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STRATEGIC MODELS FOR SOLVING RATIO AND PROPORTION PROBLEMS

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Abstract

This paper is concerned with the investigation of children's knowledge and their methods of organising it in solving ratio and proportion problems. Strategic models which represent pupils' approaches in solving different types of ratio and proportion problems and a generic model which explains the essential information for solving a set of ratio and proportion problems were identified. A total number of 499 children at an average age of 11 years sat for a ratio and proportion test. Analysis of their responses was carried out in 3 stages: (1) examining and recording different strategies used by problem solvers, (2) classifying the identified strategies in terms of techniques and methods used and the different steps in arriving at the answers and (3) linking procedural knowledge and conceptualising (a) an overall strategic model used for solving ratio and proportion problems and (b) a schematic model for solving each ratio and proportion problem. The results of the study revealed the various types of strategies used by these children and the different levels of thinking employed in solving the problems. The models are presented to show the path of each strategy by connecting sets of skills of the problem solvers. The study confirms that problem solver requires bodies of essential knowledge and well organised structure to solve mathematical problems.

Introduction

Review of literature shows that a new emphasis in the area of problem solving research has directed towards cognitive psychology. The emerging perspective is concerned with the application of psychological theories to investigate children's mental behaviours in solving mathematical problems (Greeno, 1980; Low & Over, 1989; Mayer, 1982a; Over & Low, 1990).

Two issues were raised by Mayer, a cognitive psychologist, and Silver, a mathematics educationist. Mayer (1982b) expressed concern with the poor performances of students in the 1983 and 1988 NAEP studies. He urged that there is a need to establish principles in learning, memory and cognition that are relevant to tasks in mathematics. Silver (1982) felt that more research has to be directed towards the content and organisation of a problem solver's knowledge. Specifically, the area of study has to be related to psychologists' research such as the influence of schemas to knowledge organisation. The present study examines schematic theories and their relevance to problem solving in mathematics.

The Background

Schematic Theories

Thorndyke and Hayes-Roth (1979) defined schema as a cluster of knowledge of concepts and associations between concepts. A knowledge structure is formed which comprises multiple representations of information from different contexts.

Rumelhart and Orthony (1977) also defined schema in a similar way in terms of concepts and their interrelationships. Schemas are data structures which represent the holistic concepts stored in memory. A schema contains the network of interrelationships between different sets of knowledge that constitute a concept.

Relating Schema to Problem Solving

Schematic theory is related to problem solving in that it assumes that a system is first developed in the human mind by acquiring a structure in which problem-solving procedures are integrated with some concepts. The schema is responsible for providing a structure for understanding a problem-solving situation as well as for guiding problem-solving procedures (Anderson, Greeno, Kline & Neves, 1981). To solve a problem, a problem solver is first assumed to encode information in the cognitive domain. A system of production of information is then applied to transform its encoded form to an intermediate form of the problem solving procedures. In the process, subgoals are attained until the goal of the problem is achieved.

Cognitive psychology views learning as the acquisition of knowledge. Mayer (1982d, 1987) explained problem-solving processes as employing different forms of knowledge leading to the goal of solving the problem. According to him, the types of knowledge applied in problem solving consisted of factual knowledge, schematic knowledge, algorithmic knowledge and strategic knowledge. Two processes are involved in problem solving. They are the translations of a problem statement into internal representation and the searching for a path until the goal is achieved.

Silver (1982) is more concerned about the content and organisation of a problem solver's knowledge base. He stated that knowing the background information of a problem solver could help to explain why some instructions are not effective. His view went along with Thorndyke and Hayes-Roth (1979) who explained that schemas have not only helped to interpret and encode incoming information but also to recall previously processed information. The importance of knowledge organisation is also supported by Overtoon (1990) who believed that formal reasoning involved more than just storing and retrieving actual knowledge in the memory. Schemas can influence the type of information recalled from the memory. He emphasised that one area which needs investigation is the mechanisms that integrate new information into the memory.

