A Family of Identities via Arbitrary Polynomials

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Two conjectures were posed in 2007 by Thomas Dence in this Journal [2].

Conjecture 1. Let \( n \) and \( k \) be odd positive integers with \( k \leq n \). Then

\[
\sum_{j=0}^{(n-1)/2} \binom{n}{j} (-1)^j (n-2j)^k = \begin{cases} 
0, & \text{if } k < n; \\
\frac{2^{n-1}n!}{n}, & \text{if } k = n.
\end{cases}
\]

Conjecture 2. Let \( n \) and \( k \) be even positive integers with \( k \leq n \). Then

\[
\sum_{j=0}^{(n-2)/2} \binom{n}{j} (-1)^j (n-2j)^k = \begin{cases} 
0, & \text{if } k < n; \\
\frac{2^{n-1}n!}{n}, & \text{if } k = n.
\end{cases}
\]

In 2009, Hidefumi Katsuura established the following result from which he deduced the above conjectures.

Theorem (Katsuura [3]) For any complex numbers \( x \) and \( y \), and any positive integer \( n \), we have

\[
\sum_{j=0}^{n} \binom{n}{j} (-1)^j (xj + y)^k = \begin{cases} 
0 & \text{if } k = 0, 1, \ldots, n - 1 \\
(-x)^n \times n! & \text{if } k = n.
\end{cases}
\]

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This is actually a special case of a much more general result which we prove and thus reprove Katsuura's result and conjectures 1 and 2.

First, for any complex number $z$ and non-negative integer $k$, define

$$(z)_k = \begin{cases} 
1, & \text{if } k = 0; \\
(z-1)\cdots(z-k+1), & \text{otherwise}.
\end{cases}$$

**Theorem** Assume that $(z-1)^n$ divides the complex polynomial $f(z) = \sum_{i=0}^{m} a_i z^i$, where $n$ is a positive integer. Let $P(z)$ be any complex polynomial of degree at most $n$.

(i) If the degree of $P(z)$ is less than $n$, then

$$\sum_{i=0}^{m} a_i P(i) = 0. \quad (1)$$

(ii) If the degree of $P(z)$ is $n$, then

$$\sum_{i=0}^{m} a_i P(i) = c \sum_{i=n}^{m} a_i (i)_n,$$

where $c$ is the coefficient of $z^n$ in $P(z)$.

**Proof.** (i) It suffices to show that (1) holds for $P(z) = (z)_k$ for $k = 0, 1, 2, \ldots, n-1$, because every polynomial of degree at most $n-1$ is a linear combination of the polynomials $1, z, (z)_2, \ldots, (z)_{n-1}$.

Since $(z-1)^n$ is a factor of $f(z)$, we have $f(1) = 0$ and $f^{(k)}(1) = 0$ for $k = 1, 2, \ldots, n-1$, where $f^{(k)}(z)$ is the $k$th derivative of $f(z)$, i.e.,

$$0 = f^{(k)}(1) = \sum_{i=0}^{m} a_i (i)_k.$$

Thus (1) holds for $P(z) = (z)_k$.

(ii) If $P(z)$ has degree $n$, then $P(z) - c(z)_n$ is a polynomial of degree at most $n-1$. By
(i), we have
\[ \sum_{i=0}^{m} a_i (P(i) - c(i)_n) = 0, \]
or equivalently,
\[ \sum_{i=0}^{m} a_i P(i) = c \sum_{i=0}^{m} a_i (i)_n = c \sum_{i=n}^{m} a_i (i)_n, \]
where the last equality follows from the fact that \((i)_n = 0\) when \(i \in \{0, 1, \ldots, n-1\}\).

**Corollary** Let \(n\) be any positive integer and \(P(z)\) be any complex polynomial of degree at most \(n\).

(i) If the degree of \(P(z)\) is less than \(n\), then
\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} P(i) = 0. \]

(ii) If the degree of \(P(z)\) is \(n\), then
\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} P(i) = c(-1)^n n!. \]

where \(c\) is the coefficient of the term \(z^n\) in \(P(z)\).

