Twelve Questions on Mathematics Teaching

Snapshots from A Study of the Enacted School Mathematics Curriculum in Singapore



Mathematics & Mathematics Education Academic Group

NIE NATIONAL INSTITUTE OF EDUCATION SINGAPORE

An Institute of

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

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Snapshots from A Study of the Enacted School Mathematics Curriculum in Singapore

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Foreword

The Academic Groups of the National Institute of Education are quite special in that their faculty members comprise both content experts and education experts. This symbiosis gives rise to an exceptional teacher education environment where both subject mastery and pedagogical nous are nurtured in pre-service and in-service teachers. The Mathematics and Mathematics Education (MME) Academic Group embraces this distinctive feature in its teaching and extends it into research that hopes to make an impact in the schools.

Towards this end, from 2016, nine MME faculty consisting of mathematics educators and mathematicians worked on a programmatic research project that would shed light on important questions regarding Singapore mathematics classrooms:

- What is happening in Singapore classrooms that can explain our positive results in international benchmark surveys, such as TIMSS and PISA?
- Is the curriculum enacted in Singapore classrooms anything like the curriculum in official Ministry of Education documents?
- More importantly, can we find good effective practices from our research which we can later share with other teachers?

It is fair to say that it is unfair to say that Singapore and other Asian nations achieved positive results in mathematics education because of the "Asian" way of "drill and practice". Critics have not been inside our Singapore classrooms nor have observed the classes with suitable rigorous methods. The MME programmatic research project, *A Study of the Enacted School Mathematics Curriculum*, did exactly what was required – the team went into the secondary school classrooms and painstakingly recorded video data and interviews with teachers and students. Thirty teachers and 854 students participated over 211 classroom lessons. Another 691 teachers took part in a survey that was based on the findings of the video data. The data collected were immensely rich and the team has worked hard to analyse them in various ways.

This book is one of the results of their analysis. In a similar vein to OECD's *Ten questions for mathematics teachers and how PISA can help answer them* (2016), it seeks to answer 12 questions on mathematics teaching in Singapore classrooms which will help to suggest useful and effective directions and pedagogies for all mathematics teachers.

As the Head of MME, I am proud to be the Advisor of the research project and to have been involved in its early conception. I am overwhelmed by the hard work of my colleagues and the impressive findings that they have unearthed and are sharing from their efforts. I trust that you, the reader, will find this book very useful and insightful.

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The Study

A Study of the Enacted School Mathematics Curriculum [OER 31/15 BK]

What was the study about?

The study was about the teaching and learning of mathematics in Singapore secondary schools. It had two aims. The first was to document how experienced and competent teachers enacted the school mathematics curriculum in secondary schools. It did this by examining:

- pedagogies adopted by experienced and competent teachers when enacting the curriculum, and
- experienced and competent teachers' use of instructional materials for the enactment of the curriculum.

The second was to establish how uniform these adopted pedagogies and use of instructional materials by experienced and competent teachers were practised in the mathematics classrooms of Singapore schools. Details of the study are described in Kaur et al. (2018).

How was it carried out?

The study was carried out in two phases. The first phase was the video-segment and the second was the survey-segment. The video-segment of the study adopted the complementary accounts methodology developed by Clarke (1998), a methodology which was widely used in the study of classrooms across many countries in the world as part of the Learner's Perspective Study (Clarke, Keitel & Shimizu, 2006). A three-camera (teacher camera, student camera, whole class camera) approach, shown in Figure 1, was used to collect data.

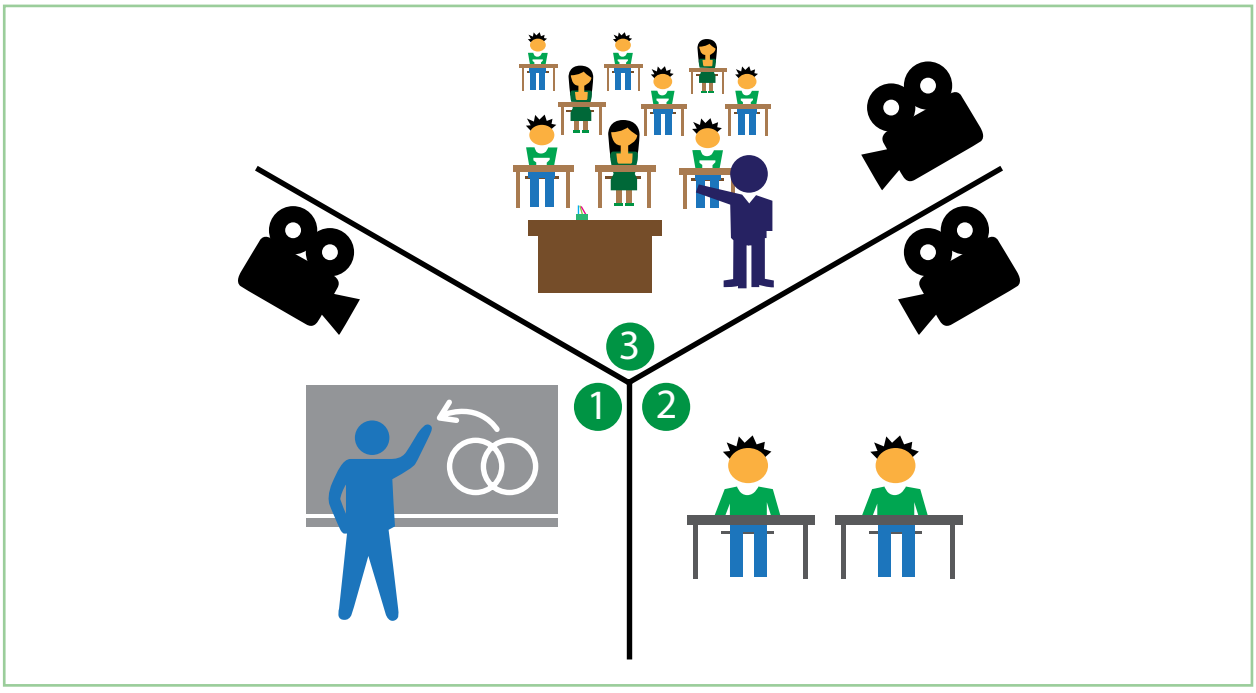


Figure 1 Three-camera approach

The teacher camera captured all gestures, tools and equipment the teacher used in the lesson. The student camera kept in view two to four students, the focus students, who were sitting adjacent to each other and focused on their actions during the lesson. Every lesson had a different group of focus students. The whole class camera captured the corporate behaviour of the class and was set at the front looking at the class such that it represented the "teacher's-eye view" of the class. Sequences of lessons (for a complete topic) taught by the participating teachers were recorded. Post-lesson video-stimulated interviews were held for the teachers and focus students. In addition, all the written work done by the focus students during and following the lesson and instructional materials used by the teachers were also documented. Researchers also kept field notes of every lesson recorded.

Findings from the video-segment of the study provided inputs for the survey that was used to establish how uniform the adopted pedagogies and use of instructional materials by experienced and competent teachers were practised in the mathematics classrooms of Singapore schools. An online survey, comprising three parts, was constructed. The first part was about the pedagogical structure of lessons and student-teacher interaction. The second part was about the enactment of the five aspects of the school mathematics curriculum framework (MOE, 2012) represented by the five sides of the "pentagon" in the framework; and the third part was about instructional materials. Participants, with three or more years of mathematics teaching experience, were asked to respond on a Likert Scale of 1 (Never/Rarely) to 4 (Mostly/Always). At the start of the survey, participants indicated the course of study (Integrated Programme (IP), Express (EX), Normal (Academic) (NA) or Normal (Technical) (NT)) and subject (Mathematics or Additional Mathematics) they were doing the survey on. For parts one and two of the survey the items were the same for all participants. However, for part three, this was not the case as the survey items were specific to instructional materials by course of study and subject. Therefore participants either attempted survey items on Additional Mathematics or Mathematics (EX) or Mathematics (NA/NT). In this book, other than drawing on the analysis of data from the video-segment of the study, Questions 1, 2 and 3 draw on items in the first part of the survey, Questions 4, 5, 6, 7 and 8 draw on items in the second part of the survey and Questions 9, 10 and 11 draw on items in the third part of the survey.

