Problem Solving  
Heuristics and Thinking Skills  

Dr Ng Wee Leng  
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Example 1

In a class, all 40 pupils are asked to indicate if they play racket games, badminton and tennis. Given that 25 of the pupils play badminton, 16 of them play tennis and 5 play neither, how many pupils play both games?

Solution 1

Suppose (make a supposition) no pupil plays both games, then total number of pupils would be 25+16+5=46 (verification). This is inconsistent with the information that there are 40 pupils in the class (verifying).

Suppose (make a supposition) only 1 pupil plays both games, then number of pupils playing only badminton is 24 and number of pupils playing only tennis is 15. This would give a total of 24+1+15+5 = 45. This is still inconsistent (verifying).

Continuing in the same way, the following table is obtained:

<table>
<thead>
<tr>
<th>Number of pupils playing both games</th>
<th>Number of pupils playing only badminton</th>
<th>Number of pupils playing only tennis</th>
<th>Number of pupils playing neither</th>
<th>Total number of pupils</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>25</td>
<td>16</td>
<td>5</td>
<td>46</td>
</tr>
<tr>
<td>1</td>
<td>24</td>
<td>15</td>
<td>5</td>
<td>45</td>
</tr>
<tr>
<td>2</td>
<td>23</td>
<td>14</td>
<td>5</td>
<td>44</td>
</tr>
<tr>
<td>3</td>
<td>22</td>
<td>13</td>
<td>5</td>
<td>43</td>
</tr>
<tr>
<td>4</td>
<td>21</td>
<td>12</td>
<td>5</td>
<td>42</td>
</tr>
</tbody>
</table>

Pupils can proceed to arrive at the solution by completing the table systematically (make a systematic list).
Alternatively, they can observe the pattern at the first and last columns (look for pattern).
Corresponding numbers in both columns add up to 46.
Since the total number of pupils in the class is 40, the number of pupils playing both games must be (46 - 40) or 6 (verification).

Solution 2

Since 5 pupils play neither, the number of pupils who play at least one of the games is 35 (verification).
The sum of 25 and 16 gives 41. The difference (41 - 35) = 6 therefore gives the number of pupils playing both games since they have been counted twice in the sum 25 + 16 = 41 (verification, spatial visualisation).

Alternatively, use a Venn diagram (as a graphic organizer) to help pupils visualize these information (use a diagram).
From the diagram, we see that the answer is 6 (verification).
**Example 2**

Investigate the results obtained when the difference between a given two-digit number and the number obtained when the same two digits are reversed.

**Solution**

Suppose (make a supposition) the number given is 28. The number with the digits reversed is 82. Difference between 28 and 82 is (82-28)=54

Trying another number, say 37, gives the difference as (73-37)=36 (guess and check).

Repeating this process (generalising) with other numbers we obtain the following table:

<table>
<thead>
<tr>
<th>Given number</th>
<th>Digits reversed</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>82</td>
<td>54</td>
</tr>
<tr>
<td>37</td>
<td>73</td>
<td>36</td>
</tr>
<tr>
<td>12</td>
<td>21</td>
<td>9</td>
</tr>
<tr>
<td>46</td>
<td>64</td>
<td>18</td>
</tr>
<tr>
<td>10</td>
<td>01</td>
<td>9</td>
</tr>
<tr>
<td>66</td>
<td>66</td>
<td>0</td>
</tr>
<tr>
<td>19</td>
<td>91</td>
<td>72</td>
</tr>
</tbody>
</table>

The pattern (looking for patterns) that can be observed indicates that all the numbers in the last column are divisible by 9.

Hence, the conclusion (induction) is:

The difference between any given two-digit number and the number when the two given digits are reversed is always a multiple of 9.

Proof:

A two-digit number $xy$ can be expressed as $10x + y$. The number, when the digits are reversed, is $yx = 10y + x$.

The difference is given by

$$(10x+y)-(10y+x)=9x-9y= 9(x-y),$$

a number which has a factor 9.

**Example 3**

A group of pupils are planning to build a triangular pyramid which is at least 1m high using identical cylindrical cans. Given that each of this cylindrical can has a height of 12 cm, calculate the minimum number of the cylindrical cans needed to build this triangular pyramid.