Research Results Related to Schema

Research results by Mayer (1982b) suggested that students' understanding of problems is influenced by their knowledge of problem schema. He defined a problem schema as a general representation for a class of problems such as 'river current' problems. His study also showed that the students bring with them a knowledge of problem schema. If the problem is unfamiliar, students tend to translate that problem into a slightly different more familiar problem.

Hinsley, Hayes and Simon (1977) carried out a study which linked schema theory and mathematical problem solving. They concluded that the subjects have schemas for standard algebra problems and that these guide the encoding and retrieval of problem information.

Numerous studies on problem-solving performance between experts and novices have been carried out (Chi, Feltovich & Glaser, 1981; Kintch & Greeno, 1985; Simon & Simon, 1978). A general finding is that people skilled in problem solving possess a large body of domain-specific knowledge and schema for the type of problem. Recognition of the pattern or schema helps them to access relevant procedures quickly.

Some studies were carried out according to Silver's thoughts on the organisation of knowledge. Ward, Byrnes and Overtoon (1990) found the importance of organised knowledge which influences reasoning. Similar results were demonstrated in favour of knowledge organisation to facilitate learning (Mannes & Kintch, 1987; Durson & Coggins, 1990 and Kuhara-Kojima & Hatano, 1991).

Relevance to the Present Study

The present study examines the mental processes of problem solvers. Schematic theory involves knowledge acquisition and their interlinking networks in the form of knowledge structure of a particular discipline. The nature of this theory seems to have some implications for the analysis of children's cognitive processes. For example, in the analysis of children's mental thinking, the types of knowledge and the ways they are linked could be identified. The knowledge includes mathematical principles, concepts, techniques, skills and specific strategies in solving a problem.

Mayer's model on the four types of knowledge also has implications to the present study in terms of the method of analysing children's strategies in solving problems. Specifically, schematic and strategic knowledge concepts have direct relevance to the analysis of children's thinking in problem solving. In solving a mathematical problem, different types of strategies are employed by children. The type of strategy applied to solve a problem depends on the existing schema available in each child. In other words, schematic knowledge contributes significantly to children's abilities to solve problems which reflect on their strategies used. Silver's concept of organised knowledge and content has similar implications for the analysis of strategies used by children. Although the present study did not involve teaching children to organise mathematics knowledge, it investigated their responses to mathematical tasks which would reflect on their abilities to organise knowledge and present them in different ways.

Identification of Strategic Models for Solving Ratio and Proportion Problems

Bearing in mind on the importance of knowledge and the ways they are organised (Silver, 1982) and the various types of knowledge required for problem solving (Mayer, 1982b), the study was conducted to achieve the following objectives using the topic ratio and proportion (r/p):

1. to identify the essential knowledge required to solve r/p problems at primary level,
2. to identify the various strategies applied by pupils to solve various r/p problems,
3. to develop a strategic model for solving each r/p problem, and
4. to identify a generic model for solving r/p problems.

Methods and Procedures

(a) Instrument

An instrument which comprises 4 r/p questions (see Appendix 1) was developed. The questions were designed which require children to retrieve different types of information such as the current knowledge acquired and information acquired prior to the current knowledge learned. The questions were first validated in terms of their hierarchical levels using the Guttman scale (reported in Fong, 1992).

(b) Schools and Subjects

The questions were administered to 499 pupils from seven different schools. Two of the schools were classified as above average, 3 were average and two were below average. They were classified into different categories on the basis of the Primary School Leaving Examination results. Out of the total number of pupils selected, 249 were boys and 250 were girls. Using the School Mathematics Examination and Chelsea Ratio and Proportion Diagnostic Test results as criteria for sampling, 149, 219 and 131 pupils were cetegorised as high, average and low ability respectively.

(c) Analysis of Pupils' Responses to the r/p Questions

Children's written responses to the four questions were analysed in a 3-stage sequence. They are:

(i) recording and classifying different strategies employed by pupils in solving the problems,
(ii) identifying the knowledge applied in solving the problems such as concepts, skills, techniques and principles and linking them in linear strategic path for each strategy, and

(iii) conceptualising the overall strategies of each question into strategic model and a conceptual model for the topic.

