Teachers Notes with Worksheet

Binomial Theorem And Its Applications (Using Maple)

Notes:
1) Teacher will alternate whiteboard teaching with CAS so as to let the students know both manual and computer methods of solving and applying binomial method.
2) Assume that students have already been introduced to Maple and are conversant with the basic commands and syntax.

Instructions
1. Open a new worksheet in Maple.
2. Make sure your floppy disk is inserted and save your class work as Binomial Theorem. (Be sure to save your work periodically and at the end to avoid losing your work).
3. Write down all the outputs from the commands keyed in in the spaces provided in the worksheet.

CLASS WORK

(Teacher will recap quadratic expansions on whiteboard and ask students to expand manually first.)

(A) Expanding powers of Binomials

Type \texttt{expand((a+b)^2);}

\[ a^2 + 2a b + b^2 \]

Type \texttt{expand((a+b)^3);}

\[ a^3 + 3a^2 b + 3a b^2 + b^3 \]

Type \texttt{expand((a+b)^4);}

\[ a^4 + 4a^3 b + 6a^2 b^2 + 4a b^3 + b^4 \]
(Teacher will point out that the above expansions are in binomial form and define what binomial means i.e. there are only two variables)

(Teacher will ask students to identify the pattern of binomial coefficients for above expansions)

(B) To find Binomial coefficients

The coefficients in the expansion of \((a + b)^n\) are called **binomial coefficients**. They can be obtained using the Maple procedure \(\text{binomial}(n,r)\).

For example, for \((a+b)^4\)

Type \(\text{binomial}(4,0)\);

1

Type \(\text{binomial}(4,1)\);

4

Type \(\text{binomial}(4,2)\);

6

Type \(\text{binomial}(4,3)\);

4

Type \(\text{binomial}(4,4)\);

1

Type \(\text{seq(binomial}(4,i),i=0..4)\);

1, 4, 6, 4, 1

Instead of \(\text{binomial}(n,r)\), a common mathematical notation for this is \(C(n,r)\), so we can assign a term \(C\) to the binomial function (defined by Maple), also known as giving it an alias.

(Note: Teacher should not delve into the combinatorial definition of \(C(n,r)\) and just ask students to accept the formula as it is in the binomial expansion)

Type \(\text{alias}(C=\text{binomial})\);
Find the sequence of coefficients for \((a+b)^5\)

Type `seq(C(5,i),i=0..5);`

`1, 5, 10, 10, 5, 1`

In general, the **coefficient** of \(a^{n-r}b^r\) in the expansion of \((a+b)^n\) is the **number of ways of selecting** \(r\) **objects from** \(n\) which is given by the formula:

\[
\frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+2)(n-r+1)}{r!}
\]

To define this function for the binomial coefficients in Maple using the alphabet \(C\), we have to first clear the function assigned to \(C\) (in this case, the binomial function defined by Maple).

(Teacher will define the binomial theorem in its full form to the students. Teacher can also mention in passing that \(n\) need not be an integer but that it is out of syllabus for secondary school)

Type `restart;`

Type `C:=(n,r)->n!/(r!*(n-r)!);`

\[
C := (n, r) \rightarrow \frac{n!}{r! (n-r)!}
\]

Try `C(5,2);`

`10`

Try `C(500,10);`

`245810588801891098700`

This is **FUN.** Try It!!!

* To calculate a few rows from
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(C) Comparing Two Loan Plans
Suppose you need a loan of $50,000 for a period of 3 years. A loanshark and a bank offer two different options from which you may choose:

(a) Loanshark - 2% monthly interest rate, with interest compounded monthly
(b) Bank - 24% annual interest rate, with interest compounded annually

Let the relevant parameters be:

$P =$ principal investment (starting amount) in dollars
$r =$ annual percentage rate (expressed as a decimal between 0 & 1)
$n =$ number of times per year interest gets compounded (so $n=12$ for monthly and $n=4$ for quarterly)
$t =$ years passed since the initial amount was invested

Given that the balance after $t$ years is:

$$B(t) = P(1+r/n)^{nt}$$

**Step 1**

We define 2 functions $BLoanshark$ and $BBank$, where

$BLoanshark(t) =$ balance after $t$ years using option (a)

$BBank(t) =$ balance after $t$ years using option (b)

Type $BLoanshark := t -> 50000*(1+0.24/12)^{(12*t)}$;

$BLoanshark := t \rightarrow 50000 1.020000000^{(12 \cdot t)}$

Type $BBank := t -> 50000*(1+0.24/1)^(1*t)$;

$BBank := t \rightarrow 50000 1.24^t$

**Step 2**

To see which plan is better, we look at the graphs of these 2 functions.

Type $plot([BLoanshark(t),BBank(t)], t=0..3, color=[red,blue]);$


Step 3

To find the exact balance at the end of 3 years:

Type `BLoanshark(3)`; for option (a)

```
101994.3672
```

Type `BBank(3)`; for option (b)

```
95331.20000
```

Step 4

To find the exact difference at the end of 3 years,

Type `BLoanshark(3)-BBank(3)`;

```
6663.16720
```

Conclusion

Option (b) is the better option because the amount you have to pay back to the bank is $6663.17 lesser than what you have to pay the loanshark after 3 years.
HOME ASSIGNMENT

Instructions

1. You may start this assignment once you have finished all the class work and complete it for homework.
2. Open a new worksheet in Maple and save your assignment as BinomialTheorem14 in your floppy disk.
3. Print out a copy of the worksheet to hand in with your diskette.
4. Your complete assignment should include your recommendation for which plan to choose. Include your reason(s) for your choice.
5. Hand in your assignment by 2.00 pm on Tuesday, 14 November 2001.

Assume that you want to make an initial investment of $500,000.

Plan 1

You invest your money in CPF. You are offered an interest rate of 4% per annum. Your returns are tax-free.

Plan 2

You invest your money in Unit Trusts. You are offered an interest rate of 8% per annum, but your returns are taxed at a rate of 33%.

Based on the codes used in the previous example, your task is to calculate the returns and recommend the better plan with reasons for the following time periods:

(i) 4 years (Manually using the Binomial Method)
(ii) 10 years (Using Maple codes with the plotted graphs)
(iii) 20 years (Using Maple codes with the plotted graphs)