

Mathematics Teachers Conference 2011

**Connecting
JC Math and University Math**
A perspective from NUS Math Lecturer

A/Prof Victor Tan
Department of Mathematics
National University of Singapore

Objective of this talk

- **For JC teachers**

- to be aware of their students' math performance in university
- to look at areas they can help to enhance their students' performance in university

- **For university lecturers**

- to better understand the background of JC students
- to identify the source of problems faced by these students in learning math

JC (H2) Math Syllabus

- The syllabus prepares students adequately for university courses including mathematics, physics and engineering, where more mathematics content is required.

Performance at university courses

- The syllabus aims to develop mathematical thinking and problem solving skills in students.

Life long skill

- Students will learn to analyse, formulate and solve different types of problems. They will also learn to work with data and perform statistical analyses.

Performance at A level exam

NUS Math Major

- Each cohort about **250**
- About **2/3** from JC (A level H2), **1/3** others
- Almost all from JC scored **A grade** for H2 math

MA1100 Module

- **Fundamental Concepts of Mathematics**
- **Essential** module for math major
- Most students take it in their very **first semester** in NUS
- Prerequisite: H2 math or equivalent
- Has been taught by this speaker for past **4 years**

MA1100 Syllabus

- Introduces the **language**, **notions**, and **methods** upon which a sound education in mathematics at the university level is built.
- Students are exposed to the language of mathematical **logic**, the idea of rigorous mathematical **proofs** and fundamental mathematical concepts such as **sets**, **relations** and **functions**.
- Major topics: **Elementary logic, mathematical statements, set operations, relations and functions, equivalence relations, elementary number theory.**

MA1100 Student Profile

Category	Percentage
NUS - H2 students	43.9%
NUS - Others	14.7%
SM2 (PRC scholars)	41.4%

Exam Q1 Question

Let A, B, C be subsets of some universal set U .

- (a) Use Element-Chasing method to show that $(A \times C) - (B \times C) = (A - B) \times C$.
- (b) Is it true that $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$? Justify your answer.

(a) Use Element-Chasing method to show that $(A \times C) - (B \times C) = (A - B) \times C$.

Q1(a) Common Mistakes

Representation of Cartesian product

Jumping to conclusion

$$(a) (A \times C) - (B \times C) = (A - B) \times C$$

Since $(A \times C) - (B \times C)$, we let $x \in A \times C$ and $x \notin B \times C$

i.e. $x \in A \times C$ and $x \in (B \times C)^c$

Thus, $x \in (A \times C) \cap (B \times C)^c$

Hence, $x \in (A \cap B^c) \times C$

Hence, $(A \times C) - (B \times C) \subseteq (A - B) \times C$

You must take a pair.

Now prove that $(A - B) \times C \subseteq (A \times C) - (B \times C)$

let $y \in (A - B) \times C$

$y \in (A \cap B^c) \times C$

$y \in (A \times C) \cap (B^c \times C)$

Then, $y \in A \times C$ and $y \in B^c \times C \Rightarrow y \notin B \times C$.

Hence $y \in (A \times C) - (B \times C)$

Hence $(A - B) \times C \subseteq (A \times C) - (B \times C)$

\therefore Since $(A - B) \times C \subseteq (A \times C) - (B \times C)$ & $(A \times C) - (B \times C) \subseteq (A - B) \times C$

$\Rightarrow (A \times C) - (B \times C) = (A - B) \times C$.

(a) Use Element-Chasing method to show that $(A \times C) - (B \times C) = (A - B) \times C$.

Q1(a) Common Mistakes

Wrong argument

Logical flaw

let $(x,y) \in (A \times C) - (B \times C)$
Then $(x,y) \in (A \times C)$ and $(x,y) \notin (B \times C) \implies x \in B$ or $y \in C$
 $x \in A$ and $x \notin B \implies x \in A - B$
 $y \in C$ and $y \notin C \implies y \in C - C \implies y \in \emptyset \subseteq C$ X
 $\therefore (x,y) \in (A - B) \times C$
Hence, $(A \times C) - (B \times C) = (A - B) \times C$.
The other

(b) Is it true that $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$? Justify your answer.

Q1(b) Common Mistakes

Mixing up conditional and bi-conditional

let $x \in \mathcal{P}(A-B)$ \Rightarrow $x \subseteq (A-B)$

Thus $x \subseteq A$ but $x \not\subseteq B$

$x \in \mathcal{P}(A)$ but $x \notin \mathcal{P}(B)$.