The following paragraphs summarise the methods of analysing pupils' responses.

(i) <u>Recording and Classifying Strategies from Pupils' Solutions</u>

The children's responses to each question were examined and then classified into different distinct strategies. Sub-categories were also identified from each strategy. The following is an example illustrating the classification of strategy (and sub-category) which was applied to solve Question 1:

Question 1

Some stamps are shared between Weichi and Yihua in the ratio 7:3. Weichi has 184 stamps more than Yihua. All the stamps are then shared equally between them. How many stamps does each person get?

Solution:

Ratio of stamps = 7:3

7-3 = 4

7+3 = 10

1u $= \frac{184}{4}$ ⟵ unitary method

10u = 46 x 10

= 460

Each one gets $\frac{460}{2}$

= 230 ⟵ concept of sharing

Classification of Strategy:

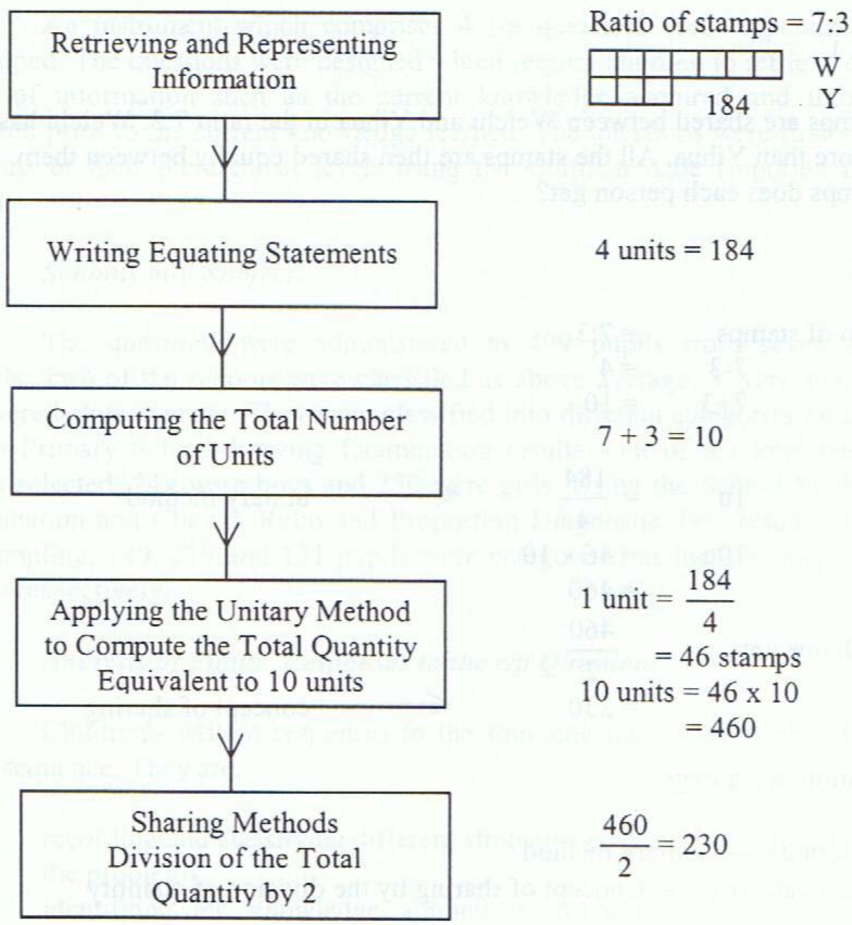
Strategy — unitary method

Sub-category — Concept of sharing by the division of quantity

(ii) Linking Knowledge in a Linear Strategic Path

From each classified strategy, the knowledge employed in solving the problem was first identified. The knowledge consisting of various skills, concepts, principles and techniques was then linked in a linear path connecting them. Pupils' written statements associated with the knowledge were placed adjacent to each item of knowledge applied. Figure 1 is an example showing the strategic path linking the knowledge in linear form.

