

Using history of mathematics in the teaching and learning of mathematics in Singapore

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Abstract

In this paper, we consider the use of history of mathematics as a methodology in the teaching and learning of mathematics in Singapore, in particular at the polytechnic and junior college level. More specifically, we investigate how the history of mathematics can be integrated into the Singapore classroom via a spectrum of well-studied and time-tested lesson designs. In the ensuing discourse, we study this methodology in terms of (1) its implementation framework, (2) its feasibility, limits and risks, as well as (3) its potential role in the existing Singapore mathematics curriculum. In the passing, we shall also briefly address the sociological dimension of such a historical approach in mathematics education, in particular, how it relates with the learning community on the whole. This study also presents a case study, based on action-research approach, which was carried out in a polytechnic setting over a twelve weeks period in 2007 to investigate the effects of the above-mentioned methodology on teaching and learning processes.

Keywords: History of mathematics, teaching and learning of mathematics, didactics, national curricula

1 Introduction

Does history of mathematics have a role to play in mathematics education? This question attracted the attention from an increasing number of researchers and mathematics educators over the past two decades. An important movement to address this was led by an ICMI (International Commission on Mathematical Instruction) study on the methodology of integrating history of mathematics in the teaching and learning of mathematics [18]. In this study, the historical methodology was extensively and thoroughly studied via (1) analyzing the impact of such a methodology on the national curricula of several countries, (2) discussing the philosophical, multi-cultural and interdisciplinary

issues related to this methodology, (3) archiving a spectrum of classroom implementation methods, and (4) debating on feasibility issues with regards to such implementations. Several authors have also addressed the same issue, including [32], [50], [47, 49, 48], [22, 23, 24, 25], [17], [52], [51] and [30].

Surprisingly, this same question received little attention in Singapore with only two exceptions. One is a recent paper [35] which examined the effects of an Ancient Chinese Mathematics Enrichment Programme (ACMEP) on secondary schools' achievement in mathematics. The other is [29], reported in the Undergraduate Research Opportunities Programme in Science (UROPS), that explored the role of the history of mathematics in fostering critical thinking and achieving deeper understanding in the learning of mathematics at the lower secondary schools. Apart from these, little has been said about history of mathematics as a tool for teaching mathematics in Singapore.

From the perspective of research systematics, we need to address issues which the ICIM study did not. This paper sets out to investigate the role of using history of mathematics in the teaching and learning of mathematics in Singapore. In view that the case studies in [29] and [35] were performed on lower secondary school levels, we choose to furnish with more examples from the *polytechnic and junior college levels*. More specifically, we investigate, how at these levels, the history of mathematics can be *integrated* into the Singaporean classroom by employing a myriad of well-studied and time-tested lesson designs.

The structure of this paper is thus as follows: In Section 2, we state the rationale of the study. In Section 3, we give an overview of the Singapore mathematics curricula. With that as the background, we propose a didactical framework upon which this methodology can be built. This is followed by exploring the different means of implementing this methodology with the help of several classroom applications. We shall then, in Section 5, deal with potentialities, limits and risks of the historical methodology as perceived by the mathematics teachers in Singapore. Section 6 presents a small-scale case study, based on an action-research approach, which was carried out in a polytechnic setting over a period of twelve weeks in 2007. There, the effects of the above-mentioned methodology on the teaching and learning processes in a linear algebra class were investigated. Finally, we discuss in our conclusion the possible impact of the historical methodology to the existing Singapore mathematics curriculum. In passing, we address the sociological dimension of such a historical approach in mathematics education, in particular, how it relates with the community as a whole.

2 Rationale

It has been put forward by several authors, notably [31], [32] and [61, 62], that employing the history of mathematics in school curricula can potentially meet the objectives of (a) increasing the students' motivation and develop a *positive attitude* towards mathematics, (b) helping explain difficulties and confusion that students encounter via an analysis of the development of mathematics, (c) en-

hancing the development of student’s mathematical reasoning skills by the use of historical problems, (d) revealing the humanistic aspects of mathematical knowledge, (e) using the lives of mathematicians as platform to introduce and inculcate good moral values such as honesty, diligence and determination, and (f) providing a guide by which teachers of mathematics may craft their lessons.

The aims of mathematics education in Singapore, clearly spelt out in [37, 38, 39], are based upon the famous pentagonal framework, comprising of five important qualities, namely, (1) Attitudes, (2) Metacognition, (3) Processes, (4) Concepts and (5) Skills, and centred about the theme of mathematical problem solving. This framework is applicable to all levels of mathematics education, from the primary to A-levels that “sets the direction for the teaching, learning, and assessment of mathematics” ([38]).

As already pointed out by Ng in [35], there seems to be an overt emphasis of honing mathematical content knowledge and in developing mathematical reasoning abilities (which are targeted to *concepts*, *skills* and *problem solving*). In contrast, little has been done to help the students develop a positive attitude towards the subject. Having a positive attitude encompasses a collection of many non-quantifiable qualities such as (i) holding onto certain beliefs and philosophy towards mathematics, such as universality of mathematical results and believing in the usefulness of mathematics, (ii) invoking genuine interest and having enjoyment in learning mathematics, (iii) developing an appreciation of the beauty and power of mathematics, (iv) building confidence in using mathematics and (v) instilling a spirit of perseverance in solving a problem as part of the mathematical training (see [39]).

In view of this, the main drive of this paper is to advocate the use of the history of mathematics to inculcate positive attitudes of the learners, as well as the teachers, towards mathematics and explain how an active interplay may occur between the Singapore mathematics curriculum and the community via this historical methodology.

3 The political context: Singapore mathematics curriculum

3.1 A brief overview of the Singapore mathematics syllabus

Singapore is a relatively young nation, having gained national independence since 1965. Compulsory education for children or primary school age (7 to 12 years old) was only enforced relatively recently in 2000 – codified under the Compulsory Education Act [36]. Since the implementation of the New Education System in 1979, English has been the medium of instruction and so mathematics has since been taught in English. Mathematics has only been taught in Singapore for only about six decades. Over these sixty years, the mathematics curriculum in Singapore has undergone much development and changes. A country’s decisions, with regards to mathematics education, as to *what* to teach,

why to teach, *who* to teach, for *whom* to teach and *how* to teach are “ultimately political, albeit influenced by a number of factors including the experience of teachers, the expectations of parents and employers, and the social context of debates about the content and style of the curriculum” ([16]). Singapore is no exception. All the sixty years of changes and development in its mathematics curriculum have been chronicled in [28] and more recently in [27], and the reader is strongly encouraged to refer to them for detailed accounts of the national mathematics curriculum in Singapore. What follows (Sections 3.1.1–3.1.5) is a point-form summary of the major changes in the Singapore mathematics curriculum as recorded in [27], with appropriate annotations of the use of history of mathematics in mathematics education for each period.

3.1.1 Early days (1945 – 1960)

- 1945 – 1946: Re-opening of schools at the end of the Second World War.
- 1950’s: Co-existence of mission and Chinese schools alongside the few government schools.
- Education largely confined to foreign textbooks.
- Absence of a unified local mathematics syllabus.
- A majority of mathematics teachers are engineers.

English schools used British mathematics textbooks by Durell, such as [8, 9, 10, 11, 12, 13, 14, 15] as their sole teaching resource, while Chinese schools used American textbooks such as *College Algebra* by Fine [19] (in Chinese translation) at the senior middle two level (Grade 11). It is worth noting these textbooks generally adopt a polished and formal ‘lecture’-style of developing the content of each topic. Virtually no mention is made of the historical background of the topics or the mathematicians responsible for inventing them, though there is enough motivation based on relevant mathematical observations. For instance, as part of the introduction of the number system in [19], the reader is called to attention objects in daily experience which are “associated in groups and assemblages” and naturally occurring objects like “ten fingers” every human has. But no mention, for example, is made of when and why numbers are essential to human civilizations.

3.1.2 First local syllabus (1960 – 1970)

- 1957: Draft of the first Singapore mathematics syllabus
- 1959: Implementation of this syllabus, covering from Year 1 to Year 13.
- Mathematics to be taught as a unified subject as opposed to a ‘many-branched’ discipline.
- Traditional classroom instruction persisted in this ten-year period.

No explicit mention of the use of history of mathematics in the mathematics curriculum during that time period was made. Several mathematics teachers (mostly graduates of the Nanyang University) did occasionally tell their classes historical folklore such as the well-worn ‘Eureka’ story of Archimedes.

3.1.3 Mathematics reforms (1970 – 1980)

- 1970s: Mathematics reforms
- Hastening of localization of syllabus and textbooks.
- Massive re-training of teachers.
- Formation of Advisory Committee on Curriculum Development (ACCD).
- 1968: Introduction of Syllabus ‘C’ for mathematics.
- 1973: Revision of Syllabus ‘C’ Mathematics.