All the teacher and student participants of the study were asked for their consent prior to participation according to the Institutional Review Board Approval which was obtained from the Nanyang Technological University's Research Integrity and Ethics Office.

Who participated in the study?

In the first phase of the study, which was the video-segment, 30 experienced and competent teachers participated. These were teachers deemed as "good mathematics teachers" in their respective schools. They had at least five years of mathematics teaching experience and were keen to participate in the study. Table 1 shows the profile of the teachers. Sixty percent of the teachers were female and 80% of the teachers had ten or more years of mathematics teaching experience.

Table 1 Number of teacher participants in video-segment of the study

Course of Study	Gender		Years of Mathematics Teaching			Total
	Male	Female	< 10	10-19	20 or more	
Integrated Programme	1	3	1	2	1	4
Express	3	7	1	3	6	10
Normal (Academic)	3	5	2	2	4	8
Normal (Technical)	5	3	2	4	2	8
Total	12	18	6	11	13	30

By virtue of the design of the study all the students in the classes of the 30 teachers were able to participate. However, only the focus students were interviewed. Table 2 shows the numbers of students by course of study who participated in the study and were interviewed. Almost half of the student participants were interviewed as they were the focus students.

Table 2 Number of student participants in video-segment of the study

Course of Study	Student Participants			Students Interviewed		
	Gender		Total	Gender		Total
	Male	Female		Male	Female	
Integrated Programme	40	70	110	23	44	67
Express	148	150	298	81	57	138
Normal (Academic)	122	95	217	80	52	132
Normal (Technical)	152	77	229	77	33	110
Total	462	392	854	261	186	447

In the second phase of the study, which was the survey-segment, 691 participants (teachers) completed the survey. In the preliminary screening of the data, some responses were removed as they did not meet the requirements of the survey. The data of 677 teachers were used for subsequent analyses. Table 3 shows the profile of these teachers. Forty percent of the teachers were male and this was representative of the demographic of the teacher population in secondary schools which was 36 % male (MOE, 2018). By course of study, 65% responded for the IP and EX courses, and 35% for the NA and NT courses. This was also coherent with the demographic of the student population in secondary schools which was 64% and 36% respectively for the IP and EX courses, and NA and NT courses (MOE, 2018) respectively. Forty-five percent of the teachers had more than three but less than ten years of mathematics teaching experience while the rest had more than ten years of experience.

Table 3 Number of teacher participants in survey-segment of the study

Course of Study	Gender		Years of Mathematics Teaching			Total	%
	Male	Female	< 10	10-19	20 or more		
Integrated Programme	21	37	17	22	19	58	64.7
Express	135	245	168	139	73	380	
Normal (Academic)	66	85	69	53	29	151	35.3
Normal (Technical)	46	42	51	30	7	88	
Total	268	409	305	244	128	677	100

What data were collected?

During the video-segment of the study, a total of 211 lessons enacted by 30 experienced and competent teachers were video-recorded. Table 4 shows the spread of the lessons across the four courses of study and grade levels.

Table 4 Number of classes and video-recorded lessons

Course of Study	Number of Classes (Number of Lessons)					
	*Sec 1	Sec 2	Sec 3	Sec 4	Sec 5	Total
Integrated Programme	0 (0)	1 (7)	3 (26)	0 (0)	0 (0)	4 (33)
Express Course	0 (0)	2 (9)	4 (24)	4 (31)	0 (0)	10 (64)
Normal (Academic)	2 (13)	0 (0)	2 (15)	3 (22)	1 (10)	8 (60)
Normal (Technical)	1 (6)	1 (8)	1 (5)	5 (35)	0 (0)	8 (54)
Total	3 (19)	4 (24)	10 (70)	12 (88)	1 (10)	30 (211)

Key - *Sec 1 means Secondary 1 which is Year 7 of schooling

Video-stimulated interviews with the focus students, and interviews (at the start, in the midst and after the sequence of lessons) with the teachers were also video-recorded. Artefacts, comprising materials used by the teachers for instruction and student work were also copied and digitised. In the survey-segment of the study, both quantitative and qualitative data were collected using an online survey system. Figure 2 summarises all the data collected by the study.

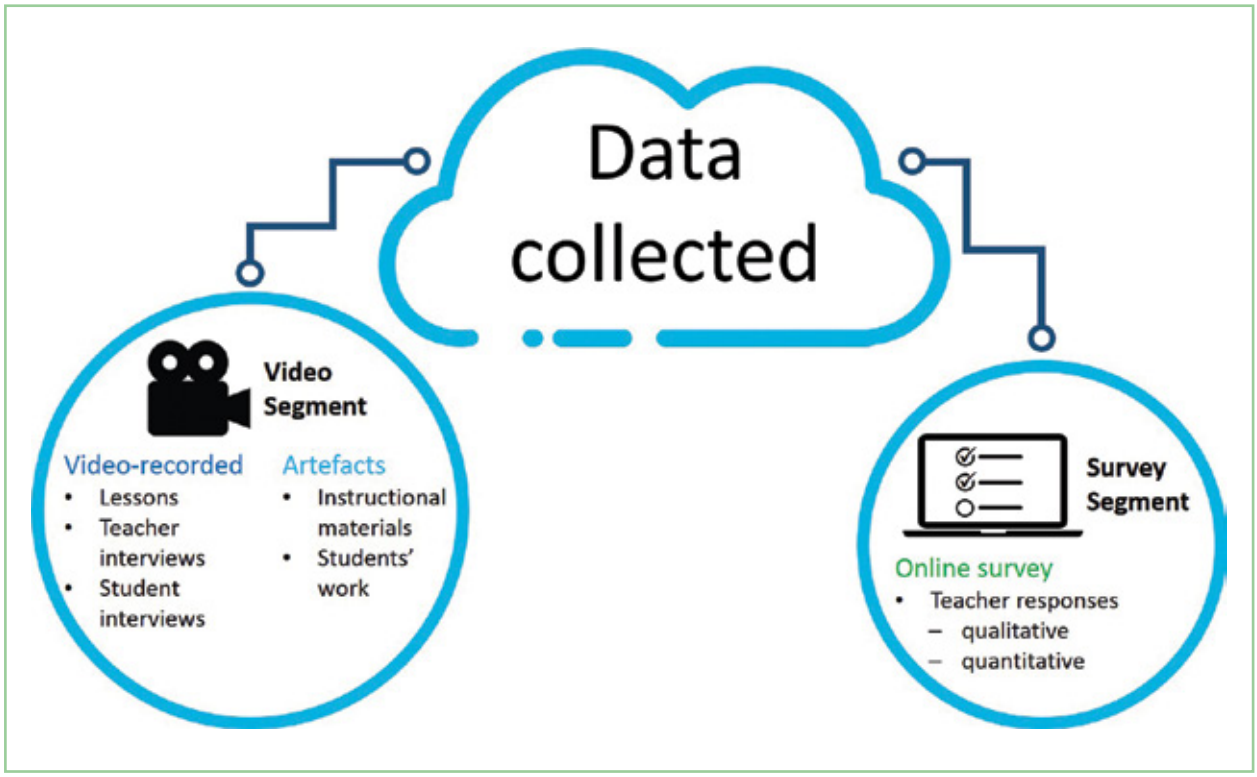


Figure 2 Summary of data collected

How can the findings of the study contribute towards the development of mathematics teachers in Singapore and elsewhere?

Findings of the study presented in this book illuminate the pedagogies of mathematics teachers in Singapore secondary schools. They also shed light on the instructional materials used by the teachers. The responses to the twelve questions addressed in this book offer mathematics teachers insights into related aspects of teaching and learning mathematics in the classrooms and also opportunities for teachers to reflect on their own practices. Some readers might be looking for features of Singapore mathematics instructional practices that contribute to Singapore students' high performance in Trends in International Mathematics and Science Study (TIMSS) and Programme for International Student Assessment (PISA). They might be able to find some answers here.

