Mathematical Modelling – From Abstract to Practical

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Outline

1 Introduction
Outline

1. Introduction

2. Key features of mathematical modelling
Outline

1. Introduction
2. Key features of mathematical modelling
3. Abstract
Outline

1. Introduction
2. Key features of mathematical modelling
3. Abstract
4. Practical
Outline

1. Introduction
2. Key features of mathematical modelling
3. Abstract
4. Practical
5. Complexity

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Mathematical Modelling – From Abstract to Practical
Question 1

What is mathematical modelling?

Do we all have the same understanding of this term?
Question 2

Why mathematical modelling?
Question 3

What is the role of mathematical modelling in mathematics?
Question 4
What is the role of mathematical modelling in mathematics education?

http://www.emcl.kit.edu/
Question 5
What are some of the ways in which mathematical modelling can be implemented?
Aims

1. Explore mathematical modelling along two different lines: abstract vs practical.
Aims

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2. Make explicit those characteristics of modelling processes associated to each type.
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1. Explore mathematical modelling along two different lines: abstract vs practical.

2. Make explicit those characteristics of modelling processes associated to each type.

3. Supply with a spectrum of modelling endeavors and seek educational relevance in these.
What is a mathematical model?

Definition

A **mathematical model** is a description of a system using mathematical concepts and language.
What is a mathematical model?

Definition

A mathematical model is a mathematical structure used to derive logic and reasoning in another mathematical system.
What is mathematical modelling?

The process of manufacturing a mathematical model is termed mathematical modelling.
What is a mathematical modelling endeavor?

A mathematical experience refers to an opportunity, a context or a task which requires mathematical modelling.
Vending machine

A typical can-drinks vending machine accepts coins of different denominations, and upon receiving the user-supplied choice of soft drinks, together with the correct payment, dispenses the correct product.

There are several different products, not necessarily equally priced.
A soft-drink company organizes a charity activity in which each participant is given a set of $N$ coins with randomly distributed denominations.
A soft-drink company organizes a charity activity in which each participant is given a set of \( N \) coins with randomly distributed denominations.

The coins are then inserted into a prototypical vending machine.
Examples

**Vending machine**

- If the amount inserted is sufficient, the participant gets to choose the product he or she desires. Otherwise, the participants tops up with his or her own money for the desired product. This top-up is then donated to charity.
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Vending machine

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*Design* a feasible distribution of the denominations for each participant, giving a suitable fixed number $N$. 
Examples

Snowplow deployment in Maryland

In the county of Wicomico in the state of Maryland, snowplows are deployed to clear snow after heavy snowstorms.
Examples

Snowplow deployment in Maryland

In a particular scenario, two snowplows are dispatched respectively from a garage about 4 miles west of each of the two points demarcated by *’s on the dual-carriageway layout map below:

Propose an efficient deployment scheme for the snowplows in clearing the snow?
Some state water-right agencies require from communities data on the rate of water use, in gallons per hour, and the total amount of water used each day. Many communities do not have equipment to measure the flow of water in or out of the municipal tank. Instead, they can measure only the level of water in the tank, within 0.5% accuracy, every hour.
Water tank flow

More importantly, whenever the level in the tank drops below some minimum level \( L \), a pump fills the tank up to the maximum level, \( H \); however, there is no measurement of the pump flow either. Thus, one cannot readily relate the level in the tank to the amount of water used while the pump is working, which occurs once or twice per day, for a couple of hours each time.
Examples

Water tank flow

*Estimate* the flow out of the tank $f(t)$ at all times, even when the pump is working, and estimate the total amount of water used during the day.
Examples