1969 saw the appearance of the first batch of locally produced secondary school textbooks, e.g., [53]. Others followed shortly. These textbooks also did not actively exploit the history of mathematics in their exposition. In terms of teaching approaches, worksheets and mathematical manipulatives evolved. In particular, due to the popularization by Caleb Gattegno (1911-1988) in many parts of the world, the Cuisenaire rods (see Figure 3.1.3) were used to teach the four operations of whole numbers in primary schools.

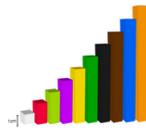


Figure 1: Cuisenaire rods

Technically speaking, the Cuisenaire rods cannot be considered as a historical artifact or an ancient mathematical instrument (see Appendix for the definition of a historical artifact). So even when teaching aids were first employed in the Singapore classroom, no historical component was evident.

3.1.4 Back to basics (1980 – 1995)

- Mathematics reforms led to a decline of the mathematical standards.
- 1979: Launch of the New Education System (NES).
- Streaming was implemented to reduce schools’ attrition rates.
- 1981: New Syllabus ‘D’ for mathematics.

- Formation of Curriculum Development Institute of Singapore (CDIS) to produce textbooks.
- Use of Scientific Calculators in secondary schools.

The more popular secondary textbooks were published commercially. While primary schools followed these textbooks closely, the secondary schools did not do so. Although secondary school teachers had the liberty of not following the textbooks closely, they still had to adhere to the schools' scheme of work, which is a set of guidelines for the topics that should be taught in specific points of time. Following a British tradition, junior colleges do not follow any textbooks but make references to textbooks such as [4, 5].

In the next time frame, changes in the national curriculum are mainly on the methodology rather than the content.

3.1.5 New initiatives (1995 – 2005)

- 1997: 'Thinking Schools and a Learning Nation' (TSLN) – a slogan put forward by Ministry of Education.
- A series of educational reform follow.
- Three new initiatives: National Education (NE), Information Technology (IT) and Thinking.
- 1997 – 2002: IT Master Plan I
- 2002 – 2008: IT Master Plan II
- 2006: Introduction of Graphic Calculators in junior colleges

By 'thinking', it meant *more learning and less teaching*. This led to a content reduction by about 10%, with more reduction in secondary schools and less so in primary schools. For mathematics, this should translate into students spending more time experiencing problem solving, rather than route learning. It was aimed to create a larger social outcome, which is to have a nation of people who are endeavored to life-long learning.

3.1.6 New 'A' level Mathematics Syllabi (2006 – 2007)

The 'AO' and 'A'-Level syllabuses before 2006 will be phased out gradually as the new 'A'-Level Curriculum Higher 1 (H1), Higher 2 (H2) and Higher 3 (H3) is introduced from 2006. The introduction of the new syllabus is also accompanied by the phasing out of the subject 'Further Mathematics' (Syllabus 9234). The majority of junior college students take H2 Mathematics (Syllabus 9740) [38], and it is intended that the new syllabus is in line with the maxim 'Teach Less and Learn More', i.e., reduction in content and focus more on independent thinking.

3.1.7 Mathematics curricula of polytechnics in Singapore

Singapore retains a system similar to that in the United Kingdom from 1969–1992, distinguishing between polytechnics and universities. Under this system, most Singapore students sit for their Singapore-Cambridge ‘O’ Level examinations after a four or five years of education in secondary school, and apply for a place at either an ITE, a polytechnic or a Pre-university centre (a junior college or the Millennia Institute, a centralized institute). Under the coinage of ‘through-train programmes’, a few secondary schools now offer a six-year programme which leads directly to university entrance. Junior college and polytechnic students fall within the same age group of 16 – 18 years old.

All polytechnics offer three year diploma courses in subjects such as information technology, engineering subjects and other vocational fields. To date, there are a total of 5 polytechnics in Singapore. Polytechnics offer a wide range of courses in various fields, including engineering, business studies, accountancy, tourism and hospitality management, mass communications, digital media and biotechnology. There are also specialized courses such as marine engineering, nautical studies, nursing, and optometry. They provide a more industry-oriented education as an alternative to junior colleges for post-secondary studies. Graduates of polytechnics with good grades can continue to pursue further tertiary education at the universities, and many overseas universities, notably those in Australia, give exemptions for modules completed in Polytechnic.

Generally, mathematics is only mandatory for students who are enrolled in engineering, chemical and life-sciences, biotechnology and business studies. Instead of one subject to be taken uniformly for all students, mathematics is usually tailor-made in the form of several progressive modules, consisting only of topics which are relevant to the respective student groups. For instance, engineering mathematics is taught to the engineering students, and business students need only take modules of business mathematics and statistics. Even among the various engineering schools, it is typical that more emphasis be placed on topics of higher relevance to their trade. A spiral approach is adopted in the curriculum planning for mathematics. For example, integration of polynomials is introduced in first year and that of rational functions in the second year of engineering mathematics. Polytechnics do not conform to one common mathematics syllabus. This is because each polytechnic offers different diploma courses of varying nature and emphasis, though the various mathematics syllabi do overlap at many common traditional topics, such as the trigonometry and calculus.

With regards to mathematics, there is an overt focus on calculational skills and rote learning, with drills and practice being central to the process of teaching and learning. Although derivations of mathematical concepts and results are provided in the students’ handouts, mathematical proofs are usually omitted in lectures and hardly any are required in tests or examinations¹. The rationale

¹One exception to this is the Certificate of Engineering Mathematics offered in Singapore Polytechnic which is targeted to selected students of a higher mathematical calibre, where advanced topics are taught with more depth and rigor.

is that for the polytechnic students, mathematics is just a computational tool in their main areas of specialization, and thus they need not be unnecessarily burdened by excessive content (e.g. history of mathematics, justifications for definitions, and formal proofs). This is in sharp contrast with the current ‘A’ level Mathematics Syllabus 9740 (H1, H2 and H3) in the junior colleges, which emphasizes on higher-order processes of critical thinking, evaluation and problem solving. In particular, the reader should be informed of the third assessment objective spelt out in [39]: ‘A’ level candidates are tested on their abilities to “solve unfamiliar problems, translate common realistic contexts into mathematics, interpret and evaluate mathematical results, and use the results to make predictions, or comment on the context.” So one expects a significant difference in the nature and the level of difficulty of the examination questions when a ‘topic-for-topic’ comparison between those of the polytechnics and of the junior colleges were to be carried out. It is also noted that the polytechnics have not taken up the use of graphic calculators in any of their mathematics courses, in contrast to that taken up by their junior college counterparts since the launch of the Syllabus 9740.

3.2 What part does history of mathematics currently occupy in the national curricula?

At present, history of mathematics does not have an official or formal placing in the mathematics education in Singapore schools, from primary to junior colleges, as well as in polytechnics. At the primary level, there is nothing to be said about history of mathematics. So one can safely say that it plays no official role whatsoever.

As for the secondary level, there was an obvious effort made by some textbook authors as early as the 1980s to include materials of a historical flavor. For instance, certain secondary one textbooks (Syllabus D) included the number systems used by respectively the ancient Egyptians and ancient Chinese. The next generation of authors (e.g., [46], [55], [54]) of secondary school textbooks (based on the New Syllabus D) exploited further the historical component by including sections typically titled as ‘Mathstory’ or ‘For your information’, where the history of mathematics is given at the side margin where appropriate. Such sections are intended to enrich students with the knowledge of how mathematics has developed over the years and provide extra information on mathematicians, mathematical history and events. A few examples of the historical snippets in Singapore mathematics textbooks and lecture notes are given in Section A.1

Although these textbooks provide historical snippets as described above, the history of mathematics does not have a place in its own right. Most of the historical notes, in the form of time history, is perceived as an extra (by both the learners and their instructors) which means that they can be omitted (and probably so during lessons).

A few individual programs, mostly research-driven, have been launched at lower secondary levels in attempt to investigate whether, by incorporating history of mathematics, there are any effects on the students’ problem solving

skills and critical thinking skills (see [29] and [35]). However, these are isolated programs and thus no continuation follows. Interestingly, both these studies produced positive experimental results, i.e., students who had undergone the lessons which incorporated the history of mathematics performed significantly better than those who did not, though it was noted in [35] that such positive results cannot be entirely credited to the use of history of mathematics.

