The Basel Problem

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Basel

Today, we shall encounter a very mesmerizing problem now known as the

Basel Problem.

The city of Basel

Basel is Switzerland's third most populous city with about 166,000 inhabitants.

Figure: Mittlere Brücke over the Rhine

The city of Basel

Located where the Swiss, French and German borders meet, Basel also has suburbs in France and Germany.

Figure: Locality map of Basel

Historic vs Modern Basel

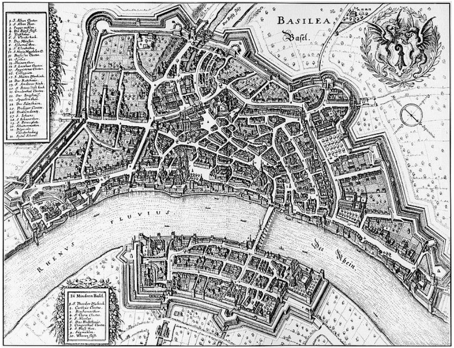

Figure: Basilea from Nuremberg chronicles and Map of Basel in 1642

Historic vs Modern Basel

Figure: Rhine river and Basel Bahnhof Train-station

The Basel Problem

The Basel problem is a famous problem in mathematical analysis with relevance to number theory, first posed by Italian mathematician Pietro Mengoli in 1644.

Figure: Pietro Mengoli (1626, Bologna – 1686, Bologna)

The Basel Problem

The Basel problem asks for the exact value of

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{n\to\infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \right)$$

i.e., the precise summation of the reciprocals of the squares of the natural numbers.

The Basel Problem

This problem survived the attacks of many people like these:

Figure: The Bernoulli family: (T1) Jacob, (T2) Johann, (T3) Nicolaus II, (B1) Daniel, (B2) Johann III and (B3) Jacob II Bernoulli

The Bernoulli family

- Jacob Bernoulli (1654-1705; also known as James or Jacques) Mathematician after whom Bernoulli numbers are named.
- Nicolaus Bernoulli (1662-1716) Painter and alderman of Basel.
- Johann Bernoulli (1667-1748; also known as Jean) Swiss mathematician and early adopter of infinitesimal calculus.
- Nicolaus I Bernoulli (1687-1759) Swiss mathematician.
- Nicolaus II Bernoulli (1695-1726) Swiss mathematician; worked on curves, differential equations, and probability.

The Bernoulli family

- Daniel Bernoulli (1700-1782) Developer of Bernoulli's principle and St. Petersburg paradox.
- Johann II Bernoulli (1710-1790; also known as Jean) Swiss mathematician and physicist.
- Johann III Bernoulli (1744-1807; also known as Jean) Swiss-German astronomer, geographer, and mathematician.
- Jacob II Bernoulli (1759-1789; also known as Jacques) Swiss-Russian physicist and mathematician.

The Basel Problem

The problem is named after the hometown of its solver:

Leonhard Euler.

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Leonhard Euler.

We shall come back to him in a moment.

The Basel Problem

Activity (5 mins)

Use your GC, find the value of

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

correcting your answer to 10 decimal places.

The Basel Problem

The correct answer is

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \approx 1.6449340668,$$

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The Basel Problem

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Your G.C. probably can't deliver this answer. Why? G.C. spoiled?

Understanding the problem

The Basel Problem demands its solver to give the exact value of the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

and the proof.

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Question

When did you first encounter an infinite series?

Understanding the problem

Definition

A sequence of real numbers

$$a_1, a_2, \cdots, a_n, \cdots$$

is a *geometric progression* if the ratio between the consecutive terms is constant, i.e., there is a constant r such that

$$a_{n+1} = a_n \cdot r$$

for all $n = 1, 2, \cdots$

Understanding the problem

It is easy to see that

$$\left(\frac{a_n}{a_{n-1}}\right)\cdot\left(\frac{a_{n-1}}{a_{n-2}}\right)\cdot\dots\cdot\left(\frac{a_2}{a_1}\right)$$

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$$=\underbrace{r\cdot r\cdot\dots\cdot r}_{n-1 \text{ copies}}$$

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$$
\begin{aligned}
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=&\underbrace{r\cdot r\cdot\dots\cdot r}_{n-1 \text{ copies}}\\
=&r^{n-1}.
\end{aligned}
$$

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$$=\underbrace{r\cdot r\cdot\cdots\cdot r}_{n-1 \text{ copies}}$$
$$=r^{n-1}.$$

It follows that

$$a_n = a_1\cdot r^{n-1}, \quad n = 1, 2, \cdots$$

Understanding the problem

Examples

Here are some geometric progressions:

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Understanding the problem

Examples

Here are some geometric progressions:

1. $1, 2, 4, 8, \cdots$
2. $1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots$
3. $1, -1, 1, -1, \cdots$

Understanding the problem

The sum to infinity of a geometric progression

$$a + ar + ar^2 + \cdots$$

exists if the limit

$$\lim_{n\to\infty} \left(a + ar + ar^2 + \cdots + ar^{n-1}\right)$$

exists.

