any measure zero set $A$ and any $\varepsilon>0$, there is an open set $U$, such that

$$(*)\qquad A\subseteq U \quad\text{and}\quad \mu(U)<\varepsilon.$$

**Lemma 5.2.** *Let $X$ be a compact Hausdorff space with a normed Borel measure $\mu$ satisfying the condition $(*)$. If $f$ is bounded and continuous almost everywhere, then $f$ is $R$-integrable.*

*Proof.* Assume that $f$ is continuous almost everywhere and $|f(x)|\leq B$ for all $x\in X$, where $B$ is a positive number. Now for each $\varepsilon>0$, by condition $(*)$ we can choose an open set $A_1$ containing $D(f)$ such that $\mu(A_1))<(\varepsilon)/(4B)$. As a closed subset of $X$, $F=X-A_1$ is compact, and $f$ is continuous at every point in $F$. Thus there is an open covering of $F$, say $\{A_2,A_2,\cdots,A_N\}$ such that $\omega(f,A_k)<(\varepsilon/2)$ for each $k=2,3,\cdots,N$. Put $\mathcal{A}=\langle A_1,A_2,\cdots,A_N\rangle$. Then

$$S^u(f,\mathcal{A})-S^l(f,\mathcal{A})=\sum_{k=1}^{N}\mu(A_k^*)\omega(f,A_K)<\frac{\varepsilon}{4B}2B+\frac{\varepsilon}{2}\sum_{k=2}^{N}\mu(A_k^*)\leq\varepsilon.$$

Hence $f$ is $R$-integrable.

**Theorem 5.3.** *Let $X$ be a compact Hausdorff space with a normed Borel measure satisfying the condition $(*)$. Then a bounded function is $R$-integrable if and only if it is continuous almost everywhere.*

It is well known that a bounded function defined on an interval $[a, b]$ is Riemann integrable if and only if it is continuous almost everywhere. And in this case the Riemann integral and the Lebesgue integral of $f$ are equal. The Lebesgue measure $\mu$ on $[a, b]$ satisfies the condition $(*)$. Thus combining the above results we obtain the following corollary which shows that the $R$-integral is a generalization of the Riemann integral.

**Corollary 5.4.** it A bounded function $f$ on $[a, b]$ is Riemann integrable if and only if it is $R$-integrable. And in this case the values of the two integrals of $f$ are equal.

*Remark* 5.5. (1) Let $X = [a, b]$ and $\mathcal{I} = \{[c, d] : a \leq c \leq d \leq b\} \cup \{\varnothing\}$. Then $\mathcal{I}$ satisfies the conditions (M1) and (M2) and we can prove that $\mathcal{I}$-integral also coincides with the Riemann integral.

(2) Suppose $\mathcal{B}$ is a basis of a topological space $X$ which includes $X$ and $\varnothing$, so $\mathcal{B}$ satisfies (M1) and (M2). It is natural to ask if $\mathcal{B}$-integral is equivalent to the $R$-integral. Since $\mathcal{B} \subseteq \mathcal{O}$, by Corollary 3.2 if $f$ is $\mathcal{B}$-integrable, then it is $R$-integrable and the values of the two integrals of $f$ are equal. Now suppose $f$ is $R$-integrable and $X$ is a compact Hausdorff space with a normed Borel measure satisfying the condition $(*)$. Then $f$ is continuous almost everywhere by Theorem 5.3. Let $B$ be a bound of $f$. For each $\varepsilon > 0$ choose $n > 0$ with $1/n < (\varepsilon/2)$. Then there is an open set $U$ of $X$ with $\mu(U) < (\varepsilon/4B)$ and $D(f; (1/n)) \subseteq U$. There exist $U_1, U_2, \cdots, U_m \in \mathcal{B}$ such that $D(f; (1/n)) \subseteq U_1 \cup U_2 \cdots \cup U_m \subseteq U$ because $D(f; (1/n))$ is a closed subset of the compact space $X$ and $\mathcal{B}$ is a basis. Let $W = U_1 \cup U_2 \cdots \cup U_m$. Now for each $x \in W^c$ we have $\omega(f; x) < (1/n)$, so there exists an open neighborhood $V$ of $x$ such that $\omega(f; V) < (1/n)$, and this $V$ can be chosen from $\mathcal{B}$. Since $W^c$ is compact it follows that there are $U_{m+1}, \cdots, U_N \in \mathcal{B}$ such that $W^c \subseteq \cup_{i=m+1}^{N} U_i$ and $\omega(f; U_i) < (1/n)$ for each $i = m + 1, \cdots, N$. Let $\mathcal{A} = \langle U_1, U_2, \cdots, U_N \rangle$. Then $\mathcal{A}$ is an ordered $\mathcal{B}$-covering, and we have