For the new ‘A’ level syllabus and the various mathematics syllabi in the polytechnics, there is again no specific mention about the use of history of mathematics. A survey (hereafter termed as *the survey*) conducted by the author in November – December 2007 among some 1000 junior college teachers and polytechnic lecturers revealed that a majority of lecturers (more than 90%) did not make use of history of mathematics (in any form) during their lectures and tutorials. Those who did mostly made use of historical snippets, historical problems and (pictures or models of) ancient mathematical/scientific instruments in their lessons. The statistics gathered from this survey are presented in the table below.

S/No.	Ways of incorporating history of mathematics in lessons	%
1	Historical snippets	9
2	Student research projects based on history texts	0
3	Primary sources	3
4	Worksheets	1
5	Historical packages	
6	Taking advantage of errors, alternative concepts, etc.	1
7	Historical problems	4
8	Mechanical instruments or ancient instruments of calculation	6
9	Experimental mathematical activities	1
10	Plays	0
11	Films and other visual means	0
12	Outdoor experiences	0
13	Internet resources	3

Figure 2: Percentage breakdown of ways of incorporating history of mathematics into lessons in current teaching practice as obtained in the survey

From this survey, one may conclude beyond doubt that the history of mathematics is employed only by a minority of teachers in the teaching and learning of mathematics in Singapore at the polytechnic and junior college level. There are many areas of concern and reasons why teachers of mathematics did not make use of the historical methodology in their teaching, and these are summarized in Section 5. Pertaining to the aspect of fostering positive attitudes of learners of mathematics, it is worth noting that

“Students’ attitudes towards mathematics are shaped by their learning experiences. Making the learning of mathematics fun, mean-

ingful and relevant goes a long way to inculcating positive attitudes towards the subject. Care and attention should be given to the design of the *learning activities*, to build confidence in and develop appreciation for the subject.”–Singapore: Ministry of Education [39]

History of mathematics, as we shall propose in the next section, can offer rich materials for these kind of activities, and so well-trained and resourceful teachers can have good classroom opportunities.

4 Didactical framework and implementation

The history of mathematics appears to present itself as a useful resource for understanding the processes of formation of mathematical thinking, for providing some insight into the development of positive attitudes towards the subject, and for translating this kind of understanding into the design of classroom activities.

In the recent decade, a number of researchers started looking into the possibility of employing the integration of history of mathematics into classroom as pedagogy of mathematics. As mentioned earlier, a few researchers in Singapore (see [29, 35]) have also considered the possibility of the historical approach. But without a sound and useful theoretical framework accounting for the general formation of mathematical knowledge, it would be impossible for mathematics educators and researchers to harness the potentials of the historical methodology. More precisely, the theoretical framework must be useful in the sense that it can be employed to *translate* the information about knowledge formation into an effective crafting of lesson designs.

It is frequently due to a lack of a suitable didactical framework of formation of mathematical knowledge, rather than a lack of teacher’s training (see Section 5), that leads the mathematics teacher to ask: “Can I implicitly assume (as it appears in most narratives in history of mathematics) that the mathematics of the past were *essentially* dealing with our modern concepts, but just did not have our modern notations and terminologies at their disposal?” This, in fact, is the very issue which many participants of the survey (mentioned in Section 3.2) are concerned. What these Singaporean mathematics teachers have in mind is to ask: “To what extent can we re-enact history in the classroom setting?” Of course, this question itself presents an immediate difficulty. How can a 21st century modern Singapore teacher ever understand (unless it had been explicitly chronicled by historians, and even so there is no guarantee that it is 100% truth), for instance, what Archimedes had in mind when he made rational approximations of π as in [2]?

There are two problems here: (1) There is an over-simplification of the way in which mathematical concepts have been developed historically. This is a question pertaining to the historical domain, which results in having the history of mathematics all too often read in an *unhistorical* way (see [43]). (2) One is enthusiastic to re-enact historical moments because one believes in psychological *recapitulationism*.

4.1 Unhistorical reading of history

Let us address the first problem raised in the preceding section. The problem of reading history the *teleological* (i.e., unhistorical) manner causes one to be disillusioned that there is to be a unique course that the historical developments just had to take. This disillusion often causes the following problem associated to even the modern mathematicians, best expressed in the words of Augustus De Morgan in his inaugural address as the first president of the London Mathematical Society, England, 16th of January 1865:

“I say that no art or science is a liberal art or a liberal science unless it be studied in connection with the mind of man in past times. It is astonishing how strangely mathematicians talk of the Mathematics, because they do not know the history of their subject. By asserting what they conceive to be facts they distort its history in this manner. There is in the idea of every one some particular sequence of propositions, which he has in his own mind, and he imagines that the sequence exists in history; that his own order is the historical order in which the propositions have successively been evolved.” –Augustus De Morgan (1806–1871)

4.1.1 The example of dot product

At this juncture, it is appropriate to give one pathological example to illustrate a situation of unhistorical reading of history. A problem which frequently confronts both the Singaporean students and their teachers is to seek for the reason why abstract mathematical definitions were formulated the way they are. This specific example is taken from the contexts of Singapore junior colleges².

Example 4.1. The dot product of two or three dimensional vectors is a notion taught in the H2 mathematics syllabus (see Section 3.1.6) and is *traditionally* defined to be

$$\mathbf{a} \cdot \mathbf{b} := |\mathbf{a}||\mathbf{b}| \cos \theta$$

where θ is the angle between \mathbf{a} and \mathbf{b} . What then follows *in most lecture notes used in many junior colleges* is a remark that $\mathbf{a} \cdot \mathbf{b}$ can also be defined in a seemingly ‘coordinates-dependent’ manner:

$$\mathbf{a} \cdot \mathbf{b} := a_1b_1 + a_2b_2 + a_3b_3$$

where $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$. Confronted by abstractness of the first definition and the presence of two different but equivalent formulations, students are often confused. If they ever raised their concern, they usually will be asked to accept the equivalence of the two definitions as a fact. Teachers find it equally frustrating because the textbook definition of the dot product is really quite unnatural. Stranger still is the fact that, among some examples

²The author does not know if this is a phenomenon peculiar to Singapore junior colleges. Further investigations need to be carried out

that follow, the famous cosine rule³ is shown to be a result of the properties of the dot-product.

The sequence of definitions and propositions involving the dot-product as defined traditionally in the classrooms are often mistaken, by many, for the historical one.

Tracking the origins of the dot-product helps demystify the situation. The dot product made its first appearance as one of the by-products of a higher concept known as the *quaternion product*, an invention of Josiah Willard Gibbs and Oliver Heaviside (independently) in attempt to overcome the cumbersome calculations involved in the original Maxwell's electromagnetism equations. For a historical account of this, the reader is referred to [7] by the celebrated mathematical historian, Florian Cajori (1859 – 1930). The second definition $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$ turns out to be the original definition. Then the equivalence of the two definitions can be obtained by means of the Cauchy-Schwarz inequality and the cosine rule.

As this case illustrates, a recourse to the history of mathematics offers the teacher a great opportunity not only to re-organize the flow of thought in a logically (and historically) coherent manner, but also to help students better appreciate the formulation of abstract definitions with the relevant historical background. Moreover such probing reveals that the cosine rule *should not* be derived as a *mathematical* consequence of the properties of the dot-product! It is also worth noting that, with the exception of the well-written guidebook [26] (see in particular, pp. 7 & 9), most textbooks and lecture notes make no mention about the origins of the dot product (or the cross product, for that matter).

4.2 From genetic epistemology to infusion of National Education in mathematics

What is *psychological recapitulation*? In a nutshell, this refers to the belief that “the *ontogenetic* (that of an individual) development of the child is but a brief repetition of the *phylogenetic* (that of mankind) evolution” ([33]). The ontogenetic development stages are termed as *psychogenetic stages*, and the phylogenetic ones as *historical stages*. Using these terms, the idea of psychological recapitulation can be rephrased as follows: Any segment of an psychogenetic stage can be seen as some time-contracted copy of a segment of a historical stage. Holding onto this simplistic view, many teachers in Singapore are still equating the historical approach of teaching mathematics to the direct import of *historical movements* into the classroom experience, thus hoping to create environments to recapitulate or repeat the corresponding *psychogenetic movements*. Thus, to them, an inability to reconstruct the past with any certainty implies an inability to recreate with confidence the desired learning environment necessary for the students to move from one psychogenetic stage to another.

³Suppose the sides of a triangle ABC are labeled as $AB = c$, $BC = a$, $CA = b$ and $\angle ACB = \theta$. Then the cosine rule states that $a^2 + b^2 - 2ab \cos \theta = c^2$.