Understanding the problem

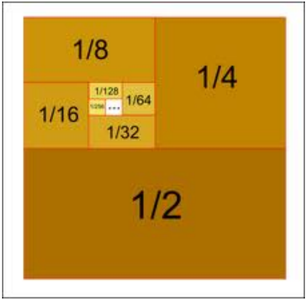

Figure: Sum to infinity of a G.P.

Understanding the problem

Do we have a formula for the finite sum

$$\sum_{k=1}^{n} ar^{k-1} = a + ar + ar^2 + \cdots + ar^{n-1}$$

so that perhaps we can better figure out the limiting value?

Understanding the problem

Exercise

Suppose we denote the finite sum by

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and multiply it by r to obtain rS_n.
By finding $S_n - rS_n$, deduce a formula for S_n in terms of n.

Understanding the problem

Now the sum to infinity of a geometric progression exists if and only if the limit

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{a}{1-r}\right)(1-r^n)$$

exists.

Understanding the problem

Now the sum to infinity of a geometric progression exists if and only if the limit

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{a}{1-r}\right)(1-r^n)$$

exists.

Theorem

$$S_\infty < \infty \iff |r| < 1.$$

Understanding the problem

Returning to the Basel problem, i.e., the infinite series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} + \cdots$$

we have two questions to ask:

Understanding the problem

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- Are we so lucky to have a closed formula for the finite sum?

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we have two questions to ask:

- Are we so lucky to have a closed formula for the finite sum?
- If not, how sure are we that the sum to infinity exists?

or maybe it does not ...

Understanding the problem

The answer to the first question is

NO, at the moment.

Understanding the problem

The answer to the second problem is

YES.

Understanding the problem

For any positive integer $k > 1$, note that

$$\frac{1}{k(k+1)} < \frac{1}{k^2} < \frac{1}{(k-1)k}.$$

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So, for any positive integer $n > 1$, we have

$$\sum_{k=2}^{n} \frac{1}{k(k+1)} < \sum_{k=2}^{n} \frac{1}{k^2} < \sum_{k=2}^{n} \frac{1}{(k-1)k}.$$

Understanding the problem

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$$\begin{aligned}\sum_{k=2}^{n} \frac{1}{k(k+1)} &= \sum_{k=2}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= \frac{1}{2} - \frac{1}{3}\end{aligned}$$

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The left hand sum simplifies to

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Understanding the problem

The right hand side simplifies to

$$\sum_{k=2}^{n} \frac{1}{(k-1)k} = 1 - \frac{1}{n}$$

likewise.

Understanding the problem

For any positive integer n, we have

$$1 + \left(\frac{1}{2} - \frac{1}{n+1}\right) < \sum_{k=1}^{n} \frac{1}{k^2} < 1 + \left(1 - \frac{1}{n}\right)$$

i.e.,

$$\frac{3}{2} - \frac{1}{n+1} < \sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}.$$

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i.e.,

$$\frac{3}{2} - \frac{1}{n+1} < \sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}.$$

Letting $n \to \infty$,

$$\frac{3}{2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2.$$

The Basel Problem

Theorem

The infinite series

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Proof.

Since the sequence $\left\{\sum_{k=1}^{n} \frac{1}{k^2}\right\}_{n=1}^{\infty}$ is monotone increasing and bounded above, the series converges. $\square$

Problem far from solved

Even if we know that

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

exists, we still have no idea what its exact value is.

The Mathematician

Our main character today is

Figure: Leonhard Euler (15 April 1707 - 18 September 1783)

Early years

Euler was born on April 15, 1707, in Basel to Paul Euler, a pastor of the Reformed Church. His mother was Marguerite Brucker, a pastor's daughter. He had two younger sisters named Anna Maria and Maria Magdalena.

Early years

Figure: A Swiss Reform church at Riehen

Soon after the birth of Leonhard, the Eulers moved from Basel to the town of Riehen, where Euler spent most of his childhood.