Water tank flow

Table 1 gives real data, from an actual small town, for one day.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Height (0.01 feet)</th>
<th>Time (s)</th>
<th>Height (0.01 feet)</th>
<th>Time (s)</th>
<th>Height (0.01 feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3175</td>
<td>35932</td>
<td>Pump operating</td>
<td>68535</td>
<td>2842</td>
</tr>
<tr>
<td>3316</td>
<td>3110</td>
<td>39332</td>
<td>Pump operating</td>
<td>71854</td>
<td>2767</td>
</tr>
<tr>
<td>6635</td>
<td>3054</td>
<td>39435</td>
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<td>75021</td>
<td>2697</td>
</tr>
<tr>
<td>10619</td>
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<td>43318</td>
<td>3445</td>
<td>79154</td>
<td>Pump operating</td>
</tr>
<tr>
<td>13937</td>
<td>2947</td>
<td>46636</td>
<td>3350</td>
<td>82649</td>
<td>Pump operating</td>
</tr>
<tr>
<td>17921</td>
<td>2892</td>
<td>49953</td>
<td>3260</td>
<td>85968</td>
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<td>2850</td>
<td>53936</td>
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<td>89953</td>
<td>3397</td>
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<td>57254</td>
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<td>93270</td>
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<td>28543</td>
<td>2752</td>
<td>60574</td>
<td>3012</td>
<td>32284</td>
<td>2697</td>
</tr>
<tr>
<td>64554</td>
<td>2927</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Examples

Water tank flow

The table gives the time, in, since the first measurement, and the level of water in the tank, in hundredths of a foot. For example, after 3316 seconds, the depth of water in the tank reached 31.10 feet. The tank is a vertical circular cylinder, with a height of 40 feet and a diameter of 57 feet. Usually, the pump starts filling the tank when the level drops to about 27.00 feet, and the pump stops when the level rises back to about 35.50 feet.
Mathematical modelling is characterized by the following cycle:

1. **Real World Problem**
2. **Mathematical Problem**
3. **Real World Solution**
4. **Mathematical Solution**

- **Formulation**: From Real World Problem to Mathematical Problem.
- **Solve**: From Mathematical Problem to Mathematical Solution.
- **Interpretation**: From Real World Solution to Real World Problem.

### Modelling cycle
- Variables and objective functions
- Classification of models
- A priori information

### Nature of Mathematical Modelling

Mathematical modelling is characterized by the following cycle:
In general, the original system in the problem involves different kind of variables:

- decision variables (independent variables)
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- decision variables (independent variables)
- input variables
- state variables (dependent variables)
- exogenous variables (parameters or constants)
- random variables, and
- output variables
Objectives and constraints of the system and its users can be represented as functions of the output variables or state variables. The objective functions will depend on the perspective of the model’s user.
There are a few variables in this modelling experience:

- The number of coins, \( N \)
- The distribution of coin denominations
- The price of each available product
- The reachability of a purchasing state
- The operational mode of the vending machine, e.g., does it give a change if the payment exceeds the price of the product?
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**Vending machine**

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Vending machine

Question

What do we aim to achieve when we solve this design problem?
Vending machine

**Question**
What do we aim to achieve when we solve this design problem?

**Objective function**
Assigning to each denomination distribution a state-reachability output together with a probabilistic measure.
Different classifications of models
Different classifications of models

1. Linear vs nonlinear
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1. Linear vs nonlinear
2. Deterministic vs probabilistic
Different classifications of models

1. Linear vs nonlinear
2. Deterministic vs probabilistic
3. Static vs dynamic
Different classifications of models

1. Linear vs nonlinear
2. Deterministic vs probabilistic
3. Static vs dynamic
4. Discrete vs continuous
Different classifications of models

1. Linear vs nonlinear
2. Deterministic vs probabilistic
3. Static vs dynamic
4. Discrete vs continuous
5. Deductive vs inductive
This problem can be formulated using a graph-theoretic model. Because the distances between junctions are important in the consideration of efficiency in this problem, it is best to employ weighted graphs. Thus, the model chosen is
Snowplow deployment in Maryland

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- discrete, and
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This problem can be formulated using a graph-theoretic model. Because the distances between junctions are important in the consideration of efficiency in this problem, it is best to employ weighted graphs. Thus, the model chosen is