But this idea was contested by Jean Piaget’s theory of *genetic epistemology* (i.e., simply put, development theory of knowledge acquisition). He argued that the understanding of the process of acquiring knowledge (and particularly, scientific and mathematical) should be based on the intellectual instruments and mechanisms allowing such a process to take place. Following this argument, Piaget and Garcia came to a conclusion in [42] that elements of knowledge acquired by the individual, as provided by the external world, can never be divorced from their social meaning. They advocated that a distinction must thus be made between the mechanisms by which knowledge is acquired and the way in which objects are conceived by the subject. So, for instance, cultural differences result in differences in the conception manner and consequently account for the differences in the knowledge acquired. However, the Russian psychologist Lev Vygotsky took a different approach in his study of the relation between ontogenesis and phylogenesis. Pertaining to the epistemological role of culture, he argued that culture “not only provides the specific forms of scientific concepts and methods of scientific enquiry but overall modifies the activity of mental functions through the use of tools – of whatever type, be they artefact used to write as clay tablets in ancient Mesopotamia, or computers in contemporary societies, or intellectual artefact such as words, language or inner speech ([60]).

This leads to the study of mathematics developed and used by different cultural groups across the world from ancient to modern times, under the name of *ethnomathematics*. One prominent study in Singapore has been pioneered by Khoo Yoong Wong in [61] and [62] regarding Singapore’s National Education (NE) initiative of adding cultural values to mathematics instruction. One of the aims of Wong’s paper is to equip the Singapore teachers with a framework by which they can infuse NE into their mathematics lessons. In addition, a number of practical suggestions, based on various elements ranging from the life stories of mathematicians, mathematics in literature and films to social impacts of mathematics, are given.

4.3 Proposed didactic framework and its implementation

The preceding subsections have clearly revealed a general principle. In process of acquiring new knowledge and later internalizing it, the learner must make *intellectual leaps* – these are precisely the passage from a lower psychogenetic stage to a higher one. Similarly, for mankind to advance in its acquisition of knowledge, there must be events which hallmark the *historical leaps*. The upshot in Piaget and Garcia’s theory is that instead of focusing on the parallelism of the contents (or elements) between psychogenetical leaps and the historical leaps, one should focus on the relationship between the mechanisms which are responsible for each of these leaps.

Therefore, a mathematics teacher who wishes to exploit the powers of the historical approach can do so in two steps. The first step deals with the ontogenetic aspect, which consists of (in the specified order)

- (1) identifying the *learning points* along the syllabus or curriculum where the

learner needs to experience a psychogenetical leap (e.g. the situation may present difficulties or confusion),

- (2) understanding the nature of the confusion or difficulties presented at these points, and
- (3) exploring the psychogenetical mechanisms which can be used to promote a smooth transition over these points.

The next step deals with the phylogenetic aspect, which consists of (in specified order)

- (1') identifying historical mechanisms which can be associated to those psychogenetic mechanisms identified above in (3),
- (2') understanding what problems these historical mechanisms were employed to tackle, and
- (3') identifying the *historical points* i.e., suitable events or elements in the history of mathematics where those problems in (2') occurred.

The reversal in the ordering of actions in the second step is intentional. So we do not expect a direct point-to-point match between the ontogenesis and the phylogenesis with regards to acquisition of mathematical knowledge. This, when translated into lesson planning, entails that the issue to be addressed in the lesson need not be matched by the same issue in history. The process of locating the historical point is called 'sourcing'. This usually involves searching for relevant data using library resources, internet resources (such as Wikipedia), and having regular discussions and sharing sessions among colleagues. Teachers and lecturers in Singapore meet regularly to share their teaching experience during local educational conferences and training workshops. All educational institutions and schools have ready access to good library and internet resources. So sourcing does not pose a big problem to Singapore teachers and lecturers.

Equally important is the process of turning the salient aspects of this historical point into actual lessons. This later process, we call 'implementation'. Together, the backward *sourcing* and the forward *implementation* constitute the classroom realization of what Luis Radford defines to be the *articulation between the psychological domain and the historical domain*, i.e., the articulation between students' learning of mathematics and conceptual development of mathematics in history (see [43]).

A good lesson design can be seen as the result of several iterations of sourcing and implementation. However, it is still possible for the process to fail since, for example, one may not be able to locate any suitable historical point which corresponds to the psychogenetical point intended in the lesson. In this case, one must remember that the historical approach is *one* of the many possible approaches.

Based on the didactical framework proposed here, several ways of implementing history in the mathematics classroom, together with examples, are given in the appendix.

Areas of concern (in the form of ‘voices of objections’)	%
Lack of teacher’s training in using history of mathematics	16.9
Lecturer’s lack of historical expertise	13.2
Lack of resources	9.4
Lack of time (e.g., existing schedules and timetables are very tight)	16.9
History is <i>not</i> mathematics.	
History is tortuous and confusing rather than enlightening.	
Hard to make any connection with the present day context.	3.8
Dislike of history by students, and possibly by lecturers	7.5
Chief emphasis should be on equipping students with routine skills	5.7
(and they already have problems do that), and why bother using history?	
Lack of appropriate assessment rubrics	13.2
If historical component is not counted towards assessment, then students will not pay attention to it.	9.4
Spending too much time retracing history & getting digressed from main topic	3.8
No faith in using history of mathematics in teaching mathematics.	

Figure 3: Percentage breakdown of areas of concern

5 Feasibility: potentialities, limits and risks

Associated to whichever teaching methodology are always three factors which are intrinsically connected with measuring the effectiveness of using history of mathematics in the classroom. They are the *potentialities*, *limits* and *risks* ([20] and [34]) involved in the methodology. In addition, these three factors will determine the feasibility of the methodology in question. With the historical approach, it is no different. Our approach here is more generic as compared to [34] in that we do not explore the potentialities, limits and risks of specific implementation models with regards to the historical approach. Instead we give a brief consideration of these three factors from the perspective of teachers and lecturers who are at the frontline in the classrooms. This overview is a collection of opinions collected from a survey which has been conducted, earlier on, by the author to investigate the areas of concern of teachers and lecturers when history of mathematics is to be integrated into the mathematics teaching. Because these are realistic opinions ‘from the ground’ (particularly, on limits) regarding the historical approach, it is hoped that their views as summarized here will be a useful source of information in the future should history of mathematics be posited as a prominent feature in the national curricula. The percentage breakdown are given in Figure 3. Next, we shall summarize the comments given by the participants of the survey which concerns the potentialities, limits and risks.

5.1 Potentialities

Based on the survey analysis, a majority of the teachers who participated in the survey do recognize the potentialities of the historical approach, in that it can (i) result in a better understanding of the topic, (ii) create a learning environment different from the traditional setting, and (iii) inculcate better attitudes of the learners as well as their teachers. There is also a general agreement that famous historical anecdotes are effective in breaking the monotony and boredom in the class.

5.2 Limits

The two main areas of concern are (1) lack of teachers training in history of mathematics, (2) lack of curriculum time and (3) lack of assessment rubrics. These are elaborated below.

5.2.1 Lack of teachers training in history of mathematics

In a recent and fairly comprehensive study [45], Gert Schubring gave an international overview of the issue of training teachers to be competent in the history of mathematics. His study indicated that practising a historical component in teacher training is no longer restricted to those countries with an extended tradition in mathematics history and a considerable mathematics community. He also observed that there has been a growing number of countries where historians of mathematics, or mathematics educators with a strong interest in mathematics history, have achieved academic positions to effect an introduction of mathematical history courses into teacher training. One striking example raised was Hong Kong. Although there are no official regulations requiring courses in mathematics history for mathematics teachers; yet at two of the well-established universities in Hong Kong such courses are regularly offered. These courses have a varying range of objectives: Some for introducing the historical element to teachers, while others dwell deeper into the development of school mathematics and the instructional use of historical materials.

Singapore and Hong Kong have many similarities: history, culture, economy and education. In contrast, Singapore suffers from a severe lack of teachers training with respect to the historical approach. This is reflected in the general lack of confidence⁴ in the teachers to incorporate a historical component in their lessons. This could, in part, be traced back to an absence of compulsory pre-service and in-service courses conducted in the teachers' training institute: the National Institute of Education (NIE)⁵. A majority of the local mathematics graduates received their training in mathematics from the National University of Singapore. The various courses taken by the mathematics undergraduates are reported in [41]. In this same report, history of Chinese mathematics was

⁴Refer to Ng's remark on the lack of confidence in the teachers who volunteered to help in the ACMEP [35].

⁵Very recently, though, history of mathematics was introduced as an elective module taught by P.M.E. Shutler.

mentioned as one of the non-traditional research areas which the university engaged in, notably in the 1970's. In 1980's, history of Chinese mathematics was included as one of the many topics in an elective module, intended for non-mathematics majors from other faculties and departments. In summary, the history of mathematics has never been offered at the university level as a compulsory module. Historical information, if it exists, has been scattered within the mathematics courses taught in the universities.