Early years

Figure: Johann Bernoulli

Paul Euler was a friend of the Bernoulli family - Johann Bernoulli, who was then regarded as Europe's foremost mathematician, would eventually be the most important influence on young Leonhard.

Early years

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- 1723: Received Saturday afternoon lessons from Johann Bernoulli

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- 1726: Completed a dissertation on the propagation of sound with the title *De Sono*

Early years

- 1727: Entered the Paris Academy Prize Problem competition, in which he won second place

Early years

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- Won this coveted annual prize 12 times in his career

St Petersburg

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- July 10, 1726: Nicolas died of appendicitis after spending a year in Russia
- Daniel assumed his brother's position in the mathematics/physics division, he recommended that the post in physiology that he had vacated be filled by his friend Euler
- November 1726: Euler eagerly accepted the offer, but delayed making the trip to St Petersburg while he unsuccessfully applied for a physics professorship at the University of Basel

St Petersburg

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- Took on an additional job as a medic in the Russian Navy.

St Petersburg

Figure: A Soviet stamp depicting Euler

- The Academy at St. Petersburg wanted to improve education in Russia and to close the scientific gap with Western Europe

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- The Academy at St. Petersburg wanted to improve education in Russia and to close the scientific gap with Western Europe
- Attract foreign scholars like Euler
- Good money and library, low enrollment to lessen the faculty's teaching burden, and the academy emphasized research

St Petersburg

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- Cut money and caused other difficulties for Euler and his colleagues.

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St Petersburg

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- 1733: Daniel Bernoulli, fed up with the censorship and hostility, left for Basel
- 1733: Euler succeeded as the head of the mathematics department

St Petersburg

Figure: The Neva River

- 7 January 1734: Married Katharina Gsell (1707-1773), a daughter of Georg Gsell, a painter from the Academy Gymnasium

St Petersburg

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- 7 January 1734: Married Katharina Gsell (1707-1773), a daughter of Georg Gsell, a painter from the Academy Gymnasium
- Young couple bought a house by the Neva River
- Of their thirteen children, only five survived childhood

Berlin

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- 1755: Elected a foreign member of the Royal Swedish Academy of Sciences.

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- Early 1760s: Wrote over 200 letters to her (Letters of Euler)
- Compilation became more widely read than any of his mathematical works
- Left Berlin because of personal conflict with Frederick The Great

The sine function

The story begins in around 1735 with an ordinary function

$$f(x) = \sin(x).$$

The sine function

We know this function looks like:

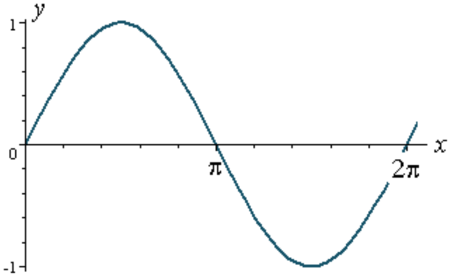

Figure: The sine curve

The sine function

We know that this function has a period of 2π, i.e.,

$$\sin(x + 2\pi) = \sin(x)$$

for all real x.

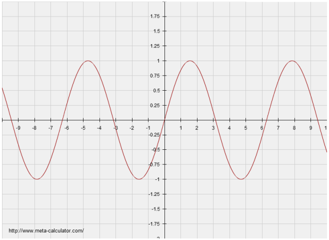

Figure: Periodic curve

The sine function

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4. $\sin(0) = 0$.
5. $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

The sine function

We do know of another family of functions which has the first two properties:

1. Continuity
2. Differentiability

The polynomials

Definition

A function of the form

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad a_n \neq 0$$

is called a *polynomial* in x.

The polynomials

Example

An example of a polynomial is

$$P(x) = (x+1)x(x-1)$$

whose graph is given by ...

The polynomials

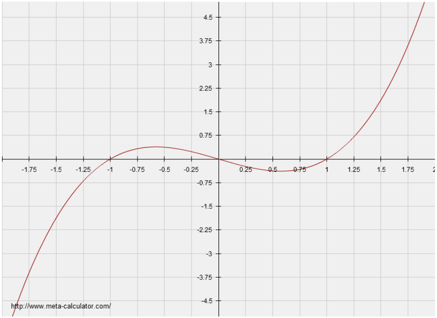

Figure: The graph of $y = (x+1)x(x-1)$

The polynomials

The cubic polynomial

$$P(x) = (x+1)x(x-1) = x^3 - x$$

has zeros

$$x = -1, 0, 1.$$

Devise a plan

Euler's 1st big idea

The sine function may be seen as an infinite polynomial with infinitely many zeros

$$\cdots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \cdots$$

Carry out the plan

Activity

Using your G.C., sketch the following graphs in succession:

1. $y = x$
2. $y = (x + \pi)x(x - \pi)$
3. $y = (x + 2\pi)(x + \pi)x(x - \pi)(x - 2\pi)$
4. $y = (x + 3\pi)(x + 2\pi)(x + \pi)x(x - \pi)(x - 2\pi)(x + 3\pi)$

Checking the solution

None of these curves look like the sine curve:

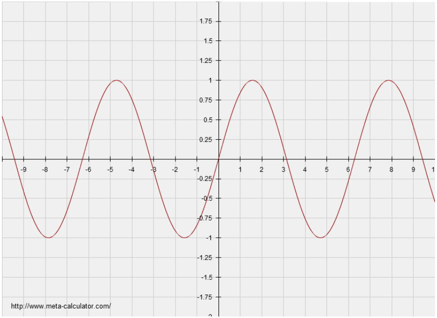

Figure: The graph of $y = \sin(x)$

Checking the solution

We know that

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

but

$$\lim_{x\to 0}\frac{(x+n\pi)\cdots x\cdots(x-n\pi)}{x} \neq 1.$$

Re-looking

But one can resort to a little trick ...

$$\lim_{x\to 0} \frac{\left(1+\frac{x}{n\pi}\right)\cdot\left(1+\frac{x}{(n-1)\pi}\right)\cdots x\cdots\left(1-\frac{x}{(n-1)\pi}\right)\left(1-\frac{x}{n\pi}\right)}{x} = 1.$$

Carry out the second plan

Activity

Using your G.C., sketch the following graphs in succession:

1. $y = x$
2. $y = \left(1 + \frac{x}{\pi}\right) x \left(x - \frac{x}{\pi}\right)$
3. $y = \left(1 + \frac{x}{2\pi}\right)\left(1 + \frac{x}{\pi}\right) x \left(x - \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)$
4. $y = \left(1 + \frac{x}{3\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 + \frac{x}{\pi}\right) x \left(x - \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)$

Re-looking

Let's do better than your G.C.'s.

Re-looking

MATLAB program

```
%% This MATLAB program plots the curve
%% of y = Q_n(x) for a user-input integer n.
clear; clc;
n = input('Enter the value of n : ');
x = [-5*pi:0.001*pi:5*pi];
y = x; z = sin(x); w = 0;
for k = 1:n
y = y.*(1-(x.^2)/(k*pi)^2);
end
plot(x,y,'-k'); hold on;
plot(x,z,'-r'); hold on;
plot(x,w,'-b');
title('Graph of Q_n(x)');
```

Re-looking

It is intended that the program outputs the graph whose equation is

$$Q_n(x) = x \cdot \prod_{k=-n}^{n} \left(1 - \frac{x}{k\pi}\right),$$

where $n = 1, 2, \cdots$.

Sample runs

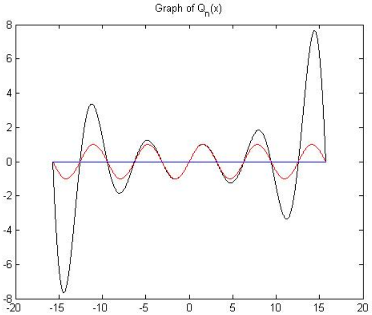

Figure: The graph of $y = Q_{10}(x)$

Sample runs

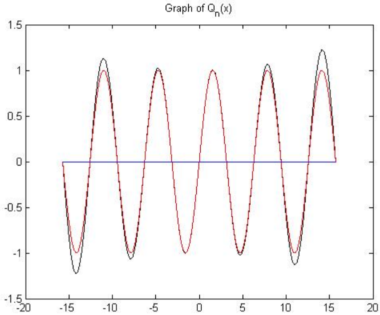

Figure: The graph of $y = Q_{100}(x)$

Sample runs

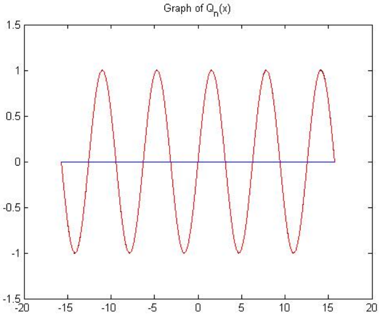

Figure: The graph of $y = Q_{1000}(x)$

Sample runs

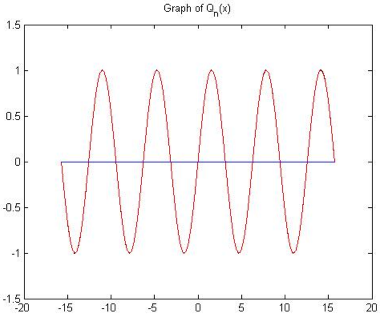

Figure: The graph of $y = Q_{10000}(x)$

Maclaurin's series

The story has an important second part which has to do with the famous

Maclaurin's series.