the following equations and inequalities:

$$\begin{aligned}
S^u(f,\mathcal{A}) - S^l(f,\mathcal{A}) &= \sum_{i=1}^{N} \mu(U_i^*)\omega(f,U_i) \\
&= \sum_{i=1}^{m} \mu(U_i^*)\omega(f,U_i) + \sum_{i=m+1}^{N} \mu(U_i^*)\omega(f,U_i) \\
&\leq 2B\sum_{i=1}^{m} \mu(U_i^*) + \frac{1}{n}\sum_{i=m+1}^{N} \mu(U_i^*) \\
&\leq 2B\mu(U) + \frac{\varepsilon}{2}\mu(X) \leq 2B\frac{\varepsilon}{4B} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Hence $f$ is $\mathcal{B}$-integrable.

*Remark* 5.6. In [**1**] Edalat defines a Riemann type integral on compact metric spaces, also called $R$-integral, by using domain theory. He also proves that a bounded function $f$ is $R$-integrable if and only if it is continuous almost everywhere [**1**, Theorem 6.5], and in this case $R$-integral of $f$ is equal to the Lebesgue integral of $f$ [**1**, Theorem 7.2]. Thus when $X$ is a compact metric space then our $R$-integral is equivalent to Edalat's $R$-integral, and the values of the two integrals coincide for every integrable function.

**6. Computability of R-integral.** Compared with the Lebesgue integral, a distinct virtue of the Riemann integral is its computability, as was pointed out by Edalat in [**1**]. In terms of the definition given in this paper, computability means that one can choose a fixed countable collection $\{\mathcal{A}_\backslash\}_{\backslash=\infty}^{\infty}$ of ordered open coverings such that for each R-integrable function $f$, we have

$$\int f\,d\mu = \lim_{n\to\infty} S^l(f,\mathcal{A}_n) = \lim_{n\to\infty} S^u(f,\mathcal{A}_n).$$

For compact metric spaces, Edalat has proved the computability of R-integral by using the domain theory. Here we provide an elementary proof for this fact.

In the following we assume that $X$ is a compact metric space with a normed Borel measure $\mu$ satisfying the condition $(*)$.

The main step in the proof is to show that if $f$ is R-integrable then for any $\varepsilon > 0$ there is $\delta > 0$ such that for each ordered open covering $\mathcal{A} = \langle A_1, A_2, \cdots, A_N \rangle$, if $\dim(A_i) < \delta$ for $i = 1, 2, \cdots, N$, then $S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) < \varepsilon$, where $\dim(A_i) = \sup\{d(x, y) : x, y \in A_i\}$.

To prove the main result we need the following lemma.

**Lemma 6.1.** *Let $f : X \to \mathcal{R}$ be a real valued function defined on a compact metric space and $\omega(f, x) < \delta$ hold for all $x \in X$. Then there is an $\varepsilon > 0$ such that*

$$|f(x) - f(y)| \leq \delta$$

*whenever $d(x, y) < \varepsilon$.*

**Lemma 6.2.** *If $f : X \to \mathcal{R}$ is R-integrable, then for each $\varepsilon > 0$ there is $\delta > 0$ such that for any ordered open covering $\mathcal{A} = \langle A_1, A_2, \cdots, A_N \rangle$ with dim $(\mathcal{A}_i) < \delta$ for each $i = 1, 2, \cdots, N$, then*

$$S^u(f, \mathcal{A}) - S^l(f, \mathcal{A}) < \varepsilon.$$

*Proof.* Suppose $|f(x)| \leq B$ for all $x \in X$. By Lemma 5.1, $f$ is continuous almost everywhere. Choose a number $r > 0$ with $r < (\varepsilon/2)$. The set $D(f; r) = \{x \in X : \omega(f, x) \geq r\}$ is closed and has zero measure. Take an open set $U \supseteq D(f; r)$ such that $\mu(U) < (\varepsilon/4B)$. Since $X$ is compact there is an open set $V$ satisfying