5.2.2 Lack of curriculum time

Many teachers raised the issue of a lack of time in the curriculum. What they meant is that a typical teacher has a relatively heavy teaching workload, ranging from 16 to 22 hours of contact time (with the students) per week. Besides teaching, teachers spend a considerable amount of time in lesson preparation, management of administrative duties, student counseling and supervision of students in co-curricular as well as enrichment activities. A considerable fraction of the teachers surveyed (16.9%) objected to the use of the history of mathematics because such an integrative approach, they feel, demands extensive resourcing, planning and re-adjustment of the existing teaching designs. In fact, it calls for a drastic change in the way the teachers (and the students) approach each mathematics topic, and the means by which students are assessed. In summary, not many are willing to take the risk for fear of a drop in the students' performance in national examinations.

5.2.3 Lack of assessment rubrics

As brought out in the previous section, there is a lack of assessment rubrics to measure the performance of the student with regards to the history of mathematics. In fact, the question that is frequently raised: Can it be measured in the first place? Even if we do not assess the students based on the historical component, will the student upon realizing this continue to pay attention to the history of mathematics in comparison to the other topics in this subject?

5.3 Risks

The simplest way to use history in teaching mathematics, as many teachers recognized, is to do so implicitly. By this, they mean to trace the origins of a mathematical concept relying on historical records, to conceive it in a different situation, and then to return to the instant in which the theory "branched out". According to [34], a mathematics teacher in doing so enters a mode of *didactical transposition*, i.e., calibrating the pedagogical processes in relation to the conceptual difficulties and complexities of a given topic. The above survey revealed that teachers who are resistant to the historical approach are afraid of going too far back into history and unable to make a relevant connection with the topic in question *within reasonably short time*. Other risks mentioned by the teachers in the survey include an overt emphasis of historical elements as

opposed to the mathematical content, unfamiliarity of the students because of the cultural differences between the past and the present, as well as the fear of using the word ‘history’ as it might lead students to think of it as a humanities subject.

6 An action-research based case study

6.1 Aim of case-study

Here we report a small-scale case study, based on an action-research approach, that was carried out by the author in the Singapore Polytechnic during Semester 2 of the academic year 2007–08. The case study consists of integrating the history of mathematics into the teaching and learning of mathematics, with the aim of understanding its effects (both in terms of motivation level and academic performance) on the students. We want to investigate whether such a methodology help the students develop (or even enhance) a positive attitude centred – around strengthening of the following aspects: (i) interest and appreciation, (ii) belief, (iii) confidence, (iv) perseverance. Notice that we choose not to study the impact of the historical methodology by employing some measurements via batteries of tests for determining the mathematical competence of students (i.e., a comparison using common tests and examinations between classes in which history of mathematics was used or not used). The justification for this is that the attainment of objectives claimed for using history cannot be measured by assessments ([44]).

6.2 Description of the experiment

A batch of 102 students in Singapore Polytechnic who enrolled for the Certificate of Engineering Mathematics (CEM) took part in this study. Upon completing the requirements of this certificate course, these students earn an extra qualification in addition to the diploma which they are majoring in. The choice of such a small group is intentional in that the performance of an otherwise larger population of students (who are taking a compulsory module) might be affected by the experiment. The reader may want to note that the certificate course is offered to academically stronger students who possess a score of 13 or less (L1R5: ‘O’ level Singapore-Cambridge examinations). Thus the content covered is more rigorous and difficult, ranging over 4 different modules: Calculus (I and II) , Differential Equations and Linear Algebra & Vectors.

The historical approach is applied to the linear algebra class taught by the author. This methodology intentionally integrated the history of mathematics into the existing syllabus laid down by the department. In the integration process, the entire syllabus was taught without any disturbance to the existing time schedule allocated to the lecturer. As such, the approach did not involve a compromise for lack of time. The first six weeks did include historical snippets but no obvious incorporation of the historical component into the class activi-

ties, and the next six weeks represent a full-fledged application of the historical programme. This is in line with the action-research approach, with regards to introducing an action of intervention, with the aim to investigate the *effects of using history of mathematics in teaching and learning mathematics on the students' attitude* in this linear algebra class. The real focus, as we explained in Section 2, is enhancing the learners' attitude towards the subject.

Each lesson from the last six weeks involve very careful planning based on the proposed didactical framework (see Section 4). This means that crucial psychogenetical leaps were identified in the Linear Algebra syllabus. The process of sourcing was carried out and this resulted in the selection of a number of appropriate historical developments. Each of the identified historical element is then crafted into a lesson plan suitable for a three hour lecture-cum-tutorial. For lack of space, we shall not dwell in detail the content of each lesson, but hopefully the table below which shows the various topics and the corresponding historical elements selected gives the reader a flavor of the lessons taught.

Week No.	Topic	Historical element
6	Gaussian elimination	Ancient Chinese Rod Numerals
7	Matrices	Life story of <i>forgetful</i> Sylvester
9,10	Eigenvalues and eigenvectors	The Invariant Subspace Problem Worksheets
11,12	Vectors	Descartes' dream in the furnace Descartes' secret notebook

6.3 Qualitative analysis

6.3.1 Student's and teacher's log

At the end of each lesson (for all the 12 weeks), every student kept a student log book in which is written one's feelings about the lesson, comments of what one has learnt (or not learnt), things they liked or disliked, remarks on the nature of the lesson. In short, the students conscientiously kept record of what went on in the classroom for the entire twelve weeks. The author also kept record of the teaching and learning processes in the form of (i) lesson plans and (ii) a teaching log. The author also had access to the students' log books and were periodically collected as a source of student feedback. In addition, the reporting officer of the author also carried out a lesson observation (in week 11) to ensure that effective teaching and learning takes place within the boundaries of the official syllabus and the limits of the classroom setting. Both the students and the reporting officer were aware of the active use of the historical approach in the lessons. The use of the student's log and the teacher's log was to provide qualitative information on the motivational aspects of both the learners and the instructor, which otherwise could not be easily quantifiable.

The following are some samples (far from being exhaustive) from the students' log which reveal the positive effect of the historical approach on their attitudes towards the subject.

1. I am motivated by Rene Descartes because every science derives from maths.
2. Besides mathematics, I learn something special: I learned to be passionate and (to) love something and put it into action.
3. History of mathematics very interesting.
4. I am motivated by the way the lecturer presents the lecture with some interesting things about past mathematicians and eigenvectors and eigenvalues.
5. I look forward to the next lesson because I am motivated and want to learn more.
6. This is a mental picture of my impression of today's lesson: Very motivational. I do not feel any barriers between students and lecturers.

6.3.2 Student survey

In addition to the student's and teacher's log, a student survey was conducted in the cohort of 102 CEM students with regards to their attitudes towards linear algebra and mathematics as a result of the lessons they attended for the Linear Algebra & Vectors course (regardless of whether they had received lessons using the historical approach).

The survey specifically include four important aspects related to the students' attitudes towards mathematics: (1) Belief, (2) interest, (3) confidence, and (4) perseverance. Figures 4–7 show four tables which record the median score (ratings from 1 to 5: 1 for lowest and 5 for highest) of the response to each of the survey items by the two different groups (the treatment group vs the control group), for the various component. We carry out, for each item, the Wilcoxon-Mann-Whitney test at 5% level of significance whether the median score for the treatment group of size 17 (TM) is higher/lower than that of the control group of size 57 (CM). The following conventions are used: N.S. = Not significant, Sig. = Significant; > means that $TM > CM$; < means that $TM < CM$.

In a nutshell, the test results indicate that the historical approach is more effective in the components of belief and perseverance. However, one must note that the case study considered here is of a very small size for the result to be conclusive in general.

7 Conclusion

Scientific advancement is of great importance to mankind and thus scientific research is keenly sought after by countries in all parts of the world. As a result, one can safely say that science is an issue of politic interest. The emphasis which Singapore places on scientific research and development has always been

No	Survey item	Sig. > or <?
	Belief	
1.	I believe that linear algebra is useful.	N.S. >
2.	I believe, as a result, that mathematics is useful.	Sig. >
3.	I believe, independent of my experience in the linear algebra module, that mathematics on the whole is useful.	Sig. >
4.	I believe that the mathematical theorems of linear algebra are universally true.	N.S. >
5.	I believe, as a result, that all mathematical theorems are universally true.	N.S. >
6.	I believe, independent of my experience in the linear algebra module, that mathematical theorems are universally true.	Sig. >

Figure 4: Student survey: Belief component

strong. Ultimately, mathematics education is also a political issue since it is the foundation of science. The history of mathematics bears a strong relationship with the subject of mathematics, with mathematics education as well as the general conceptions of people at each time and at each place. The content of the national mathematics curriculum is one of intentional choice, and rightly so a political one, and one which has been made in line with the country's vision and direction. Thus the decision of what constitutes the mathematics curriculum and how it is to be implemented is of great significance and demands delicate care in handling. With the existing curriculum, and suppose one has to intention of incorporating history of mathematics into the teaching of mathematics in Singapore, the education authority must have at its disposal a variety of strategies with which to deploy.