More findings of the study are available elsewhere, specifically in the following publications.

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QUESTION

1

What is the instructional core that guides mathematics teaching in the classrooms of Singapore secondary schools?



Question 1

What is the instructional core that guides mathematics teaching in the classrooms of Singapore secondary schools?

What is an instructional core?

The instructional core of a lesson comprises interactions between the teacher, the student/s and the content. Through activities designed by teachers, and participation by student/s the curriculum is enacted. Like the DNA of the human body, it unfolds the character of a lesson. The DNA of a human body is made up of building blocks. Similarly, mathematics lessons can also comprise instructional components (building blocks) that illuminate interactions between teacher, student and content, and characterise them.

How does it look like in the classrooms of experienced and competent mathematics teachers in Singapore secondary schools?

A study of mathematics lessons enacted by 30 experienced and competent teachers in Singapore secondary schools shows that mathematics lessons are driven by instructional objectives, for example, such as:

> At the end of the lesson, students will be able to:
> 1. Verify and state Pythagoras theorem.
> 2. Find the unknown sides of a right-angled triangle using Pythagoras theorem.

To achieve the objectives, teachers detail micro-instructional objectives. When enacting the micro-instructional objectives an instructional core appears to drive the teaching and learning of mathematics in their lessons. The components of the instructional core are:

> D → teacher develops concepts/demonstrates skills
> S → teacher sets students work to do [students apply the concepts/practise skills]
> R → teacher reviews student work, drawing the attention of the whole class to errors, misconceptions, correct solutions, good presentations, etc…

Figure 1.1 shows the *DNA* of *mathematics lessons* made up of the instructional components, D, S and R (akin to the building blocks of the human DNA), constituting the instructional core that drives the teaching and learning of mathematics in Singapore secondary schools. The D component develops a concept or introduces a skill. Teachers may show, tell, explain or guide students to uncover/make sense of new concept/s. They may also introduce and demonstrate skill/s. The following S component involves students working on mathematical task/s during classwork, homework or assessment. Students may do the work individually or in groups. The tasks involve application of new knowledge or skill building that was developed in the D component. The R component, a critical one, is where student understanding is monitored. Almost always student work during the S component is used for whole class discussion during the R component. Good presentations are showcased, alternative solutions are discussed, and erroneous solutions are examined and corrected with inputs from the class. This component also includes the review of past knowledge that is necessary for the lesson or a task students are set to do. Many a time when a teacher is not satisfied with students' grasp of a concept or proficiency of a skill, the teacher engages students in more rounds of **S → R** until he/she is satisfied with students' mastery of knowledge.

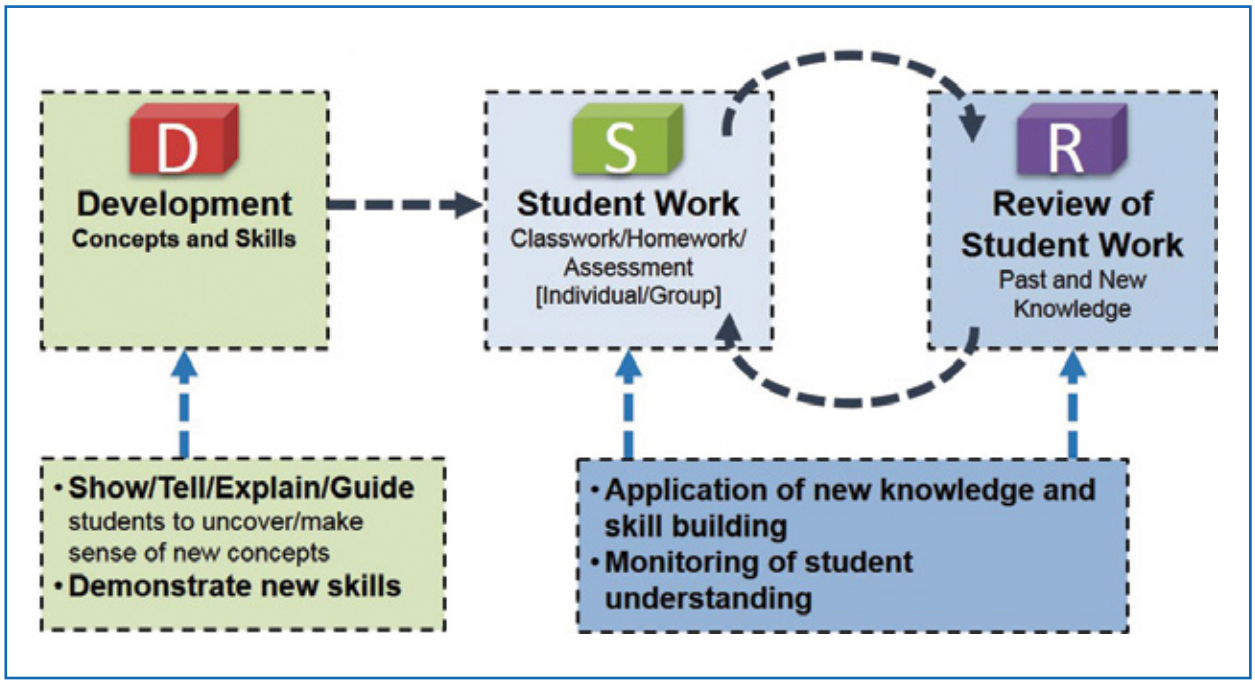


Figure 1.1 DNA of mathematics lessons

Through the varied combinations of D, S and R, as shown in Episodes 1.1 and 1.2, teachers enact micro-instructional objectives to achieve their instructional objectives. Episode 1.1 shows that to achieve the objective – To verify and state Pythagoras theorem, a teacher did it through six micro-instructional objectives that involved components of the instructional core, D, S and R. Episode 1.2, further exemplifies how the teacher achieved the next objective in a similar manner.

Episode 1.1

Objective: To verify and state Pythagoras theorem			
Cycle 1	Core*	Micro-instructional objective	Teacher/Student activities
	R	Review past knowledge	Teacher drew a right angled triangle and with inputs from students reviewed its properties.
	D	Verify Pythagoras theorem	Teacher explained the activity (guided investigation) and gave out the activity kits.
	S	Verify Pythagoras theorem	Students carried out the activity verifying that the area of the square on the longest side of the triangle = sum of the squares on the other two sides of the triangle.
	R	Verify Pythagoras theorem	With inputs from students related to the activity, teacher formalised the relationship between the sides of a right-angled triangle.
	R	Develop the vocabulary related to the sides of a right-angled triangle	Teacher labelled the sides of the right angled triangle on the board; named the longest side as the hypotenuse and drew the attention of the whole class to the 'new' word (must use it and spell it correctly).
	R	State Pythagoras theorem	With inputs from students teacher wrote the theorem on the board: $a^2 = b^2 + c^2$ (where a is the length of the hypotenuse).

* components of instructional core

Episode 1.2

Objective: To find the length of the hypotenuse given the other two sides of a right-angled triangle			
Cycle 2	Core*	Micro-instructional objective	Teacher/Student activities
	D	Demonstrate how to find the length of hypotenuse of a right angled triangle given the other two sides	Teacher worked an example of how to find the hypotenuse of a right angled triangle with sides 3 cm and 4 cm.
	S	Engage students in applying new knowledge and skill building	Students worked individually and found the hypotenuse of a given right-angled triangle. Teacher walked around the class noting student work for 'review'.
	R	Monitor student understanding	Teacher drew on samples of student work (correct and incorrect solutions) and invited inputs from students. For the incorrect solutions, errors and their causes were identified. As some computational errors were due to incorrect use of the calculator, teacher did a quick review of how to find squares and square roots using a calculator.
	S	Reinforce – application of new knowledge and skill building	Students worked individually and found the hypotenuse of a given right-angled triangle. Teacher walked around the class noting student work for 'review'.
	R	Monitor student understanding	Teacher drew on samples of student work (mostly correct solutions) and invited inputs from students. Teacher highlighted correct and logical presentations.