This follows directly by taking \(f(z) = (-1)^n (z - 1)^n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} z^i.\)

Katsuura’s theorem now follows directly by letting \(P(z) = (xz + y)^k\), where \(x\) and \(y\) are complex numbers. We can also apply the corollary to prove conjectures 1 and 2 directly.

Let \(P(x) = (n - 2x)^k\). From the corollary, if \(k < n\),
\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n - 2j)^k = 0, \]
and if \(k = n\),
\[ \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n - 2j)^k = (-2)^n(-1)^n n! = 2^n n!. \]
Observe that if both $k$ and $n$ are odd, then

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} (n - 2j)^k
= \sum_{j=0}^{(n-1)/2} (-1)^j \binom{n}{j} (n - 2j)^k + \sum_{j=(n+1)/2}^{n} (-1)^j \binom{n}{j} (n - 2j)^k
= \sum_{j=0}^{(n-1)/2} (-1)^j \binom{n}{j} (n - 2j)^k + \sum_{i=0}^{(n-1)/2} (-1)^{n-i} \binom{n}{n-i} (2i - n)^k
= 2 \sum_{j=0}^{(n-1)/2} (-1)^j \binom{n}{j} (n - 2j)^k.
\]

This is Conjecture 1. If both $k$ and $n$ are even, then

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} (n - 2j)^k
= \sum_{j=0}^{(n-2)/2} (-1)^j \binom{n}{j} (n - 2j)^k + (-1)^{n/2} \binom{n}{n/2} (n - 2 \times \frac{n}{2})^k + \sum_{j=(n+2)/2}^{n} (-1)^j \binom{n}{j} (n - 2j)^k
= \sum_{j=0}^{(n-2)/2} (-1)^j \binom{n}{j} (n - 2j)^k + \sum_{i=0}^{(n-2)/2} (-1)^{n-i} \binom{n}{n-i} (2i - n)^k
= 2 \sum_{j=0}^{(n-2)/2} (-1)^j \binom{n}{j} (n - 2j)^k.
\]

This is Conjecture 2.

Many well-known combinatorial identities (see, for example, [1 pp106, 159]) are instances of our corollary, i.e., for special choices of the polynomial $P(z)$. Here are some examples.

(a) The following identity can be obtained by choosing $P(z)$ to be $z^k$:

\[
\sum_{r=0}^{n} (-1)^r r^k \binom{n}{r} = \begin{cases} 
0, & \text{if } 0 \leq k < n; \\
(-1)^n n!, & \text{if } 0 < k = n.
\end{cases}
\]
(b) The following result can be obtained from our Corollary by choosing $P(z)$ to be $\binom{z}{k} = \frac{(z)_k}{k!}$:

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} (n - r)^k = \begin{cases} 0, & \text{if } 0 \leq k < n; \\ (-1)^n, & \text{if } 0 < k = n. \end{cases}$$

(c) The following result follows by choosing $P(z)$ to be $(n - z)^k$:

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} (n - r)^k = \begin{cases} 0, & \text{if } 0 \leq k < n; \\ n!, & \text{if } 0 < k = n. \end{cases}$$

(d) For any positive integer $q$, the following result can be from our theorem by choosing $f(z)$ to be $(1 - z^q)^n$ and $P(z)$ to be $z^k$:

$$\sum_{r=0}^{n} (-1)^r \binom{n}{r} (qr)^k = \begin{cases} 0, & \text{if } 0 \leq k < n; \\ (-1)^n q^n n!, & \text{if } 0 < k = n. \end{cases}$$

Summary

In this short article, we prove an identity from which a theorem of Katsuura and two conjectures previously posed in this JOURNAL follow directly.

References


3. H. Katsuura, Summations Involving Binomial Coefficients, this JOURNAL 40 (2009), 275-278.