Thus so $x \in (\mathcal{P}(A) - \mathcal{P}(B))$

Thus it is true #

Exam Q2 Question

(a) Let $f_1, f_2, \dots, f_n, \dots$ denote the Fibonacci sequence defined by

$$f_1 = 1, \quad f_2 = 1, \quad \text{and} \quad f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 3.$$

Show by induction that, for all $n \in \mathbb{N}$, $f_2 + f_4 + \dots + f_{2n} = f_{2n+1} - 1$.

(b) If you have proven: (i) $P(3)$ is true; and (ii) $P(k) \Rightarrow P(k+4)$ for all $k \geq 3$ is true.

For which universal set U have you proven $(\forall n \in U) P(n)$? Justify your answer.

(c) Suppose you want to prove $P(n)$ is true for every natural number $n \geq 5$ by induction and you manage to show the inductive step $P(k) \wedge P(k+2) \Rightarrow P(k+4)$ for all $k \geq 7$. What are the base cases that you need to establish (i.e. which additional values a that you need to show that $P(a)$ is true separately)? Justify your answer.

(b) If you have proven: (i) $P(3)$ is true; and (ii) $P(k) \Rightarrow P(k+4)$ for all $k \geq 3$ is true.

For which universal set U have you proven $(\forall n \in U) P(n)$? Justify your answer.

Q2(b) Common Mistakes

Misconception about PMI

3 4 5 6
7 8 9 10
11 12

$P(3) \Rightarrow P(7) \Rightarrow P(11) \Rightarrow \dots$
$P(4) \Rightarrow P(8) \Rightarrow P(12) \Rightarrow \dots$
$P(5) \Rightarrow P(9) \Rightarrow P(13) \Rightarrow \dots$
$P(6) \Rightarrow P(10) \Rightarrow P(14) \Rightarrow \dots$

U is the set $\mathbb{N} - \{1, 2\}$

(b) If you have proven: (i) $P(3)$ is true; and (ii) $P(k) \Rightarrow P(k+4)$ for all $k \geq 3$ is true.

For which universal set U have you proven $(\forall n \in U) P(n)$? Justify your answer.

Q2(b) Common Mistakes

Misconception about PMI

Bad set notation

$P(3) \Rightarrow P(7)$ $P(4) \Rightarrow P(8)$ $P(5) \Rightarrow P(9)$ $P(6) \Rightarrow P(10)$

universal set $U = \{(n+2) - \{6\} \mid n \in \mathbb{Z}^+\}$

$P(1)$ and $P(2)$ and $P(6)$ not proven.

- (c) Suppose you want to prove $P(n)$ is true for every natural number $n \geq 5$ by induction and you manage to show the inductive step $P(k) \wedge P(k+2) \Rightarrow P(k+4)$ for all $k \geq 7$. What are the base cases that you need to establish? Justify your answer.

Q2(c) Common Mistakes

Having problem writing proper justification

We need to show base cases $P(5)$, $P(6)$ and $P(7)$ and ?

Reason: Inductive step is $P(k) \wedge P(k+2) \Rightarrow P(k+4)$

$$P(5) \wedge P(7) \Rightarrow P(9)$$

To get $P(9)$, we need to find $P(7)$ as well.

$$P(5) \Rightarrow P(6) \Rightarrow P(7)$$

Hence, we need to find the 3 bases, $n=5$, $n=6$ and $n=7$.

(c) Suppose you want to prove $P(n)$ is true for every natural number $n \geq 5$ by induction and you manage to show the inductive step $P(k) \wedge P(k+2) \Rightarrow P(k+4)$ for all $k \geq 7$. What are the base cases that you need to establish? Justify your answer.

Q2(c) Common Mistakes

Having problem writing proper justification

Base case $P(5)$, $P(6)$, $P(7)$, $P(8)$, ~~$P(9)$~~ , $P(10)$

if Base case $P(5)$, $P(6)$ are not true.
 ~~$P(7)$~~ , $P(8)$, $P(9)$, $P(10)$

then $P(11)$ is not true.

Thus need to pro ?

(c) Suppose you want to prove $P(n)$ is true for every natural number $n \geq 5$ by induction and you manage to show the inductive step $P(k) \wedge P(k+2) \Rightarrow P(k+4)$ for all $k \geq 7$. What are the base cases that you need to establish? Justify your answer.

Q2(c) Common Mistakes

Having problem writing proper justification

$$\begin{aligned} P(7) \wedge P(9) &\Rightarrow P(11) \\ P(8) \wedge P(10) &\Rightarrow P(12) \\ P(9) \wedge P(11) &\Rightarrow P(13) \\ P(10) \wedge P(12) &\Rightarrow P(14) \\ P(11) \wedge P(13) &\Rightarrow P(15) \\ P(12) \wedge P(14) &\Rightarrow P(16) \end{aligned}$$

$$P(5)$$

Base cases: $P(5)$, $P(6)$, $P(7)$
 $P(8)$, $P(9)$, $P(10)$

Exam Q5 Question

(a) Give two examples of denumerable sets (i.e. countable sets that are infinite) $S_1, S_2 \subseteq \mathbb{R}$ that satisfy the following conditions:

(i) all the elements in S_1 are irrational numbers;

(ii) all the elements in S_2 are within the open interval $(0, 1)$.