* components of instructional core

A lesson often comprises one or more cycles of instruction depending on the number of objectives. A cycle comprises combinations of D, S and R such as R-D-S-R-R-R, D-S-R-S-R, …, as shown in Figure 1.2.

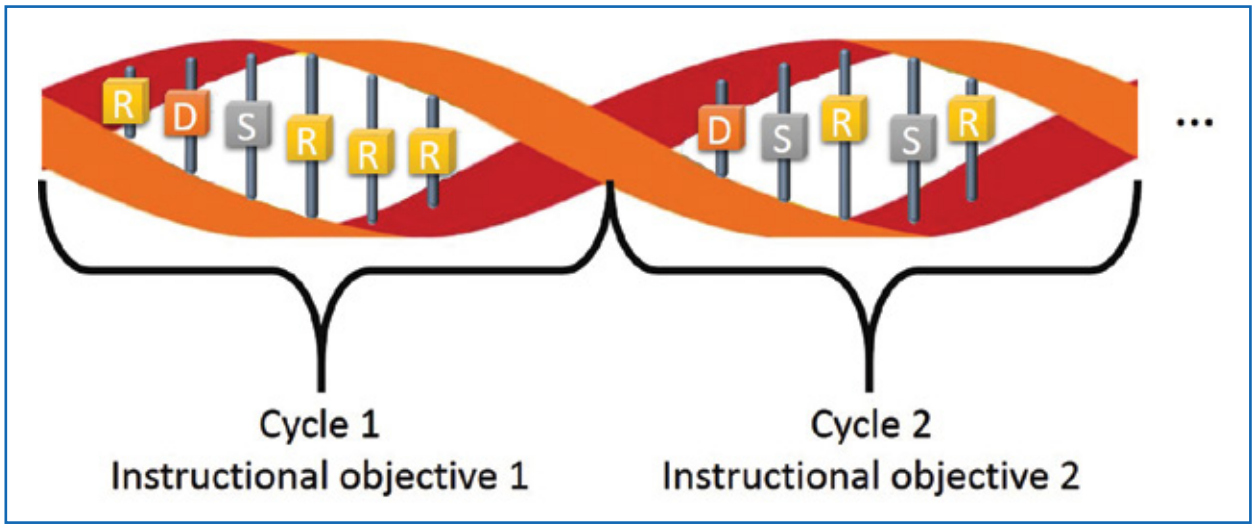


Figure 1.2 Instructional cycles

How does it look like in the classrooms of mathematics teachers in Singapore secondary schools, in general?

In the survey, the participants (teachers) were asked if their lessons

i. were also guided by lesson objectives that were enacted through micro-instructional objectives?
ii. comprised the instructional components **D**, **S** and **R**?

Figure 1.3, shows the responses of the participants. 94.8% of the participants reported that their lesson objectives comprised micro-instructional objectives and 96.8% of them affirmed that their lessons comprised the instructional components **D**, **S** and **R**.

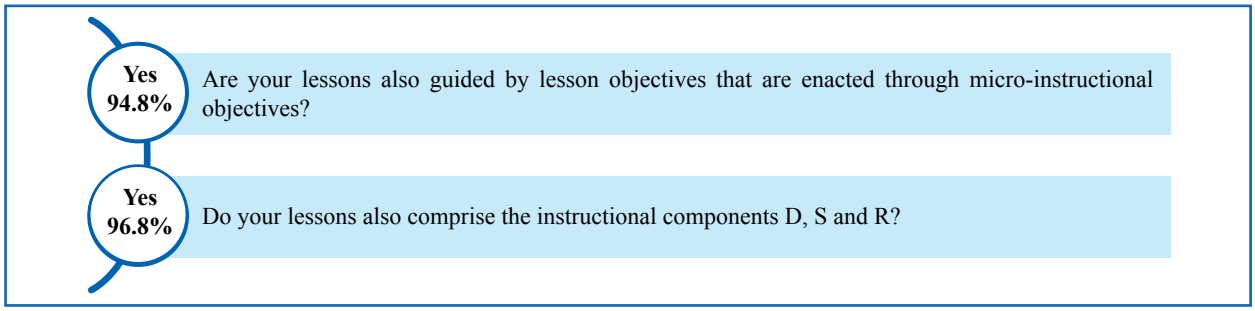


Figure 1.3 Responses of survey participants on the instructional core of their lessons

What are the implications for mathematics teachers in Singapore secondary schools?

Knowledge of what comprises the instructional core provides us with an insight about the character of mathematics lessons in Singapore secondary schools.

Traditionally teachers may talk about their lessons in terms of phases, such as Introduction, Development, Consolidation and Conclusion. Components of the instructional core, D, S and R, shown in Figure 1.1, and instructional cycles, shown in Figure 1.2 provide teachers with an alternative micro-level perspective (lens) that they may draw on when dissecting enacted lessons for review and refinement.

Of significance, is the S-R loop, shown in Figure 1.1. It exemplifies a critical activity in almost all of our mathematics lessons. This activity is often labelled as 'practice'. However, knowledge of the instructional cycles detailing sequences such as D-S-R-S-R helps us examine the form and function of activities that shape conceptual understanding and procedural fluency in our lessons. This loop is often the crucible where understanding takes place. Students' work is foregrounded in the classroom discourse and it may be said that though the discourse is teacher-led it is student-centred.

What is the instructional core in your mathematics lessons like?



My notes

Contributor
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QUESTION

2

Which models of instruction guide mathematics teaching in the classrooms of Singapore secondary schools?



Question 2

Which models of instruction guide mathematics teaching in the classrooms of Singapore secondary schools?

What is a model of instruction?

A model of instruction is a set of strategies that guide teachers in their instructional practice. Often such models are based on learning theories.

What models of instruction guide mathematics teaching in the classrooms of experienced and competent mathematics teachers in Singapore secondary schools?

A study of mathematics lessons enacted by 30 experienced and competent teachers in Singapore secondary schools shows that teacher and student actions from three main models of instruction guide them in their instructional practice. We elaborate the models and provide examples of teacher and student actions that were observed in the lessons of the experienced and competent teachers (which are marked *) as well as those that were not but were included in the survey. For actions that are marked * we also indicate the respective courses of study which are Integrated Programme (IP), Express Course (EX), Normal (Academic) Course (NA) and Normal (Technical) Course (NT) where the actions were observed.

The models were:
Traditional Instruction (TI) – A method of instruction that is teacher-centred in which the focus is on rote-learning and memorisation. In the context of Asian classrooms it is often associated with drill and practice (Hogan et al., 2013; Biggs & Watkins, 2001; Leung, 2006). Examples of actions that belong to this model of instruction are:

Teacher –
- using the textbook to introduce concepts/skills
- emphasising basic facts/steps and asking students to memorise them
- *asking students direct questions to stimulate students' recall of past knowledge/check for understanding of concepts being developed in the lesson (EX, NA)
- telling students how to do it when they face difficulty with a mathematical task they are doing
- *providing students with sufficient questions from textbooks/workbooks/other sources to practise so as to develop procedural fluency (EX, NA, NT)
- *getting students to automatise steps leading to a solution through repetitive exercises (NT)
- engaging students in practising past exam papers

Direct Instruction (DI) – A method of instruction that involves an explicit step-by-step strategy, often teacher-centred, with checks for mastery of procedural or conceptual knowledge (Hogan et al., 2013; Hattie, 2003; Good & Brophy, 2003). Examples of actions that belong to this model of instruction are:

Teacher –
- *using the "I do, We do, You do" strategy, i.e.
  - demonstrating how to apply a concept/carry out a skill on the board [I do]
  - demonstrating again the same using another similar example but with inputs from students [We do]
  - asking the students to do a similar question by themselves [You do] (EX, NA)
- *using exposition (Teacher at the front talking to whole class) to explain mathematical ideas, facts, generalisations (EX)
- *using a set of teacher customised notes cum worksheets to introduce concept/skill (IP)
- *using a set of teacher customised notes cum worksheets to engage students in practice (application of concepts/development of procedural fluency) (IP, NT)