- deterministic
- static
- discrete, and
- deductive.
Water tank flow

This problem demands a solution for the rate of water flow $f(t)$, from which the volume $V(t)$ of water in the tank at any time $t$ can be determined. A fairly direct approach is to figure out the variational plot of the height of the water level against time.
Water tank flow

Such a mathematical model is non-linear (depends on the context) dynamic, continuous, and inductive.
Water tank flow

Such a mathematical model is
- non-linear (depends on the context)
Such a mathematical model is

- non-linear (depends on the context)
- dynamic
Such a mathematical model is

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- dynamic
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Such a mathematical model is
- non-linear (depends on the context)
- dynamic
- continuous, and
- inductive.
Black box vs white box

Different models work on varying degree of a priori information, i.e., available information which is independent of experience.

Black box model: no a prior information at all
White box model: full a priori information
We start with the white box model.
Modelling in abstract contexts

We start with the white box model.
We begin with a model of computation.
We start with the white box model. We begin with a model of computation.

**Term model for typed \( \lambda \)-calculus**

The raw terms of the calculus can be given by the *Backus Naur Form*, i.e., BNF below:

\[
M ::= x \mid \lambda x. M \mid M(M) \mid n \mid \\
    \text{succ}(M) \mid \text{pred}(M) \mid \text{ifz}(M, M, M) \mid \text{fix}(M)
\]
The operational semantics of this language is given by

\[
\begin{align*}
\text{(VAR)} & \quad x \Downarrow x \\
\text{(VAL)} & \quad V \Downarrow V \\
\text{(APPL)} & \quad M \Downarrow \lambda x. T \quad T[N/x] \Downarrow V \\
\text{(Succ)} & \quad M \Downarrow n \\
\text{(Pred)} & \quad M \Downarrow 0 \\
\text{(Pred)} & \quad M \Downarrow n + 1 
\end{align*}
\]
The operational semantics of this language is given by

\[
\begin{align*}
M \Downarrow 0 & \quad M_1 \Downarrow V \quad (\text{IF-ZERO}) \\
\text{ifz}(M, M_1, M_2) \Downarrow V & \quad M \Downarrow n + 1 \quad M_2 \Downarrow V \quad (\text{IF-ZERO}) \\
M(\text{fix}(M)) \Downarrow V & \\
\text{fix}(M) \Downarrow V \quad (\text{FIX})
\end{align*}
\]
Addition & Factorial

The term-model can be implemented on the functional language Haskell as follows:

```haskell
add :: Int -> Int -> Int
add x y = add (x + 1) (y - 1)

fact :: Int -> Int
fact n = if (n == 0) then 1 else fact (n-1)
```

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Mathematical Modelling – From Abstract to Practical
Program correctness and equivalence

- How can one verify if the given program works?
Program correctness and equivalence

- How can one verify if the given program works?
- How can one tell if two programs are operationally equivalent?
The term model has too much a priori information; in fact, the full information describes how the machine runs.
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**Solution**

- Manufacture a mathematical model for the term calculus
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- Manufacture a mathematical model for the term calculus
- Model must be fairly abstract, i.e., does not depend on the operational semantics ☩
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Solution

- Manufacture a mathematical model for the term calculus
- Model must be fairly abstract, i.e., does not depend on the operational semantics
- Model must be accurate enough to capture the essential aspects of the calculus
D.S. Scott (late 1960’s) invented the Scott model to interpret data types as topological orders (called domains)
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1. data types as topological orders (called **domains**)
2. data elements as the elements of domains
Denotational semantics

D.S. Scott (late 1960’s) invented the Scott model to interpret

1. data types as topological orders (called domains)
2. data elements as the elements of domains
3. functional programs as continuous functions between domains
Denotational semantics

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1. data types as topological orders (called domains)
2. data elements as the elements of domains
3. functional programs as continuous functions between domains

This is an example of a ‘grey’ and deductive model (based on Scott’s theory).
Modelling computation

In any computation, every finite part of the output depends on only a finite part of the input. The central theme is:

**Continuity.**
Surprising fact

For any terms $M, N : \sigma$,

$$\llbracket M \rrbracket = \llbracket N \rrbracket \implies M = N.$$  

The converse does not hold.
Question
What is a real number?
Real exact arithmetic

There are nine-thousand answers to this question. Amongst these, we pick out three:
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**Dedekind cuts**

In mathematics, a Dedekind cut, named after Richard Dedekind, is a partition of the rational numbers into two non-empty parts $A$ and $B$, such that all elements of $A$ are less than all elements of $B$, and $A$ contains no greatest element.
Every real number can be represented by a floating point decimal.

\[ \sqrt{2} = 1.41421356237309504880168872420969807856967187... \]
Real exact arithmetic

Decimal

Every real number can be represented by a floating point decimal.

$$\sqrt{2} = 1.41421356237309504880168872420969807856967187...$$

Warning

Problematic representation for computers!
Theorem

The simple scaling function

\[ f : \mathbb{R} \rightarrow \mathbb{R}, \ x \mapsto 9x \]

can never be realized by any program

\[ \overline{f} : (10)^{\omega} \rightarrow (10)^{\omega}. \]
Real exact arithmetic

Signed bit streams

Every real number can be represented by signed bit stream:

\[ \{-1, 0, 1\}^\omega. \]
Real exact arithmetic

Example

\[
\left[0: -1: 1^\omega\right] = 0 \cdot \frac{1}{2^1} + (-1) \cdot \frac{1}{2^2} + 1 \cdot \sum_{k=3}^{\infty} \frac{1}{2^k} = 0
\]
The negative truth

In actual fact, what we are proposing to be $\mathbb{R}$ are not $\mathbb{R}$!
The negative truth

In actual fact, what we are proposing to be $\mathbb{R}$ are not $\mathbb{R}$!

The positive truth

They are all models for $\mathbb{R}$!
Real exact arithmetic

**Surprising fact**

Derivatives are not computable but Riemann integrals are!
Rubik’s cube

This physical gadget needs no introduction.
Modelling the moves

Name each 90° clockwise turn of these faces to be

\[ T, B, F, P, L, R. \]
To understand the configurations of the Rubik’s cube (i.e., the position of each of the faces of the ‘cubies’), it is essential to model the numerous possible moves.
Modelling the moves of a Rubik’s cube

The mathematical model here is the algebraic structure of a group.
Modelling the moves of a Rubik’s cube

The mathematical model here is the algebraic structure of a group. Denote the set

\[ S := \{T, B, F, P, L, R\} \]

and the group generated by the set \( S \),

\[ (G, *) = \langle S \rangle. \]

where the group operation \( * \) is that of composition of moves.
Surprising fact

Every scrambled Rubik’s cube configuration can be returned to its home state in at most 26 moves.
The subject of study is itself a fully-defined mathematical entity in the real world.
Characteristics of modelling in abstract contexts

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- Its behavior is very complex and thus, reasoning directly in/with this entity is difficult/tedious.
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Characteristics of modelling in abstract contexts

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- Its behavior is very complex and thus, reasoning directly in/with this entity is difficult/tedious.
- Based on experience with the entity, certain important behavioral aspects are extracted into a theoretical model.
- Interpretation to the real world entity will only be accurate up to those identified salient features.
Unlike a priori information, subjective information derives from some of the following:

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- Intuition
- Experience
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- Intuition
- Experience
- Expert opinion
Subjective information

Unlike a priori information, subjective information derives from some of the following:

- Intuition
- Experience
- Expert opinion
- Convenience of mathematical form.
For approximately 15 years, a Midwestern state has stored salt used on roads in the winter in circular domes. Figure 1 shows how salt has been stored in the past. The salt is brought into and removed from the domes by driving front-end loaders up ramps of salt leading into the domes. The salt is piled 25 to 30 ft high (i.e., 7.6 m to 9.2 m), using the buckets on the front-end loaders.
Salt storage problem

- door clearance: 19ft 9in
- front-end loader: 10ft 9in high
- salt ramp
- dome height: 50 ft
- retaining wall: 4 ft high
- 103 ft diameter
Engineering problem

Salt storage problem

Recently, a panel determined that this practice is unsafe. If the front-end loader gets too close to the edge of the salt pile, the salt might shift, and the loader could be thrown against the retaining walls that reinforce the dome. The panel recommended that if the salt is to be piled with the use of the loaders, then the piles should be restricted to a maximum height of 15 ft (i.e., 4.6 m).