**Solution**

Since the height of the pyramid must be at least 1m, we shall first find out the minimum number of tiers of cans required (solve part of the problem).

$$(\text{height of pyramid}) \div (\text{height of each cylindrical can}) = 100/12 = 8.3$$

Therefore, minimum number of tiers required = 9
Consider a simpler case of a 4-tier triangular pyramid (simplify the problem). Using some buttons or coins to represent the cans (act it out), the following is observed:

[If you draw the diagram above instead of using concrete objects, you are using the heuristic use a diagram.]
Thus, we have the pattern (look for patterns):

\[
1 + 3 + 6 + 10 + 15 + 21 + 28 + 36 + 45 = 165
\]
Thus, the total number of cans required to build the pyramid = 165

**Example 4**

Three coin boxes A, B and C contain a total of $243. When some money from box A is transferred to box B, the amount of money in box B increases by 50%. Next, some money is transferred from box B to box C so that the money in box C increases by 50%. Finally, when some money from box C is transferred to box A, the money in box A also increases by 50%. In the end, all three boxes contain the same amount of money. How much money does each box contain originally?

**Solution**

The last row in the table is completed first. Since the three boxes contain the same amount of money after the transfer, the amount in each box is \( \$ \frac{243}{3} = \$ \text{81} \). This is then followed by the completion of the 2nd last row.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Start</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>After A → B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Before B → C</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>After B → C</td>
<td>$54</td>
<td>$81</td>
<td>$108</td>
</tr>
<tr>
<td>Before C → A</td>
<td>$\frac{2}{3} \times $81 = $54</td>
<td>$81</td>
<td>$108</td>
</tr>
<tr>
<td>After C → A</td>
<td>$81</td>
<td>$81</td>
<td>$81</td>
</tr>
</tbody>
</table>

After transferring money from box C to box A, amount in box A increases by 50%. Before the transfer, box A contains $54 and box C contains $108. This situation is the same as after money has been transferred from box B to box C. Following this line of reasoning (use before-after concept and work backwards), the problem can be solved.

**Example 5**

Five school teams (Team A, Team B, Team C, Team D and Team E) took part in a round-robin badminton tournament (i.e. every team played against every other team exactly once). There were no draws in this tournament. Suppose that Team A won \( a \) games, Team B won \( b \) games, and so on, as shown below:

<table>
<thead>
<tr>
<th>Team</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of games won</td>
<td>( a )</td>
<td>( b )</td>
<td>( c )</td>
<td>( d )</td>
<td>( e )</td>
</tr>
</tbody>
</table>

Find the value of \( a + b + c + d + e \).
Solution

Since there was no draw in any of the games, each win corresponds to a game played. Hence, the problem of finding the value of \(a+b+c+d+e\) can be restated as finding the total number of games played in the tournament (restate the problem in another way).

To find the total number of games played, the games between each pair of teams are listed systematically and represented as follows:

- \(AB, AC, AD, AE\) Number of games = 4
- \(BC, BD, BE\) Number of games = 3
- \(CD, CE\) Number of games = 2
- \(DE\) Number of games = 1

Total number of games played in the tournament = \(4+3+2+1=10\)

Note: This problem can be extended to the general case when \(n\) teams play in the tournament. The solution in this general case is equivalent to finding the algebraic form, in terms of \(n\), of the sum \(1+2+3+4+5+6+7+\ldots+n\)

Thinking Skills in Our Mathematics Curriculum

- Classifying
  - Using relevant attributes to sort, organize and group information
  - list the similarities between two or more items, group items according to their commonalities
  - identify accurately the attributes of each group
  - label the groups
  - explore the different ways the items can be grouped

- Comparing
  - Using common attributes to identify commonalities and discrepancies across numerous sets of information
  - Identify significant characteristics as the basis of comparison.
  - List similarities / differences between two or more items.
  - Draw conclusions from similarities and difference noted.

- Sequencing
  - Placing items in a hierarchical order according to a quantifiable value
  - assign a quantifiable value to items
  - compare items
  - arrange items in hierarchical order