- providing students with directed guidance (asking close-ended questions) when they face difficulty with a mathematical task they are doing, focussing them on the concept/skill necessary to do the task
- *walking around the class and providing students with between desk instruction (i.e. helping them with their difficulties) when they are doing their work at their desks (EX)
- *walking around the class noting student work that teacher would draw on to provide the class feedback during whole class review (IP, EX)
- *explaining what exemplary solutions of mathematics problems must contain (logical steps and clear statements and/or how marks are given for such work during examinations) (IP, EX, NA)
- *only progressing to the next objective of the lesson when teacher is confident that students have grasped the one before (IP, EX, NA, NT)

Students –
- *asking questions when they do not understand (IP, EX, NA, NT)
- *practising a similar problem after teacher has shown them how to do it on the board (IP, EX, NA, NT)

Teaching for Understanding (TfU) – A method of instruction that places student learning at the core. Teacher facilitates, monitors and regulates student learning through student-centred approaches (Hogan et al., 2013; Perkins, 1993; Good & Brophy, 2003). Examples of actions that belong to this model of instruction are:

Teacher –
- *focusing on mathematical vocabulary (such as equations, expressions) to help students build mathematical concepts (IP, EX, NA, NT)
- *focusing on mathematical vocabulary (such as factorise, solve) to help students adopt the correct skills needed to work on mathematical tasks (IP, EX, NA, NT)
- *asking questions to encourage reasoning and speculation, not just eliciting right answers (IP, EX)
- *using examples and non-examples to engage students in discussion to make sense of a concept (IP, EX)
- *focusing on mathematical processes (such as compare and contrast, logical reasoning), to facilitate the development of concepts or student understanding (IP)
- *leading whole class discussion (with guided questions) to facilitate the development of concepts (IP)
- *exchanging ideas with students on how to solve a problem (IP, EX)
- asking students open-ended questions and allowing them to build on each other's responses to develop concepts or clarify their understanding
- *building on students' responses rather than merely receiving them (IP, EX)
- *providing students with probing guidance (open-ended questions about their thinking and why they are considering certain approaches) when they face difficulty with a mathematical task they are doing (IP, EX)
- *providing feedback to individuals for in-class work and homework to serve as information and diagnosis so that students can correct their errors and improve (IP, EX)
- *providing collective feedback to whole class for common mistakes and misconceptions related to in-class work and homework (IP, EX, NA, NT)
- reviewing student performance by providing the class detailed comments on tests and examinations

Students –
- *explaining how their solutions or how their answers are obtained (IP, EX, NA, NT)
- *analysing why a procedure (that teacher has shown on the board) works or why a solution method makes sense (EX)
- *predicting ideas and justifying them (IP)
- developing concepts through exploratory/investigative activities

- comparing multiple procedures/solution methods the teacher has shown on the board and evaluating them for efficiency, appropriateness, ease of use, or other advantages or disadvantages
- developing concepts together with teacher through class discussion
- *offering alternative method(s) or solution(s) to a problem teacher has shown on the board (IP)
- justifying why their answer to a problem is different from the one teacher has put on the board
- *asking questions (such as 'what if') to probe further or deepen understanding (IP)
- teaching/explaining to a classmate while doing individual assigned seatwork
- *working collaboratively as a group on mathematical tasks and discussing the solution before presenting to the whole class (IP, EX, NA)
- *discussing and helping each other while doing individual seatwork (IP, EX, NA, NT)
- *defending and explaining to classmate(s) why their approach/method to solve a problem is better (more efficient or more elegant) (IP)
- *critiquing one another's work presented on board/screen so as to improve their understanding of concepts or elegance in their presentation/solution (IP)
- *discussing features of a problem that teacher has shown on the board so as to select a particular procedure or solution method (IP)

What models of instruction guide mathematics teaching in the classrooms of mathematics teachers in Singapore secondary schools, in general?

In the survey, participants (teachers) were asked to reflect on their lessons for a course (IP, EX, NA or NT) they were teaching, and respond to the items indicating the frequency of their actions. Table 2.1 shows the mean for each of the three models of instruction for all the teachers as well as for the four courses of study (IP, EX, NA, NT).

Table 2.1 Mean for each of the models of instruction

Course of Study	**Mean* Model of Instruction**		
	TI	DI	TfU
All (n=677)	2.8	3.1	2.9
Intergrated Programme (n=58)	2.4	3.1	3.0
Express (n=380)	2.8	3.1	2.9
Normal (Academic) (n=151)	2.8	3.1	2.9
Normal (Technical) (n=88)	2.9	3.2	2.8

*maximum = 4; minimum = 1.

Table 2.1 shows that participants (teachers) appear to draw on teaching moves from all the three models of instruction, though with differing emphasis to enact their lessons. Direct Instruction appears to be the dominant model that teachers draw on in all the four courses of study. In the NT classes, Direct Instruction and Traditional Instruction are apparently more prevalent whilst in the IP, EX and NA classes Direct Instruction and Teaching for Understanding are apparently more prevalent. We next examine the survey items for each course of study that had a mean greater than 3 and a standard deviation of less than or equal to 0.7. Tables 2.2, 2.3 and 2.4 show teacher/student actions from the three models of instruction that were frequent in the classrooms of the teachers who participated in the survey.

Table 2.2 Traditional Instruction actions that were frequent in mathematics lessons

Survey Item	IP	EX	NA	NT
Teacher asking direct questions to stimulate students' recall of past knowledge/check for understanding of concepts being developed in the lesson		*	*	*
Teacher providing students with sufficient questions from textbooks/workbooks/other sources to practise so as to develop procedural fluency	*	*	*	*
Teacher getting students to automatise steps leading to a solution through repetitive exercises				*

Table 2.3 Direct Instruction actions that were frequent in mathematics lessons

Survey Item	IP	EX	NA	NT
Teacher using the "I do, We do, You do" strategy, i.e. - demonstrating how to apply a concept/carry out a skill on the board [I do] - demonstrating again the same using another similar example but with inputs from students [We do] - asking the students to do a similar question by themselves [You do]		*	*	*
Students practising a similar problem after teacher has shown them how to do it on the board		*	*	*
Students asking questions when they do not understand	*	*	*	*
Teacher walking around the class and providing students with between desk instruction (i.e. helping them with their difficulties) when they are doing their work at their desks	*	*	*	*
Teacher walking around the class noting student work that teacher would draw on to provide the class feedback during whole class review	*	*	*	*
Teacher using exposition (Teacher at the front talking to whole class) to explain mathematical ideas, facts, generalisations		*		
Teacher using a set of teacher customised notes cum worksheets to introduce concept/skill	*			
Teacher using a set of teacher customised notes cum worksheets to engage students in practice (application of concepts/development of procedural fluency)	*			
Teacher explaining what exemplary solutions of mathematics problems must contain (logical steps and clear statements and/or how marks are given for such work during examinations)	*	*	*	
Teacher only progressing to the next objective of the lesson when teacher is confident that students have grasped the one before	*	*	*	*

Table 2.4 Teaching for Understanding actions that were frequent in mathematics lessons