Fig. 1: Linear Strategic Path Linking a Set of Knowledge



From the example, notice the list of items of knowledge identified are retrieving and representing information from an external source, writing equating statements, computing a number of certain units from the given units, applying the unitary method to compute the total quantity and the concept of sharing using division. An example is shown adjacent to each item of the knowledge identified. For example, 4 units = 184 is an equating statement linking the computed units and the given quantity 184.

(iii) Strategic Models for Solving Ratio and Proportion Problems

From the list of identified knowledge of Question 1, an analysis was carried out to check if a common pattern could be deduced which represented the knowledge in the form of a network of skills. This was first done by assigning a symbol to each skill identified. Table 1 shows the use of symbols representing all the skills identified in Question 1 and the paths of strategies connecting the sets of skills.

Table 1. Path of Each Strategy Connecting a Set of Skills

a-b-d-e	a-Retrieving & representing information
a-c-b-d	b-Computing ratio units
a-d-b-e	c-Sharing of parts/units
a-d-e-g	d-Writing equating statements
	e-Applying ratio & proportion concept
	g-Concept of sharing

The linear path "a-b-d-e" refers to a specific strategy used to solve Question 1. The letters a, b, d and e are skills applied in sequence to solve the problem. The example above shows that different sets of skills may be used to solve the same problem.

Results of Analysis

Similar method of analysis (i.e. i, ii and iii above) was applied to analyse Questions 2, 3 and 4. The analysis of pupils' solutions revealed that various strategies were employed in solving each question. In each strategy, some variations were also observed. Table 2 summarises all the strategies and sub-categories of strategies identified in Questions 1 to 4 of the r/p problems.

Table 2. Strategies and Sub-categories of Strategies

Identified in Questions 1 to 4

Question 1

Strategy	Sub-category
1. Unitary Method	a. Concept of sharing by quantity (of stamps). b. Part-whole concept (difference) in finding the total quantity. c. Concept of sharing by units. d. Part-whole concept (addition) in finding the total quantity. e. Sharing of the difference of parts approach.
2. Proportion Method	a. Part-whole concept (difference) in finding the quantity. b. Concept of sharing by quantity (of stamps). c. Concept of sharing by units. d. Part-whole concept (addition) in finding the total quantity. e. Sharing of the difference of parts approach.
3. The Building-up Strategy	

Question 2

Strategy	Sub-category
1. Unitary Method	a. Using the symbol π b. Using the value of π as 3.14. c. Using the value of π as $\frac{22}{7}$.
2. Proportion Method	a. Using the symbol π. b. Using the value of π as 3.14.

3. Multiplicative Proportion Method	a.	Using the building-up strategy.
	b.	Using the building-up strategy and the inverse multiplication strategy.
	c.	Using the equivalent fraction strategy.

Question 3

Strategy		Sub-category
1. The Multiplicative Method	a.	Using the ratio approach.
	b.	Using the fraction approach.
2. Division of Parts	a.	Using the unitary method.
	b.	Using the proportion method.
	c.	Using the two equivalent ratios approach.
3. A Short-cut Strategy	a.	Using the unitary method.
	b.	Using the building-up strategy.
4. Intuitive (insight) Strategy		
5. LCM Concept Method		

Question 4

Strategy		Sub-category
1. Finding the Area of Whole Circle Strategy	a.	Using the unitary method, % and proportion concept.
	b.	Using the unitary method and set concept.
	c.	Using the proportion method and % concept.
	d.	Using proportion method and set concept.
	e.	Using the part-whole concept in % context.
2. Finding the Area of Part-circle Strategy	a.	Finding the % of non-overlapping part of first circle.
	b.	Finding the % of non-overlapping part of second circle.

3. Equivalent Ratio Strategy
   a. Increasing the building-up ratio approach.
   b. Reducing the building-up ratio approach.