The question of why a historical approach should be infused in the official mathematics curriculum of Singapore is one which must be answered before such a decision is to be taken. Even if one adopts this approach, how can it be realized? As the previous sections shows, there are many means by which a historical dimension can be injected into the teaching and learning processes. These different entry points into the education system range from the choice of an appropriate didactical framework, the actual implementation means, the influence on the curriculum, teachers training, and the inculcation of positive attitudes (both the teachers and the learners) towards the subject. In the presence of so many possibilities and aspects of this historical approach, there are many avenues where confusion lurks in and so any decision to make use of it

No	Survey item	Sig. > or <?
	Interest	
1.	Genuine interest is developed in me.	N.S. >
2.	I am keen and enthusiastic about the development of the theory.	N.S. >
3.	I am keen and enthusiastic to learn about the applications of the theory.	N.S. >

Figure 5: Student survey: Interest component

No	Survey item	Sig. > or <?
	Confidence	
1.	When explaining to a fellow classmate, I feel I am up to the job.	N.S. >
2.	I am convinced that I understand the various topics of linear algebra.	N.S. >
3.	When asked to work out a problem in front of the class, I am not up to it.	N.S. <

Figure 6: Student survey: Confidence component

must call for clarity in the advice given to policy-makers.

Taking a careful stance is crucial because of the limitations and the risks as discussed in earlier. Policy proposals should be carefully analyzed before they are carried out in the schools. In order to anticipate what problems and concerns are likely to surface, the present study hopes to gather some relevant preliminaries for future studies and action. In Singapore, various groups of people are concerned over what mathematics are taught in schools: the teachers, the principals, the ministry of education, the mathematics educators and researchers, students and their parents, politicians, publishers, the media and the employers. The historical approach, if implemented, will have an effect on these groups of people.

Teachers Many difficulties can be expected even when one speaks of the implementation at the classroom level, not mentioning the other social levels. As reflected in Section 5, incorporating history of mathematics into the mathematics curriculum is thought of, by many, to incur an expensive cost of time, effort and money. Some teachers are even worried that it might be distracting the students from gaining basic mastery of mathematical skills and problem solving. Moreover, as the survey reveals, Singapore teachers are quite resistant to this approach for lack of the necessary expertise and training even if they were

No	Survey item	Sig. > or <?
	Perseverance	
1.	When I meet a routine but tedious problem, I give up.	N.S. <
2.	When I meet an unseen problem, I keep trying until it is solved.	N.S. >
3.	In group discussions, when challenged by a difficult problem, I advise the group to evade the problem.	N.S. <
4.	When challenged with a question by the teacher, I put some thought to the problem and try my best to answer.	Sig. >

Figure 7: Student survey: Perseverance component

to pursue this approach. To some of these teachers, it might even be the case that they have chosen to read mathematics and teach it since it did not involve the humanities aspect of writing essays and dealing with history. In fact, there is great danger in enforcing this methodology as a new pedagogical directive – doing more harm than good. Ultimately, the effectiveness of teaching relies more on the enthusiasm of the teacher, rather than the manner of delivering the subject matter.

Principals In the capacity of the headmaster, to what extent should the history of mathematics be implemented given the existing staff profile? Like in other methodologies, principals and possibly heads of departments must decide on the degree of emphasis in requiring teachers to be sufficiently trained in the historical methodology. Singapore schools compete among themselves in various national and international mathematical competitions. These usually call for participants who are mathematically capable. Perhaps the history of mathematics can be an avenue to explore for students who are more inclined towards the humanities. School management can then take advantage of this to create avenues for this group of students to contribute towards mathematics, through outdoor experience and historical exhibitions, etc.

Ministry of Education It cannot be further stressed the importance of the ministry and its role in the implementation of such a historical approach nationwide. After all it is the official body governing the direction of mathematics education in Singapore. Till now, there is no clear direction how schools can positively fortify the learners' attitude towards mathematics.

With all that has been explored within the space of this article, we are still far

from being able to give foolproof advice to the policy makers. However, what we have attempted to do is to introduce to the learning community an interesting and rich alternative methodology in teaching and learning mathematics.

Employers *How competent are my prospective employees in problem solving?* This is one important question any employer will ask. Is there any assurance that students who received mathematics education via the historical methodology are truly those who have understood the mathematics as a language of conveying abstract ideas and have mastered the problem skills intended to be developed therein? Does the historical approach necessarily create a higher quality workforce?

Parents Most Singaporean parents are concerned with the education of their children. There is a growing trend for parents to engage private mathematics tutors for their children and to enrol their children for mathematics enrichment courses outside their school hours. What does the historical approach then mean to them? Since the parents themselves did not learn mathematics via this approach, how much faith do they have in this methodology? How can they help in encouraging their children to learn mathematics the historical way?

Acknowledgment

I would like to thank several people, without them, there would not be this paper. My gratitude goes to A/P Lee Peng Yee for reading my first draft and giving me several pointers, to Dr Jaya Shreelam for his expertise in performing the many tedious statistical tests in this study, and to the teachers and lecturers who participated in the survey. Thanks also goes to my lovely Linear Algebra class (PCEM/FT/2B/01) who was always there for their lessons with me. I also take this opportunity to thank my department for supporting me in this research project.

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A Implementation methods

The methods of implementation recorded in this section come from four different sources: (1) Those employed by the author in the recent case study (see appendix) in one of the polytechnics, (2) those employed by Ng in [35] and Lim & Pang in [29], (3) those which were recorded by the author during lesson observations or in the capacity of a student, and (4) those activities known to the author via private communication with other research colleagues. Thus they are far from being exhaustive.

A.1 Historical snippets

Historical snippets are pieces of historical information which are incorporated into the main text. As mentioned in Section 3.2, several commercially produced secondary school textbooks already make use of historical snippets. In this section, we shall characterize historical snippets according to their format and content, with examples drawn from textbooks for the New Mathematics Syllabus D as used in Singapore schools, lecture notes in polytechnics and junior colleges. The characterization and classification we employ here model closely after those used in [59]

Under *format*, we consider the

1. positioning in the text in relation with the mathematical exposition to which it refers: whether the snippet occurs before the text, in the midst of the text (e.g., in the footnotes), in parallel to the main text but separated from it, or after the text.

Example A.1. Teh & Looi, in [55, 56, 57], position all the historical snippets (as well as all other interesting activities and information) alongside the main text, i.e., parallel but separated from the main exposition which they are referring to.

2. pedagogical approach: whether the snippet is merely expository, does it engage the reader into active involvement such as solving a problem, identifying a notation or other projects or activities, or a combination of some of these.

Example A.2. (Teh & Looi [55], p.264)

For over a thousand years, Greek mathematicians used to consider the tangent of a circle as a line touching the circle at only one point. To them, a circle was a stagnant figure and no more than a geometrical shape. However, Isaac Newton, an Englishman, considered the tangent at a point to be the limiting position of a line passing through that point, as well as another point, which was running closer and closer to the former point. Newton's circle was a circular path which allowed motion and thus, a dynamic one. With his idea of a running point on the circle, he established the theory of calculus.

The narrative is categorized under the heading of “Back in Time” with an obvious intention of telling a historical story or providing historical information of mathematical concepts. The snippet is expository and points towards an alternative conception of a tangent to a curve. Note that calculus is not covered in the New Mathematics Syllabus D.

Example A.3. (Teh & Looi [55], p.65)

The ancient Egyptians were the first to use fractions. However, they only used fractions with a numerator of one. Thus they write $\frac{3}{8}$ as $\frac{1}{4} + \frac{1}{8}$, etc. What do you think the Egyptians would write for the fractions $\frac{3}{5}$, $\frac{9}{20}$, $\frac{2}{3}$ and $\frac{7}{12}$?

Under the meaningful heading of “Investigate”, students are invited to provide their opinion to a given question about Egyptian fractions. So this snippet not only contains relevant historical facts about the origin of fractions but most importantly engages the reader actively.

3. substantiality: to what extent are details of historical facts provided, e.g., besides the dates given to a mathematician, what are details of his life is relevant to the mathematical exposition.