Survey Item	IP	EX	NA	NT
Teacher providing feedback to individuals for in-class work and homework to serve as information and diagnosis so that students can correct their errors and improve	*	*	*	*
Teacher providing collective feedback to whole class for common mistakes and misconceptions related to in-class work and homework	*	*	*	*
Teacher focusing on mathematical vocabulary (such as factorise, solve) to help students adopt the correct skills needed to work on mathematical tasks	*	*	*	*
Teacher focussing on mathematical vocabulary (such as equations, expressions) to help students build mathematical concepts	*	*	*	
Teacher asking questions to encourage reasoning and speculation, not just eliciting right answers	*	*	*	
Teacher using examples and non-examples to engage students in discussion to make sense of a concept	*			
Teacher leading whole class discussion (with guided questions) to facilitate the development of concepts	*			
Teacher building on students' responses rather than merely receiving them	*			
Students predicting ideas and justifying them	*			
Students asking questions (such as 'what if') to probe further or deepen understanding	*			
Teacher providing students with probing guidance (open-ended questions about their thinking and why they are considering certain approaches) when they face difficulty with a mathematical task they are doing	*	*		
Teacher focusing on mathematical processes (such as compare and contrast, logical reasoning), to facilitate the development of concepts or student understanding	*	*		
Students analysing why a procedure (that teacher has shown on the board) works or why a solution method makes sense	*	*		
Students explaining how their solutions or how their answers are obtained	*	*	*	*

From Tables 2.2, 2.3 and 2.4, it is apparent that teachers from all four courses of study drew on a multitude of teaching actions from the three models when enacting their lessons. It appears that though ability of students may be a limiting factor at times for teacher actions such as teacher using examples and non-examples to engage students in discussion to make sense of a concept, there are several teacher and student actions that are consistent in their mathematics lessons irrespective of student abilities. These are:

- Teacher providing students with sufficient questions from textbooks/workbooks/other sources to practise so as to develop procedural fluency
- Students asking questions when they do not understand
- Teacher walking around the class and providing students with between desk instruction (i.e. helping them with their difficulties) when they are doing their work at their desks
- Teacher walking around the class noting student work that teacher would draw on to provide the class feedback during whole class review
- Teacher only progressing to the next objective of the lesson when teacher is confident that students have grasped the one before
- Teacher providing feedback to individuals for in-class work and homework to serve as information and diagnosis so that students can correct their errors and improve
- Teacher providing collective feedback to whole class for common mistakes and misconceptions related to in-class work and homework
- Teacher focusing on mathematical vocabulary (such as factorise, solve) to help students adopt the correct skills needed to work on mathematical tasks
- Students explaining how their solutions or how their answers are obtained

What are the implications for mathematics teachers in Singapore secondary schools?

There was little evidence from the video-recorded lessons about a fourth model of instruction: Co-Regulated Learning Strategies (CRLS), which basically involves self-directed learning, self-assessment and peer-assessment (Hogan et al., 2013; Black et al, 2003; Wiliam, 2007). The only student actions that were occasionally present in the lessons of the 30 experienced and competent teachers were students explaining how to correct an error or a misconception that the teacher has put on the board and exploring alternative solution methods for a problem besides the one the teacher has shown on the board. Nevertheless, we included teacher/student actions that belonged to this model of instruction in the survey to explore how teachers in general would respond to them. Examples of actions that belonged to this model of instruction were:

Teacher –

- getting students to set their own learning goals for mathematics at the beginning of each school term/ semester
- getting students to make a plan to revise their work and correct their mistakes
- getting students to work with peers to make a plan for revision and correction of mistakes
- getting students to grade their own mathematics work (with the marking scheme/rubric provided and teach them how to use it)
- helping students identify strategies that would help them achieve their learning goals for mathematics
- encouraging students to show him/her their work and review their progress for mathematics

Students –

- explaining how to correct an error or a misconception that the teacher has put on the board
- reviewing their mistakes and identifying possible causes by themselves
- exploring alternative solution methods for a problem besides the one the teacher has shown on the board
- working with peers to review their mistakes, identifying and justifying possible causes
- asking self/classmate questions to check their understanding

Table 2.5 shows the mean for CRLS model of instruction for all the participants (teachers). From Table 2.5, it is apparent that on the average participants (teachers) in all the four courses of study do sometimes guide students in self-directed learning and engage them in self and peer assessment.

Table 2.5 Means for CRLS model

Course of Study	Mean*
All (n=677)	2.5
Intergrated Programme (n=58)	2.5
Express (n=380)	2.6
Normal (Academic) (n=151)	2.5
Normal (Technical) (n=88)	2.6

*maximum = 4, minimum = 1

Traditional and Direct Instructions provide students with mainly teacher-centred learning experiences. For students to also engage with student-centred learning and be empowered to take ownership of their learning, Teaching for Understanding and Co-Regulated Learning Strategies teacher/student actions must also pervade our instructional approaches. To do this we must shift our role from being mainly a disseminator of knowledge to a facilitator of knowledge construction. In addition, we must also believe that students are capable of taking charge of their learning and provide them with opportunities to do so.

Which models of instruction guide teaching and learning of mathematics in your lessons?



My notes

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QUESTION

3

What kinds of math talk are enacted in the classrooms of Singapore secondary schools?



Question 3

What kinds of math talk are enacted in the classrooms of Singapore secondary schools?

What is meaningful math talk?

Meaningful math talk can enhance learning. Such talk is crucial for students to think conceptually about mathematics. Talk may be used to convey meaning or to generate meaning (Lotman, 1988). Univocal talk conveys meaning whilst dialogic talk generates meaning (Wertsch, 1991). In a univocal talk, the listener receives the "exact" message that the speaker intends for the listener to receive, and once the speaker's intention has been conveyed, the talk ends. In contrast, in a dialogic talk, there is a give-and-take communication that extends beyond the conveyance of an exact message leading to generation of meaning through dialogue as a "thinking device" (Lotman, 1988).

We can examine math talk from the perspectives of the teacher (teaching talk) and the students (learning talk). According to Alexander's (2018) dialogic teaching framework, the spectrum of talk strategies specific to teaching are:

- Rote: the drilling of facts, ideas and routines through constant repetition;
- Recitation: the accumulation of knowledge and understanding through questions designed to test or stimulate recall of what has been previously encountered, or to cue pupils to work out the answer from clues provided in the question;
- Instruction/Exposition: telling the pupil what to do, and/or imparting information, and/or explaining facts, principles or procedures;
- Discussion: the exchange of ideas with a view to sharing information or solving problems;
- Dialogue: achieving common understanding through structure, cumulative questioning and discussion which guide and prompt, reduce choices, minimise risk and error, and expedite 'handover' of concepts and principles; and

the spectrum of talk strategies specific to student learning are those when teachers engage students to narrate, explain, speculate, imagine, explore, analyse, evaluate, discuss, argue, justify and question.

What kinds of math talk are enacted in the classrooms of experienced and competent mathematics teachers in Singapore secondary schools?

The three teaching episodes in this section illustrate the kinds of math talk (teaching and learning talks) enacted in the interactions between an experienced and competent teacher, Mr R, and his students. The learning goal of this lesson is for students to develop their understanding of vectors and their representations.

Episode 3.1

Line	Teaching Episode	Teaching/ Learning Talks
(1)	T: You may have heard of vectors when you're studying physics. Now, can you give me an example of what you already studied in physics which you understand as vectors? What did you already know about vectors?	Recitation
(2)	S: Gravity.	Narrate
(3)	T: Gravity, what else? I'm not going to correct you now. I'm just letting you tell me what you understand about vectors. Tell me what else you know about vectors.	
(4)	S: Got direction.	Narrate
(5)	T: So vectors have direction. Is that correct? [Students nod.] Give me an example.	Rote Exposition
(6)	S: Velocity.	Narrate
(7)	T: Velocity. Does velocity have direction?	Exposition
(8)	S: Yes.	
(9)	T: If I run towards Sean [pointing at Sean] at a speed of 4 km/h from here. Then I ask Hadi to run towards Sean also at a speed of 4 km/h from there [pointing at Hadi], are the two of us travelling at the same velocity? Hadi, let us run towards Sean now. [Both T and Hadi move towards Sean.] We are both running towards Sean but are we running in the same direction?	Exposition
(10)	S: No. Towards the same direction, yes. Ay? So same direction?	Narrate
(11)	T: We are both running TOWARDS Sean but are we running in the same direction? I don't know. Yes or no, I'm not sure. You discuss.	
	[Students discuss among themselves.]	
(12)	T: So, are Hadi and I running in the same direction?	
(13)	S: No.	
(14)	T: We are not running in the same direction. Why?	Exposition
(15)	S: Because one person is pointed this way and the other person is pointed the other way.	Explain
(16)	T: So we are running in different directions although we are both running at the same speed.	Rote

Up to line 16 in Episode 3.1, the math talk enacted is primarily univocal because the teacher's intention is to convey the message that vectors have directions. Mr R ensures that his intended message for this lesson is adequately conveyed by using a live demonstration of two persons running at the same speed but in different directions. His focus thus far is on how well everyone understands his perspective rather than on making sense of the students'. The teaching talks invoked are *rote, recitation*, and *instruction/exposition;* while the learning talks invoked are *narrate* and *explain*.