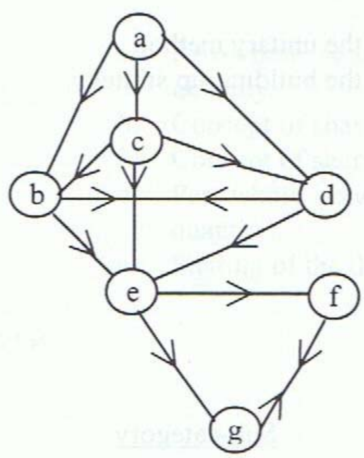
4. Mixed Strategy

5. Algebraic Strategy

Symbolic letters were used to represent the skills of all the strategies. Pooling all the strategies together which were in the form of linking paths, a network of skills connected by paths was drawn to represent the strategic model for the question. Figure 2 shows the strategic models for solving Questions 1 to 4.

Fig. 2 : Strategic Models Represented by Network Paths

<u>Strategic Model of Question 1</u>

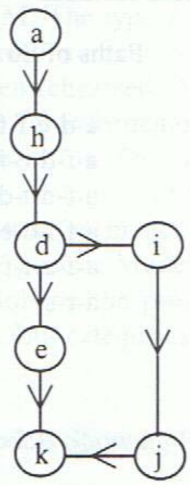


Paths of Strategies

a-b-d-e
a-c-b-d
a-d-b-e
d-b-e
a-d-e-g
d-e-f-g
a-d-b-e-g
a-d-e-f-g
a-d-e-f-f-g
a-d-e-g-f
a-d-c-e-e

Fig. 2 (cont'd)

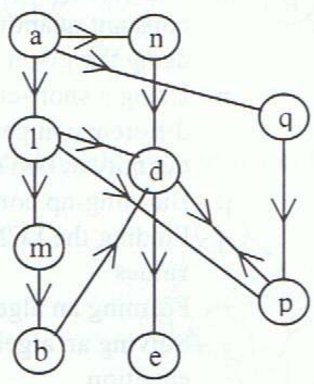
Strategic Model of Question 2



Paths of Strategies

a-h-d-e-k
a-h-d-i-j-k

Strategic Model of Question 3

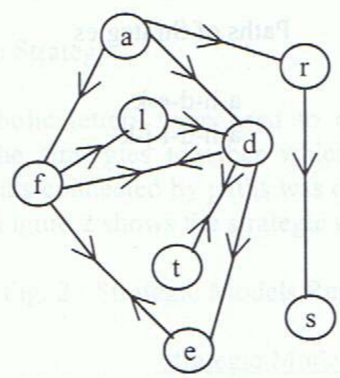


Paths of Strategies

a-l-m-b-d-e
a-n-d-e
a-l-p-d-b
a-q-p-d-e

Fig. 2 (cont'd)

Strategic Model of Question 4



Paths of Strategies

a-d-e-f-f
a-f-d-e-f
a-f-d-t-d-f-f
a-f-l-d-e-f
a-f-e-f-f
a-r-s

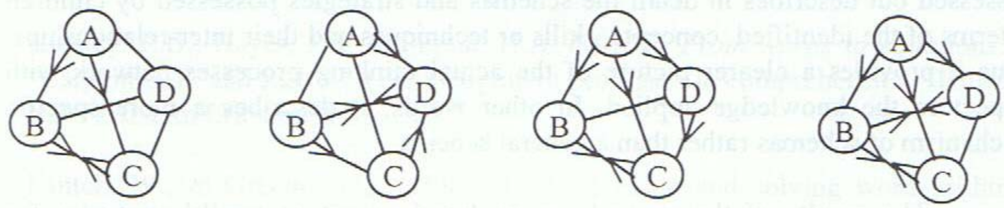
Note:

a - Retrieving & representing information
b - Computing ratio units
c - Sharing parts/units
d - Writing equating statements
e - Applying ratio and proportion concept
f - Using addition in part-whole concept
g - Concept of sharing
h - Computing the area of a circle
i - Finding the multiplication or inverse multiplication factor from a proportion statement
j - Applying the multiplication factors or its inverse to compute the ratios of 2 quantities
k - Finding the radius of a circle
l - Expressing ratios of equivalent ratios, fractons or parts
m - Comparing 2 ratios and fixing a constant quantity between them using the given data
n - Using a short-cut to compute the difference in parts between a quantity at two situations
p - Building-up concept
q - Finding the LCM of 2 ratio values
r - Forming an algebraic equation
s - Solving an algebraic linear equation
t - Building up % concept