Example A.4. (Teh & Looi [56], p.156)

*Carl Friedrich Gauss, a German mathematician (1777 – 1855), is often considered to be the greatest mathematician of all time. He proved that every algebraic equation in one unknown has a root. This is known as the **fundamental theorem of algebra**.*

The above snippet contains dates of the mathematician but only in years. The main focus here is the most important achievement of this mathematician. This snippet is included in parallel to mathematical exposition on plotting graphs of polynomials, up to the cubics. An interesting point to take note is that this snippet is slightly inaccurate in its content since the fundamental theorem of algebra states that every algebraic equation with complex coefficients in one unknown *has a root in the field of complex numbers*.

4. style and design: whether the narrative is formal and accessible to student readers, whether it is relevant and distinguishable from the main text (e.g., using different colors, backgrounds, fonts), whether it is visually aesthetic.

Example A.5. Textbooks such as [54] and [55, 56, 57] make use of relatively formal language (in English) in their snippets. The language is pegged at an elementary level so that it is easily accessible by secondary students. At this juncture, the reader is reminded of the fact that English has been the medium of instruction for mathematics since 1979 (see Section 3.1) and students have attained a certain level of mastery of the English language sufficient to understand the content of the snippets. Notably, the snippets contained in these textbooks are printed in a smaller font and a different color (blue for [55], brown for [56], etc) so that visually the snippet is distinguishable from the main text.

Under *content*, we consider what the snippet consists of, and which aspects of history it emphasizes:

1. factual data: photographs, facsimiles of pages of books, biographies, attribution of authorship and priorities, anecdotes, dates and chronologies, pictures of mechanical instruments, architectural, artistic and cultural designs.

Example A.6. (Raffles Junior College [40])

In the comment box (intended for lecturers), it is written:

Comment [m9]: Can tell the famous story of 10 year-old Gauss who was asked to find $1 + 2 + 3 + \dots + 100$ during maths class.

The target audience of the above example are the Year 5's (i.e., junior college first year students). At this point of the lecture, the lecturer can choose to relate the historical anecdote of Gauss' mathematical ingenuity at a young age.

2. conceptual issues: snippet contains text which touches on motivation, origins and evolution of ideas, ways of noting and representing ideas as opposed to modern ones, arguments (which may contain errors, provide alternative conceptions, etc), problems of historical origin, ancient methods of calculations, etc.

Example A.7. (Singapore Polytechnic)

In the slides designed for the lectures on the solution of non-homogeneous second order differential equations with constant coefficients using Laplace transforms, the author used the following snippet to introduce the difference and similarity between the modern method of employing Laplace transform and that used by English engineer and mathematician, Oliver Heaviside, exploiting the differential operator D . This is done by using a snippet located at [58], which reads:

Between 1880 and 1887, Heaviside developed the operational calculus (involving the D notation for the differential operator, which he is credited with creating), a method of solving differential equations by transforming them into ordinary algebraic equations which caused a great deal of controversy when first introduced, owing to the lack of rigor in his derivation of it. He famously said, "Mathematics is an experimental science, and definitions do not come first, but later on." He was replying to criticism over his use of operators that were not clearly defined. On another occasion he stated somewhat more defensively, "I do not refuse my dinner simply because I do not understand the process of digestion."

A.2 Primary sources

By far one of the strongest feelings about using (historical) primary sources has already been articulated two centuries ago by Norway's greatest mathematician, Niels Henrik Abel (1802–1829), in the margin of one of his notebooks:

"It appears to me that if one wants to make progress in mathematics one should study the masters."

A report by Hans Niels Jahnke [21] expresses that "the study of original sources is the most ambitious ways in which history might be integrated into the teaching of mathematics, but also one of the most rewarding for students both at school and at teacher training institutions."

Typically, an original source derives from appropriate excerpts of *original* writings by mathematicians of the past. Of course, original copy does not mean authentic copy, and thus refers to photocopies of manuscripts. In most cases, when an original source is used in a lesson, it *replaces* the usual with something different. This allows the participants realize that mathematics is an intellectual activity which *evolves over time*, rather than as a meaningless body of knowledge or techniques. The learners are frequently alerted to the difference in quality of publication, handwritings, notations, the language used, the intended audience and the illustrations used.

In teaching, instructors often fall into the trap of thinking that concepts appear as if it is existing. For instance, the notion of function were invented and did not naturally occur by itself. History reminds us of this, and challenges one to look at familiar things the unfamiliar way. This thus requires, according to [21], a state of *reorientation* of the mind.

Most crucially, perhaps, is that by integrating the original source in the lesson, we invite the teachers and the learners to go on a cultural trip back in time. The objective is then to allow the learners experience the cultural component of mathematics. In Singapore, this would tie in nicely with the *culture* component of mathematics instruction as proposed by Wong [61] – allowing mathematics to be viewed from a more humanistic perspective.

Example A.8. (Singapore Polytechnic) In the linear algebra class taught during September 2007 to January 2008, an original source was employed in a class activity. The original source is a copy of the page from Gottfried Wilhelm Leibniz’s handwritten copy of Descartes’ secret notebook, together with an excerpt of a modern commentary written by Amir D. Aczel (p. 226–228 of [1]). The teaching package which spans about three weeks was centred on the marriage of geometry and algebra, i.e., coordinates geometry – a crucial contribution by French mathematician and philosopher, René Descartes (1596–1650). This particular class activity was positioned during the second week when the relation of geometry and algebra was explored using vectors. The students were given time to study the original source, noting the difference in writing, notations and language used in the manuscript as compared with modern times. They were asked to pay attention to two number sequences 4, 6, 8, 12, 20 and 4, 8, 6, 20, 12 found on Leibniz’s copy of Descartes’ notebook. During this part of the activity, the students were invited to explore what these sequences could possibly mean. Finally, with further help from the commentary and the author, they were able to come out with the conjecture

$$F + V - E = 2$$

where F is the number of faces, V that of vertices and E that of edges of all regular three dimensional solid. This topological invariant is usually credited to Leonhard Euler (1707–1783). The learning outcome intended for this classroom activity is for the learners (1) to understand a non-trivial connection of algebra and geometry that goes deeper than the current vectorial treatment, (2) to be acquainted with the cultural differences as reflected from the mathematical writing, and (3) to bring out the spirit of determination as displayed by Leibniz in the quest for mathematical knowledge.

A.3 Worksheets

The use of worksheets is a common teaching tool worldwide and, notably, Singapore schools and educational institutions first made active use of this tool during the period of mathematics reforms (1970 – 1980) (see Section 3.1.3).

Such worksheets are intended for either group work or individual learning, and may take place in the presence or absence of information technology.

Worksheets are used to help students master a mathematical algorithm and summarize learning points in a lesson. They can also be employed as bring-home exercises for students to self assess their learning. Worksheets nowadays also come in a collection of questions or small activities, intended to guide students into group discussion and to explore and discover new knowledge, possibly based on some existing knowledge. For this more recent version of worksheets, a historical component can be integrated so as to develop critical thinking and meaningful understanding.

Example A.9. (Singapore Polytechnic) In the same linear algebra class, worksheets have been employed. These are worksheets which help students consolidate their observation and learning points in conjunction with an online software used for teaching two- and three-dimensional vectors, and linear transformations. This tutorial was positioned after the students were taught the basic definition of a linear transformation. In the first part of the lesson, they were asked to investigate the (geometrical) effects of pre-multiplying a vector by a matrix and to record their findings in the first worksheet. After this round of investigation, the students were introduced to the German mathematician, David Hilbert (1862–1943) and his 1900 presentation of a collection of important mathematical problems. The invariant subspace problem was presented to the class. Students were given an opportunity to try to understand the statement of the problem. This serves as an introduction to the eigenvector problem of linear algebra. The second worksheet took off from this problem and invited the students to investigate, with the aid of the computer software (and trial-and-error), which vectors remain invariant in their directions under a given linear transformation. The two worksheets were intended to precede the lesson where the standard algorithm of finding eigenvalues and eigenvectors were taught. Copies of these worksheets appear in the appendix.

A.4 Historical packages and enrichment programmes

By a historical package, I mean a collection of materials used with a focused intention of teaching a small but crucial topic which has strong links with other parts of the syllabus. Such packages, according to [6], are most suitable for two to three periods, and their nature is that they are ready for use by teachers in their classrooms. It is intended that the depth of a package exceeds that of a historical snippet (which usually is used as a historical aside) but falls short of a fully developed historical treatment. A package usually consists of an induction (or introduction) via suitable historical quotations, anecdotes or historical snippet. The presentation of this part is done by the teacher. What follows is, based on this introduction, a series of activities or questions proposed for groups to carry out. Of course, the discussion is centred on some mathematical concepts to be learned and wherever possible anchored around fragments of primary sources. A well-designed package is easy and ready to use as it should

provide the teacher with a folder containing the detailed text of the activity, historical and pedagogical background, guidelines for classroom implementations, expected learners' outcome and illustrative teaching aids (such as lecture slides, mechanical instruments, original sources). An enrichment programme is one which is intended to equip students with additional skills or knowledge that are not normally taught in their schools. In such enrichment programmes, history of mathematics can be either be taught as a subject on its own right or used as a means to enhance students' achievement in mathematics. Usually, the time span of an enrichment program varies from a few weeks to as long as a few months, but usually at a low frequency, say once a week.