However, the following Episode 3.2 reveals how the teacher carries on to leverage students' responses (line 10 in Episode 3.1) as generators of meaning, illustrating the essence of a dialogic math talk. The teaching talks invoked are still *recitation* and *exposition*; while the learning talks have included *justify*.

Episode 3.2

Line	Teaching Episode	Teaching/ Learning Talks
(17)	T: But some of you have this idea that we're running in the same direction because we are running towards the same person? How to disprove that? How can we show that both of us are not running in the same direction?	Recitation
(18)	S: Bearings.	Narrate
(19)	T: Bearings? Can you show me how?	Exposition
(20)	S: This is Sean. [S draws a point.] Teacher is running towards Sean in this direction. [S draws an arrow to the point.] Hadi is also running towards Sean in this direction. [S draws another arrow to the point.] North is in this direction. [S draws another arrow to denote North.] We measure the bearing of the two of you from Sean. So we can see that the two bearings are not the same.	Justify

The following Episode 3.3 further illustrates a math talk that embodies the dialogic characteristics. Again, Mr R first attempts to see/hear what the students understand of equal vectors and uses the students' talk to generate meaning for both himself and the students. He creates the learning moment by using student's response (line 30) to incept the idea of representing a vector horizontally and vertically. Rather than telling the class directly how a vector can be represented in a column vector, Mr R turns to the whole class for inquiry and discussion. He prompts students to use peers' responses as thinking devices and provides opportunities for students to construct their own knowledge.

Episode 3.3 is primarily dialogic. A significant mark of Mr R's classroom is the degree to which the students took ownership of the learning situation. The student-generated responses that emerged during the lesson encouraged dialogues and discussions in a productive manner. The teacher encourages students to build ideas on the basis of one another's insights. Student collaboration is evident as students attempt to refine one another's ideas, help one another explain, and verify one another's claims. Mr R does not attempt to convey a particular message by engaging his students in a specific approach. Instead, he is open to his students' ideas and allows his students to pursue approaches, that may be quite unexpected to him, to generate new mathematical understanding, and this is the essence of a dialogic math talk.

Episode 3.3

Line	Teaching Episode	Teaching/ Learning Talks
(21)	T: Let's work in pairs. I want Partner A to draw any vector and label it $\overrightarrow{AB}$. Now Partner B, how are you going to draw a vector $\overrightarrow{PQ}$ such that $\overrightarrow{AB} = \overrightarrow{PQ}$, that is, to replicate exactly the same vector your partner has drawn? [Students discuss in pairs.]	Instruction
(22)	S: Use a protractor.	Narrate
(23)	T: How to use a protractor to draw another vector that is equal to this vector?	Exposition
(24)	S: Draw another vector of same length and is parallel.	Narrate
(25)	T: So how do we make sure the two vectors are parallel?	Exposition
(26)	S: Use bearing. I use the protractor to measure the bearing like this.	Explain
(27)	T: If you do not have a protractor, then how? Is there another way?	Dialogue
(28)	S: Use tracing paper.	Imagine
(29)	T: That's a good idea. What if I make it difficult for you and say cannot use tracing paper? What other paper will you use?	Dialogue
(30)	S: I draw horizontal lines like lines in the exercise book. I also draw vertical lines. Then I count how many lines and then draw my vector like this.	Explore
(31)	T: Oh, that's smart! So if the vector is drawn on grid like this, [T shows a vector drawn on grid], is it easier now to draw another vector that is equal to this? Shane, show us how. [S draws a vector on the board.]	Discussion
(32)	T: How do the others ensure that the lengths of the two vectors are the same?	Dialogue
(33)	S: Use the boxes.	Analyse
(34)	T: How to use the boxes?	Dialogue
(35)	S: The vector you draw is between 6 boxes, so you find another 6 boxes and draw the vector.	Explain
(36)	T: I don't quite understand what you're saying. Can anyone help to explain?	Dialogue
(37)	S: The vector cuts across these 6 boxes. [S points the 6 boxes.] So I copy and draw my vector that cuts same 6 boxes like this. [S points the other 6 boxes.]	Explain
(38)	T: Is everyone convinced that the two vectors are equal? How are you so sure that they are equal?	Dialogue
(39)	S: The vector is from here to here. [S points at initial and terminal points of the vector.] I start from here, it goes down by 2 boxes and then goes left by 3 boxes.	Analyse
(40)	T: So how are you counting?	Dialogue
(41)	S: Vertically and horizontally.	Analyse
(42)	T: So if each box is a unit, the vector represents a movement of 3 units to the left and 2 units down.	Rote
(43)	T: How can we express this vector in a form that represent 3 units to the left and 2 units down?	Discussion

How does it look like in the classrooms of mathematics teachers in Singapore secondary schools, in general?

In the survey, participants (teachers) were asked to reflect on their lessons and the frequency of their use of the kinds of teaching talk. The aggregated data is shown in Table 3.1.

Table 3.1 Use of teaching talk

Kind of Talk / Survey Items	Percentage of Participants			
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always
Rote - I use "I do, We do, You do" strategy - I emphasise basic facts/steps for students to memorise them - I provide students with sufficient questions to practise so as to develop procedural fluency - I ask students to recall past knowledge - I get students to automatise steps leading to a solution through repetitive exercises	2.5	15.5	47.6	34.4
Recitation - I ask direct questions to stimulate students' recall of past knowledge/ check for understanding of concepts being developed in the lesson - I provide students with directed guidance (close-ended questions) when they face difficulty with a mathematical task they are doing, focussing them on the concept/skill necessary to do the task	1.2	18.8	54.0	26.0
Instruction/Exposition - I use the textbook to introduce concepts/skills - I use exposition (teacher at the front talking to whole class) to explain mathematical ideas - I tell students how to do it when they face difficulty with a mathematical task they are doing - I explain what exemplary solutions of mathematics problems must contain - I ask questions to encourage reasoning, not just to elicit right answers	6.3	27.9	45.9	19.9
Discussion - I use examples and non-examples to engage students in discussion to make sense of a concept - I focus on mathematical processes (such as compare and contrast, logical reasoning) to facilitate the development of concepts or student understanding - I lead whole class discussion (with guided questions) to facilitate the development of concepts	1.6	22.2	54.2	22.0
Dialogue - I exchange ideas with students on how to solve a problem - I ask students open-ended questions and allow them to build on one another's responses to develop concepts or clarify their understanding - I build on students' responses rather than merely receiving them	2.7	31.4	50.1	15.8

From the data in Table 3.1, based on the percentage of responses for "frequently" and "mostly/always", it is apparent that most teaching draws on the basic repertoire of teaching talk: rote, recitation and instruction/ exposition; and less frequently, on dialogue. It is observed that the percentage of responses on the use of discussion, is highest for "frequently", and interestingly, comparable for "sometimes" and "mostly/always". To this, we are mindful of the participants' perception of the nature of discussion and the involvement of students' voice in the discussion. Nonetheless, we are not arguing that rote, recitation and instruction/ exposition should be abandoned. All five have their place and they are not mutually exclusive. Discussion and dialogue are larger oral repertoires that harness the power of talk to engage students, stimulate and extend their thinking, and advance their learning and understanding.

In the survey, participants (teachers) were also asked to reflect on their lessons and kinds of learning talk they engaged their students in, detailing the frequencies. Table 3.2 shows the aggregated data.