Construct a mathematical model for this situation and find a recommended maximum height for salt in the domes.
Hospitals generally categorize surgical operations as major, standard, and trivial cases. Major cases involve complex procedures such as heart bypasses; standard ones cover gastronomical and intestinal procedures; and trivial cases include removal of small external growths. Each category calls for a number of senior (attending) surgeons, anesthesiologists, resident surgeons, operating nurses, and non-operating nurses; as well as time and costs.
Logistics problem

Personnel management system

<table>
<thead>
<tr>
<th>Category</th>
<th>S</th>
<th>A</th>
<th>RS</th>
<th>ON</th>
<th>NN</th>
<th>Time</th>
<th>Aver. costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>1 day</td>
<td>$30,000</td>
</tr>
<tr>
<td>S</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1/2 day</td>
<td>$16,000</td>
</tr>
<tr>
<td>T</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1/5 day</td>
<td>$3,000</td>
</tr>
</tbody>
</table>
Logistics problem

Personnel management system

A certain hospital is staffed with

1. 21 senior doctors (only these can be senior attending surgeons)
A certain hospital is staffed with

1. 21 senior doctors (only these can be senior attending surgeons)
2. 44 junior doctors
Personnel management system

A certain hospital is staffed with

1. **21** senior doctors (only these can be senior attending surgeons)
2. **44** junior doctors
3. **60** senior nurses (only these can be operating nurses)
Logistics problem

Personnel management system

A certain hospital is staffed with

1. 21 senior doctors (only these can be senior attending surgeons)
2. 44 junior doctors
3. 60 senior nurses (only these can be operating nurses)
4. 40 junior nurses
Personnel management system

A certain hospital is staffed with

1. 21 senior doctors (only these can be senior attending surgeons)
2. 44 junior doctors
3. 60 senior nurses (only these can be operating nurses)
4. 40 junior nurses
5. 30 anesthesiologists

Design an efficient and effective personnel management system for daily operations.
Characteristics of modelling in practical contexts

- The real world object is not a mathematically defined object.
Characteristics of modelling in practical contexts

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- Its behavior is very complex, woven together as a fabric of several variables.
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- Interpretation to the real world situation is not guaranteed to be accurate.
Characteristics of modelling in practical contexts

- The real world object is not a mathematically defined object.
- Its behavior is very complex, woven together as a fabric of several variables.
- Based on the user’s need, certain important behavioral aspects are formulated to a theoretical model.
- Interpretation to the real world situation is not guaranteed to be accurate.
- Cyclically improving the model, with every added availability of subjective information.
Model complexity involves a trade-off between
Model complexity involves a trade-off between

- simplicity, and
Model complexity involves a trade-off between

- simplicity, and

- accuracy

of the model.
Occam’s razor

... refers to the principle of choosing, amongst all models of approximately equal predictive power, the simplest one.
Occam’s razor

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As for what is relevant, it is up to the user and the model builder!
Implications

- What can be considered a mathematical modelling task?
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- What can be considered a mathematical modelling task?
- How can we design more meaningful mathematical modelling task?
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- What can be considered a mathematical modelling task?
- How can we design more meaningful mathematical modelling task?
- What opportunities are there to turn real world problems into mathematical modelling experiences?
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- What can be considered a mathematical modelling task?
- How can we design more meaningful mathematical modelling task?
- What opportunities are there to turn real world problems into mathematical modelling experiences?
- Are we mindful of the applications of Information Technology in mathematical modelling?
1. Plus teacher and student package: Mathematical Modelling at http://plus.maths.org/content/os/issue44/package/index