Example A.10. The lessons in the linear algebra class (conducted by the author) which cover the connection of geometry and algebra using a vectorial approach span over three to six weeks. This in turn was divided into five parts. The first part is being carried out using a teaching package. The lessons span over a three-period session. Since a connection is to be built between algebra and geometry, the terms 'algebra' and 'geometry' must be explained. This was done using suitable historical works. For algebra, we use the narrative on Al-Kitab al-Jabr wa-l-Muqabala which is a compendium of balancing and solving equations of a Persian/Arabic origin; for geometry, we mention Euclid's Elements. Pictures of copies of such works were shown in the lesson as snippets, and the importance of each work explained. This leads to the definition of the Euclidean space, following which is a classroom activity which was intended for learners to discover other kinds of geometry besides the Euclidean one. In particular, students were asked to investigate the shortest path that can be traced on the different curved surfaces (ranging from apples, oranges, balls, etc).

Example A.11. Both the ACMEP (Ancient Chinese Mathematics Enrichment Programme) conducted by Ng (see [35]) and CAIUH (Classroom Activities Involving the Use of History) conducted by Lim (see [29]) are considered enrichment programs which are based on history of mathematics, and are intended to improve on the students' performance in and attitude towards the subject.

A.5 Experiential activities using ancient instruments and artefact

By an ancient instrument, we mean a historical tool (usually mechanical or calculational) that is employed to perform mathematical calculations, carry out geometrical constructions (which is beyond the abilities of the straightedge and compass). An artefact is a piece of archeological evidence which gives information about the mathematical activities during the ancient times. For both ancient instruments and artefact, usually pictures of which are included in snippets to illustrate or provide extra information to the main mathematical exposition in the text. Ancient instruments can be *re-constructed* and used as interesting objects for 'hands-on', giving students a taste of how people of the past carry out certain mathematical procedures.

Example A.12. (Singapore Polytechnic) While teaching the standard algebraic technique of *completing the square*, the author employs a snippet of an ancient artefact (as shown below) proving that this technique came from the ancient Babylonians – a Babylonian tablet (now kept in the British Museum) records this algebraic procedure.



Figure 8: Ancient Babylonian evidence of completing the square

Example A.13. The author employs the Ancient Chinese Rod Numerals (Counting Rods) to teach Gaussian Elimination Method, in accordance with the *Jiuzhang Suanshu* (translated as the Nine Chapters on the Mathematical Art). In that lesson, the Chinese Rod Numerals system (see figure below) was explained to the students of the same linear algebra class.

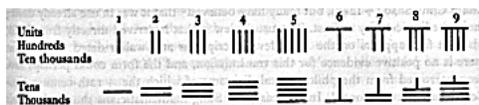


Figure 9: Ancient Chinese Rod Numerals

In short, the Chinese counting rods were ancient calculators. Representing numbers, these rods were laid on the ground and manipulated (i.e., re-positioned) during calculations. Algorithms for the four operations and solution of equations were well-recorded in literature like *Sunzi Suanjing* (translated as Sun’s Mathematical Manual). The rods which they used during the lessons were drinking straws of two different colors (the blue ones for positive numbers, the red ones for negative numbers). Using the straws, the author demonstrated how the rod numerals were used to performing basic arithmetic calculations. Then an ancient rod numeral algorithm equivalent to the modern day *Gaussian Elimination Method* was explained. Based on their existing knowledge of the Gaussian elimination method, students are then invited to practice the elimination algorithm *using the Ancient Chinese Rod Numerals*. Intrigued by the efficiency of this ancient method and the ingenuity of the ancient Chinese for inventing the Gaussian elimination method hundred of years before Gauss did, students appeared motivated since this approach *replaced* the usual, *re-orientated* their mathematical perspective and promoted *cultural understanding*.

Example A.14. (Singapore Polytechnic) This example used by a colleague, L.S. Leow, in her lessons on magnetism (an application of mathematics in

physics) involves the ancient water compass (see figure below) used by the Chinese long ago. A reconstructed version was shown to the students who were then invited to explain (1) how the instrument was possibly used and (2) how it worked. This method aroused the students' interest and allowed an appreciation of the cultural element present.



Figure 10: Ancient Chinese water compass

A.6 Outdoor experiences

Learning mathematics through outdoor experiences provides the instructor a fresh alternative. Using outdoor ancient mechanical instruments for land survey exploits trigonometry and is one example of such experience. This can also take the form of a series of games played outdoors, where players (in teams) are required to accomplish certain tasks using mathematics (and its instruments). A version of this, called the *Maths Trail*, was organized by the National Institute of Education in Singapore at the old Bukit Timah campus for several years (e.g., in 1997). Outdoor experiences can also include visits to museums which display mathematical exhibits, and explanation given to the historical background. It is worth noting that the Singapore Science Centre, a place of interest to the public as well as an ambassador of popular science and mathematics to Singaporeans, offers a number of mathematics-related programmes for school groups. With regards to the use of history of mathematics, the Singapore Science Centre has recently launched an Euler Exhibition, in which the life story of the famous mathematician Leonhard Euler and his contributions were systematically organized and exhibited. This exhibition was also accompanied by a series of talks held under the name of *Science in the Caf e*, where distinguished speakers were invited to share with the audience (from the general public) about their works on certain mathematical topics. These talks were very well received⁶ and audience gave positive feedback. The reader is referred to the table below for the titles of these talks, the speakers and the numbers attended.

⁶The official targeted audience is of size 30 to 40.

S/No.	Presenter (Origin/Sponsor) and title of talk	Dates and attendance
1	Pierre-Louis Lions (French Embassy) Why Is Mathematics Needed!	21/09/2005 84
2	Rolf Jeltsch (Swiss House) Leonhard Euler - His Life, His Discoveries And Their Impact Today	25/04/2007 99
3	Hanspeter Kraft (Swiss House) Leonhard Euler - A Man to be Reckoned With!	18/07/2007 103
4	A/P Helmer Aslaksen (NUS) Heavenly Mathematics and the Armillary Spheres	22/08/2007 88
5	Prof Walter Gautschi (Swiss House) Leonhard Euler - His Life, The Man, and His Work	29/08/2007 76
6	Prof Gerhard Wanner (Swiss House) From Euler's Discoveries in Trigonometry to Fourier Analysis, MP3, & JPEG	20/09/2007 220

A.7 Integration into modes of assessment

One possibility of history of mathematics entering into assessment is to inject a historical dimension into the design of a test or examination question. In particular, one can craft the question by either (i) borrowing directly from a historical problem, or (ii) asking for a modern investigation of an ancient algorithm or method. There can be a certain degree of risk involved in that (1) more effort need to be made to exploit this historical approach, and (2) students may need other literacy skills to understand the development of the question, such as scanning for relevant data and understanding the problem stated.

The following example appears in one of the common test used in Anderson Junior College in 2007, and illustrate such an integration of history of mathematics into existing modes of assessment can be implemented.

Example A.15. This question is intended to test the students on the topic of binomial series:

- (i) Find, up to the term in x^2 , the binomial expansion of $(1+x)^{\frac{1}{2}}$.
- (ii) State the range of x for which the above expansion is valid.

An ancient algorithm for finding the square root of a positive integer S is given in the *Bakhshali manuscript*. This is described as follows:

- (1) Let the perfect square nearest to S be N^2 .
- (2) Calculate $d = S - N^2$.
- (3) Calculate $P = \frac{d}{2N}$.
- (4) Calculate $A = N + P$.

(5) Approximate \sqrt{S} by the formula:

$$\sqrt{S} \approx A - \frac{P^2}{2A}.$$

In summary, this amounts to

$$\sqrt{S} \approx N + \frac{d}{2N} - \frac{d^2}{8N^2} \left(N + \frac{d}{2N}\right)^{-1}.$$

The following parts explore how the ancient algorithm works.

(iii) Using (i) and assuming that $\left|\frac{d}{N^2}\right| < 1$, show that

$$(N^2 + d)^{\frac{1}{2}} \approx N + \frac{d}{2N} - \frac{d^2}{8N^3}.$$

(iv) Hence explain how the ancient algorithm works.