For each of your examples, briefly explain why it is denumerable.

(b) True or false: If the intersection of two uncountable sets is infinite, then the intersection is uncountable. Justify your answer.

(i) all the elements in S_1 are irrational numbers;

(ii) all the elements in S_2 are within the open interval $(0, 1)$.

Q5(a) Common Mistakes

Wrong examples

$$S_1 = \{x \in \mathbb{N} \mid x \text{ is irrational}\} = \emptyset$$

since irrational numbers is an infinite subset of natural numbers, and natural numbers are denumerable,

by definition, the set of irrational numbers is denumerable.

This is due to every infinite subset of a denumerable set is denumerable. \triangleright

$$S_1 = \{x \in \mathbb{N} \mid 0 < x < 1\} = \emptyset$$

since the open interval $(0, 1)$ is infinite, and it is a subset of \mathbb{N} , by definition, the open interval $(0, 1)$ is also denumerable

(i) all the elements in S_1 are irrational numbers;

(ii) all the elements in S_2 are within the open interval $(0, 1)$.

Q5(a) Common Mistakes

Bad set notation

i) $S_1 = \{(-(\mathbb{N}-\mathbb{Q}), (\mathbb{N}-\mathbb{Q}))\}$?
ii) $S_2 = \{\mathbb{Q} \setminus ((-\infty, 0) \cup (1, \infty))\}$?

We can count and order elements in S_1 and S_2

(b) True or false: If the intersection of two uncountable sets is infinite, then the intersection is uncountable. Justify your answer.

Q5(b) Common Mistakes

“Proof” by example

True:

choose $S_1 = \mathbb{R}$, $S_2 = \mathbb{I}$.

Both S_1 and S_2 are uncountable.

$S_1 \cap S_2 = \mathbb{I}$ which is infinite.

\mathbb{I} is also uncountable.

(b) True or false: If the intersection of two uncountable sets is infinite, then the intersection is uncountable. Justify your answer.

Q5(b) Common Mistakes

Wrong example

Bad set notation

If the set is uncountable, it is both uncountable and infinite

Let (a, b) and (b, c) be 2 uncountable sets where $a < b < c$.

$$(a, b) \cap (b, c) = b.$$

which is countable.

Hence the statement is false. ✓

(b) True or false: If the intersection of two uncountable sets is infinite, then the intersection is uncountable. Justify your answer.

Q5(b) Common Mistakes

Wrong proof – converse error

If the intersection of 2 uncountable sets is infinite, then the intersection is uncountable. Let A & B be the 2 uncountable sets.

Assume if intersection is countable, being a subset of A & B , then A and B should be countable. Therefore, there is a contradiction to the hypothesis whereby 2 uncountable sets is infinite. Hence the intersection is uncountable by proof of contradiction.

Therefore it is a ~~true~~ statement.

Exam Q7 Question

- (a) Use Euclidean Algorithm to find integers x and y such that $\gcd(33, 54) = 33x + 54y$.
- (b) Show that, if p is an odd prime, then $\gcd(p^2 + 1, (p + 1)^2) = 2$.
- (c) Show that we cannot find three consecutive odd numbers that are all primes, except 3, 5 and 7.

(b) Show that, if p is an odd prime, then $\gcd(p^2 + 1, (p + 1)^2) = 2$.

Q7(b) Common Mistakes

Things done right:

- Express $p = 2k + 1$
- Let $\gcd(\dots) = d$
- Try to show $2 \mid d$ and $d \mid 2$

Things done wrong:

- $d \mid 2p \implies d \mid 2$ or $d \mid p$
- $d \mid p \implies d = p$
- $2 \mid d \implies 2 = d$
- $a, b \in \mathbf{Z} \implies \gcd(2a, 2b) = 2$
- $2n = dm \implies d \mid 2$

Converse errors!

(b) Show that, if p is an odd prime, then $\gcd(p^2 + 1, (p + 1)^2) = 2$.