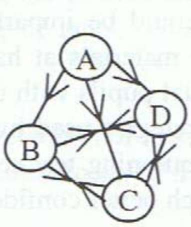
Re-examining all the skills (from skill a to skill t) identified in all the four ratio and proportion questions, four common categories of information were further re-categorised from them. The four categories were: (A), the retrieval and representation of information from an external source; (B), skills pertaining to a directly related topic (ratio and proportion); (C), skills classified under type A information of the Long Term Memory (LTM) and (D), skills classified under type B information of the LTM. The type A information refers to the set of knowledge which is derived from the content of area of a topic under consideration or information which is often rehearsed. Any other related topics are classified under type B operation. Type A information is assumed to be readily retrieved as compared to type B information. The conversion was done by transforming all the skills (a to t) from all the strategies of the four r/p questions to the four categories A to D. As a result, a strategic model of each ratio and proportion question was developed followed by a generic Model for solving ratio and proportion problems (See Figure 3 below). Solving the problems involved the interlinking of relevant information among these four categories.

Fig. 3 : Pyramidal Models Showing Paths Connecting Categorical Information

Question 1 Question 2 Question 3 Question 4



General Model for Ratio and Proportion Problems



Discussion

A set of strategic models was identified from the analysis of the four r/p problems. Each model represents all the identified strategies applied by the subject to a question. They are described as a network of interlinking knowledge or strategic schemas. Results of the analysis of each model show that a variety of types of knowledge were applied such as skills, concepts and techniques. One essential skill which is found to be pivotal to the problem is forming equating statements. This is applied in all the strategies for solving the four r/p problems. Other knowledge applied are classified as the type A and type B information. The analysis seems to imply that domain-specific knowledge is indispensable in problem solving. This appears to support the theory of knowledge which describes the essence of knowledge and its organisation in problem solving (Greeno, 1980; Silver, 1982; Simon & Simon, 1980).

The schematic and strategic knowledge described by Mayer had some weaknesses as they could not probe more deeply into the internal functioning of children. The present study does not talk about what Mayer's knowledge they possessed but describes in detail the schemas and strategies possessed by children in terms of the identified concepts, skills or techniques and their inter-relationships. Thus it provides a clearer picture of the actual thinking processes network with respect to the knowledge applied. In other words, it describes a more specific mechanism of schemas rather than a general schema.

The results of the study have produced a set of pupils' strategies for solving each of the r/p problems. For each strategy, the knowledge retrieved in solving the problem is also identified showing how they are linked in a form of strategic path. The sets of model could be imparted to trainee teachers studying mathematics education. With these materials at hand, they have the flexibility to apply the right strategies to individual pupils with different abilities. Besides, some teachers may encounter different strategies used by pupils but are quite hesitant to accept their responses. However, equipping teachers with the strategic models for solving r/p problems gives them much better confidence to handle similar incidents.

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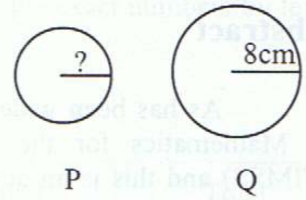
Appendix 1

Ratio and Proportion Questions:

1. Some stamps are shared between Weichi and Yihua in the ratio 7:3. Weichi has 184 stamps more than Yihua. All the stamps are then shared equally between them. How many stamps does each person get?

2. P and Q are two circles. The ratio of the area of circle P to the area of circle Q is 9:16. The radius of circle Q is 8 cm. What is the radius of circle P?



3. In January, the ratio of the weight of Fei-fei to Tin-tin is 5:2. One year later, Tin-tin's weight has increased by 14 kg while Fei-fei's weight remains the same. If Fei-fei's weight to that of Tin-tin's becomes 4:3, what is Fei-fei's weight?

4. Circles P and Q overlap at R. The overlapping part represents 36% of circle P and 18% of circle Q. If the area of R is 54 sq cm, what is the total area of the non-overlapping parts?