$$D(f; r) \subseteq V \subseteq \mathrm{cl}\,(V) \subseteq U, \mathrm{cl}\,(V) \neq U,$$

where $\mathrm{cl}\,(V)$ is the closure of $V$. Let $\delta_1 = \inf\{d(x, y) : x \in \mathrm{cl}\,(V), y \in X - U\}$. Then $\delta_1 > 0$ and $\omega(f, x) < r$ for all $x \notin V$. By Lemma 6.1, it follows that there exists $\delta_2 > 0$ such that for any $x, y \in V^c$, if $d(x, y) < \delta_2$ then $|f(x) - f(y)| \leq r < (\varepsilon/2)$. Let $\delta = \min\{\delta_1, \delta_2\}$. Now suppose $\mathcal{A} = \langle A_1, A_2, \cdots, A_N \rangle$ is an ordered open covering such that $\dim(A_i) < \delta$ for $i = 1, 2, \cdots, N$. Then each $A_i$ is either contained in $U$ or is contained in $V^c$. Assume that $A_{i_1}, A_{i_2}, \cdots, A_{i_m}$ are contained in $U$ and the rest of them are contained in $V^c$. Then

$$\begin{aligned} S^u(f,\mathcal{A}) - S^l(f,\mathcal{A}) &= \sum_{j=1}^{m} \mu({A_{i_j}}^*)(\sup\, f(A_{i_j}) - \inf\, f(A_{i_j})) \\ &\quad + \sum_{k\neq i_j} \mu(A_k^*)(\sup\, f(A_k) - \inf\, f(A_k)) \\ &\leq 2B\sum_{j=1}^{m} \mu({A_{i_j}}^*) + \frac{\varepsilon}{2}\sum_{k\neq i_j} \mu((A_k)^*). \end{aligned}$$

Note that the sets ${A_i}^*$ are pairwise disjoint sets, so

$$\sum_{j=1}^{m} \mu(A_{i_j}^*) = \mu(\cup_{j=1} A_{i_j}^*) \leq \mu(U).$$

Similarly

$$\sum_{k\neq i_j} \mu(A_k^*) \leq \mu(V^c).$$

Hence

$$S^u(f,\mathcal{A}) - S^l(f,\mathcal{A}) \leq 2B\mu(U) + \frac{\varepsilon}{2}\mu(V^c) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

The proof is complete.

**Theorem 6.3.** *For each $n \in N$ choose an ordered open covering $\mathcal{A}_n$ such that each $A_i$ in $\mathcal{A}_n$ has diameter less than $1/n$. Then a bounded function $f$ is R-integrable if and only if*

$$\lim_{n\to\infty} S^l(f,\mathcal{A}_n) = \lim_{n\to\infty} S^u(f,\mathcal{A}_n),$$

*and in this case*

$$\int f\, d\mu = \lim_{n\to\infty} S^l(f,\mathcal{A}_n) = \lim_{n\to\infty} S^u(f,\mathcal{A}_n).$$

*Proof.* By Lemma 3.4 the condition is evidently sufficient. The necessity follows from Lemma 6.2. The equations

$$\int f\, d\mu = \lim_{n\to\infty} S^l(f,\mathcal{A}_n) = \lim_{n\to\infty} S^u(f,\mathcal{A}_n)$$

obviously hold. □

*Remark* 6.4. Since $X$ is a compact metric space, for each $n > 0$ there exists an ordered open covering $\mathcal{A}_n$ such that for each $A_i$ in $\mathcal{A}$, $\dim(A_i) < (1/n)$. Also by Lemma 2.2, if we define $\mathcal{B}_{n+1} = \mathcal{A}_{n+1} * \mathcal{B}_n$ for $n = 1, 2, \cdots$, then $\{\mathcal{B}_n\}_{n=1}^{\infty}$ is a sequence of ordered open coverings that can replace $\{\mathcal{A}_n\}_{n=1}^{\infty}$. In addition, for each bounded real valued function $f$ we have two monotone sequences $S^l(f, \mathcal{B}_n) \nearrow$ and $S^u(f, \mathcal{B}_n) \searrow$, which converge to the same number when $f$ is R-integrable.

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