Q7(b) Common Mistakes

Abusing of notation

If p is odd, $p = 2x + 1$

$$p^2 + 1 = (2x + 1)^2 + 1 = 4x^2 + 4x + 2 = 2(2x^2 + 2x + 1) = 2m, \text{ for some } m \in \mathbb{Z}.$$

$$(p + 1)^2 = (2x + 2)^2 = 4x^2 + 8x + 4 = 2(2x^2 + 4x + 2) = 2n, \text{ for some } n \in \mathbb{Z}.$$

Let $\gcd(p^2 + 1, (p + 1)^2) = d$.

$$d \mid p^2 + 1 \Rightarrow p^2 + 1 = dm, \text{ for some } m \in \mathbb{Z}.$$

$$d \mid (p + 1)^2 \Rightarrow (p + 1)^2 = dn, \text{ for some } n \in \mathbb{Z}.$$

$$\text{Since } p^2 + 1 = 2m, \quad d = 2.$$

$$\text{Since } (p + 1)^2 = 2n, \quad d = 2.$$

$$\therefore \gcd(p^2 + 1, (p + 1)^2) = 2 \quad (\text{shown}).$$

(b) Show that, if p is an odd prime, then $\gcd(p^2 + 1, (p + 1)^2) = 2$.

Q7(b) Common Mistakes

Bits and pieces; having problem connecting them

$$\gcd(p^2+1, (p+1)^2) = \gcd(4a^2+4a+2, 4a^2+8a+4)$$

$$\text{Let } k = \gcd(4a^2+4a+2, 4a^2+8a+4)$$

$$\Rightarrow k \mid 4a^2+4a+2 \quad \text{and} \quad k \mid 4a^2+8a+4$$

$$\Rightarrow 4a^2+4a+2 = km \quad 4a^2+8a+4 = kn$$

Also $(4a^2+4a+2)x + (4a^2+8a+4)y = k$

$$kmx + kny = k$$

$$mx + ny = 1$$

(b) Show that, if p is an odd prime, then $\gcd(p^2 + 1, (p + 1)^2) = 2$.

Q7(b) Common Mistakes

Unnecessary algebraic manipulation

$$\text{let } \gcd(p^2+1, (p+1)^2) = d$$

$$\text{Then } d \mid p^2+1 \text{ and } d \mid (p+1)^2 \Rightarrow d \mid p^2+1 + (p+1)^2 \text{ and } d \mid p^2+1 - (p+1)^2$$

$$p^2+1 + (p+1)^2 = p^2+1 + p^2+2p+1 \\ = 2p^2+2p+2$$

$$= 2(p^2+p+1) \Rightarrow 2 \mid p^2+1 + (p+1)^2$$

$$p^2+1 - (p+1)^2 = p^2+1 - p^2-2p-1$$

$$= -2p$$

$$= 2(-p) \Rightarrow 2 \mid p^2+1 - (p+1)^2$$

$$\therefore 2 \mid d$$

(c) Show that we cannot find three consecutive odd numbers that are all primes, except 3, 5 and 7.

Q7(c) Common Mistakes

Things done right:

- Write the three consecutive odd as: $2k+1$, $2k+3$, $2k+5$
- Try to prove by contradiction: suppose we can find ...
- Try to prove by cases: (i) k is even; (ii) k is odd

(c) Show that we cannot find three consecutive odd numbers that are all primes, except 3, 5 and 7.

Q7(c) Common Mistakes

Prove by listing

If there are 3 consecutive odd numbers that are prime.

$$a, a+2, a+4,$$

when $a > 5$,

$$a, a+2, a+4.$$

5	7	9
7	9	11
9	11	13
11	13	15
13	15	17
.	.	.
.	.	.

Justify.

there is always a
multiple of 3
 \therefore not prime.

- (c) Show that we cannot find three consecutive odd numbers that are all primes, except 3, 5 and 7.

Q7(c) Common Mistakes

Having difficulties developing the proof

Prove by contradiction.
Assume we can find 3 consecutive odd numbers that are all primes.
Let the 3 numbers be : $2x+1$, $2x+3$, $2x+5$; $x \in \mathbb{Z}$.

$$2x+1 = a$$

$$2x+3 = b$$

$$2x+5 = c$$

$$ab = (2x+1)(2x+3) = \text{composite number}$$

$$= 4x^2 + 6x + 2x + 3$$

$$= 4x^2 + 8x + 3$$



Weaknesses of H2 students

- **Proficiency in mathematical language**
 - Reading
 - Listening
 - Writing
 - Thinking
- (Set) notation - Mathematical Alphabet
- Logic - Mathematical Grammar
- Mathematical statement
- Mathematical thinking

How JC math can help?

- Familiarize students with **written mathematical statements**
- Emphasize to students:
 - **If-then statement vs biconditional statement**
 - **Universal statement vs existential statement**
- Get students to use **proper set notations** in workings
- Encourage students to **write more words** in workings other than algebraic manipulation
- Set questions that require students to **justify their answers** instead of just getting the answers



Thank you!
Questions and feedback
are welcome!