Maclaurin's series

That a function can be seen as an infinite polynomial is not new.

Maclaurin's series

That a function can be seen as an infinite polynomial is not new. The Maclaurin's series expansion of an infinitely-differentiable function f is given by:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k,$$

where $f^{(k)}(0)$ denotes the kth derivative evaluated at $x = 0$.

Maclaurin's series

Suppose that a function has all derivatives, and it can be expressed as an infinite polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots.$$

Maclaurin's series

Suppose that a function has all derivatives, and it can be expressed as an infinite polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots.$$

Then, by substituting $x = 0$ into this equation, we have

$$f(0) = a_0$$

so that

$$a_0 = f(0) = \frac{f^{(0)}(0)}{0!}.$$

Maclaurin's series

Differentiating, w.r.t. x, the original series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

yields:

$$f^{(1)}(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots.$$

Maclaurin's series

Differentiating, w.r.t. x, the original series

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots$$

yields:

$$f^{(1)}(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots.$$

Again by substituting $x = 0$, we have:

$$f'(0) = a_1$$

so that

$$a_1 = f'(1) = \frac{f^{(1)}(0)}{1!}.$$

Maclaurin's series

Going on this way, it is not difficult to see that

$$f^{(k)}(0) = k! \cdot a_k,$$

i.e.,

$$a_k = \frac{f^{(k)}(0)}{k!}.$$

Maclaurin's series

The Maclaurin's series expansion for the sine function is given by

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Maclaurin's series

Euler's 2nd big idea

The infinite product representation and the infinite sum representation of the sine function as an infinite polynomial must be the same!

The 'Eureka' moment

Since the two representations are equal, the coefficient of each x^k must agree.

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Since the two representations are equal, the coefficient of each x^k must agree.
Let us say, we compare the coefficients of x^3.
For the infinite product

$$\cdots\left(1+\frac{x}{3\pi}\right)\left(1+\frac{x}{2\pi}\right)\left(1+\frac{x}{\pi}\right)x\left(1-\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1-\frac{x}{3\pi}\right)\cdots,$$

we expand systematically:

$$-\frac{1}{\pi^2}-\frac{1}{2^2\pi^2}-\frac{1}{3^2\pi^2}-\cdots$$

The 'Eureka' moment

For the infinite sum

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

the coefficient of x^3 is

$$-\frac{1}{3!} = -\frac{1}{6}.$$

The 'Eureka' moment

Equating the coefficients of x^3 yields:

$$-\frac{1}{\pi^2}-\frac{1}{2^2\pi^2}-\frac{1}{3^2\pi^2}-\cdots=-\frac{1}{6}$$

which gives

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$

Search for a rigorous proof

For years to come, Euler searched for a rigorous proof that justifies the infinite product formula for the sine function. He found a rigorous proof in 1741.

Other developments

Euler found the exact values for

$$\sum_{k=1}^{\infty} \frac{1}{k^{2m}}, \quad m \in \mathbb{N}.$$

Other developments

Euler found the exact values for

$$\sum_{k=1}^{\infty} \frac{1}{k^{2m}}, \quad m \in \mathbb{N}.$$

In fact, each of these are of the form

$$r \cdot \pi^{2m}$$

where $r = \frac{(-1)^{n-1} 2^{2n-1} B_{2n}}{(2n)!}$.

Zeta function

Definition

The zeta function, defined by Bernard Riemann, is

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad s \in \mathbb{C}.$$

Zeta at odd integral arguments

Open problems

Find the exact value of

$$\zeta(2m+1), \quad m \in \mathbb{N}.$$

Zeta at odd integral arguments

It has been proven that

$$\zeta(3) \notin \mathbb{Q}.$$

Open problems

Are these zeta-values

$$\zeta(2m+3), \quad m \in \mathbb{N}.$$

irrational?