Table 3.2 Use of learning talk

Kind of Talk / Survey Items	Percentage of Participants			
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always
Narrate - I get my students to provide answers or solutions (without any explanation) to my questions - I get my students to practise a similar problem after I have shown them how to do it on the board - I get my students to state/list what they have learnt at the beginning/ end of the lesson	16.6	29.4	34.3	19.7
Explain - I get my students to explain how their solutions or how their answers are obtained - I get my students to teach/explain to another classmate while doing individual assigned seatwork - I get my students to explain how they would correct an error or a misconception that I have put on the board	2.5	27.0	54.4	16.1
Speculate - I get my students to predict outcomes	4.3	30.4	50.5	14.8
Explore - I get my students to develop concepts through exploratory/ investigate activities - I get my students to work collaboratively as a group on mathematical tasks and discuss the solution method before presenting to the whole class - I get my students to explore alternative solution methods for a problem besides the one I have shown on the board	6.5	48.5	37.5	7.5
Analyse - I get my students to analyse why a procedure (that I have shown on the board) works or why a solution method makes sense - I get my students to review their mistakes and identify possible causes by themselves	2.6	25.7	55.5	16.2
Evaluate - I get my students to compare multiple procedures/solution methods I have shown on the board and evaluate the efficiency, appropriateness, ease of use, or other advantages/disadvantages - I get my students to critique one another's work presented on board so as to improve their understanding of concepts or elegance in their presentation/solution	4.4	40.4	43.3	11.9
Discuss - I get my students to develop concepts together with me through class discussion - I get my students to discuss features of a problem that I have shown on the board	4.9	38.2	46.7	10.2
Argue - I get my students to offer alternative solution method(s) to a problem I have shown on the board	1.0	36.5	49.9	12.6
Justify - I get my students to justify why their solution(s) to a problem is different from the one I have put on the board - I get my students to defend and explain to classmate(s) why their approach/method to solve a problem is better	9.1	38.9	43.4	8.6
Question - I get my students to ask questions when they do not understand - I get my students to ask self/classmate questions to check their understanding - I get my students to ask questions (such as "what if") to probe further or for deeper understanding	5.6	28.5	46.9	19.0

The data in Table 3.2 tells us that students are mostly engaged in the basic repertoire of learning talk: narrate and explain; and are given less opportunities to further develop their repertoire of learning talk, such as speculate, imagine, explore, analyse, evaluate, discuss, argue, justify, and question.

What are the implications for mathematics teachers in Singapore secondary schools?

The distinction between univocal and dialogic math talks is at times difficult to discern. A math talk can be a continuum between univocal and dialogic. Both univocal and dialogic can be appropriate forms of math talk, depending on the instructional goals. We have to acknowledge the reality of a teacher's classroom, which includes the competing demands of depth versus breadth in content coverage, the students' differing abilities and interests, and time constraints. These factors often influence the learning goals, which in turn influence the kinds of math talk. Our data reveals that the kinds of math talk enacted in our Singapore classrooms are more often univocal than dialogic. We encourage teachers to strive to engage students in more dialogic math talk so that they can acquire a deeper understanding of mathematics when they use their own responses, as well as those of their peers and teacher, as thinking devices.

So what kinds of math talk are enacted in your mathematics classroom?



My notes

Contributors
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QUESTION

4

How do mathematics teachers in Singapore secondary schools introduce concepts to students or engage students in constructing concepts?



Question 4

How do mathematics teachers in Singapore secondary schools introduce concepts to students or engage students in constructing concepts?

What does it mean for students to learn concepts or construct concepts?

It is deemed that a student has grasped a concept or constructed one if he/she could communicate the construct through the use of external representations. Since a particular mode of external representation cannot embody an abstract concept completely, it is necessary to have more than one external representation for a concept. Cleaves (2008) classified six categories of mathematical representations: numerical/tabular, pictorial, graphical, verbal, symbolic (equations or expressions), and physical/concrete. Lesh, Post and Behr (1987) stressed that a student who "understands" a mathematical concept "can (1) recognise the idea embedded in a variety of qualitatively different representational systems, (2) flexibly manipulate the idea within given representational systems, and (3) accurately translate the idea from one system to another" (p. 36).

How do experienced and competent mathematics teachers in Singapore secondary schools introduce concepts to students or engage students in constructing concepts?

Our study of mathematics lessons enacted by 30 experienced and competent teachers in Singapore secondary schools revealed that they introduce concepts to students or engage students in constructing concepts through seven approaches. As illustrated in Figure 4.1, the analysis of 211 lessons showed that 80.0% of them explain the concept/formula/property to the whole class while asking students questions along the way, 46.7% of them guide the whole class to discover the concept/formula/property, and 40.0% of the teachers assign tasks for students to discover the concept/formula/property. Other approaches that are relatively less commonly observed are also shown in the figure.

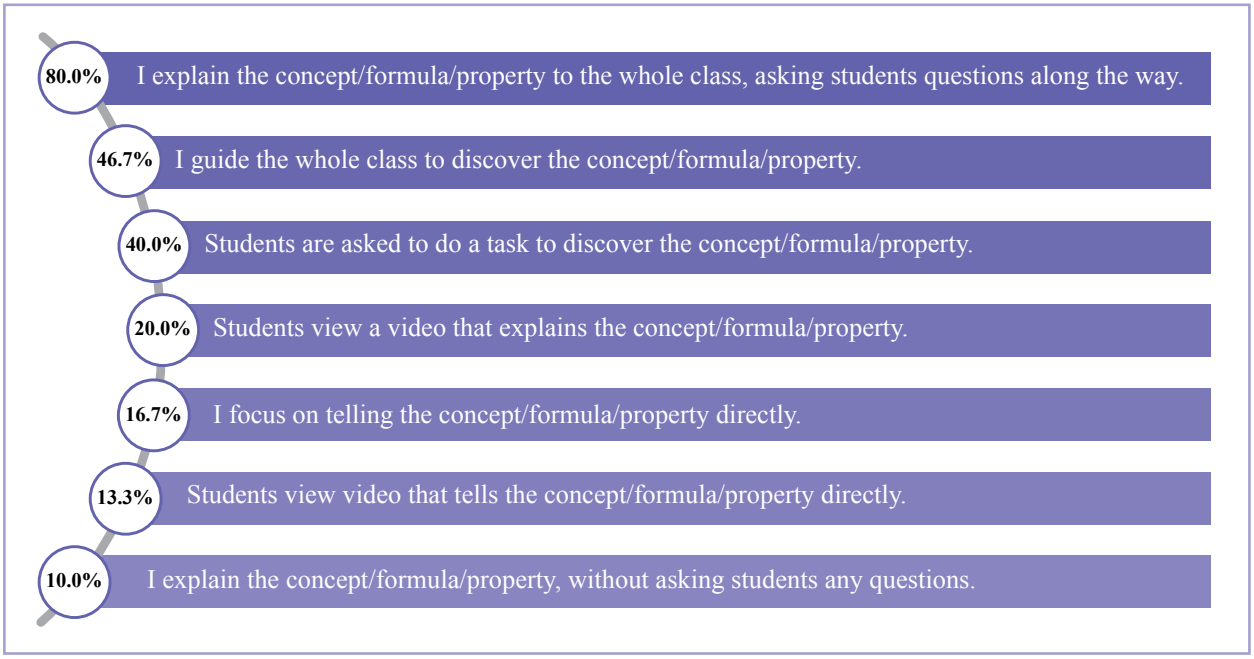


Figure 4.1 Approaches adopted by experienced and competent teachers when introducing concepts or engaging students in constructing concepts

Three key approaches used by the experienced and competent teachers are:

- Explaining the concept/formula/property to the whole class while asking questions along the way
- Guiding the whole class to discover the concept/formula/property
- Asking students to do a task to discover the concept/formula/property

The following three episodes illustrate how three experienced and competent teachers introduced concepts to students or engaged them in constructing concepts using the key approaches.

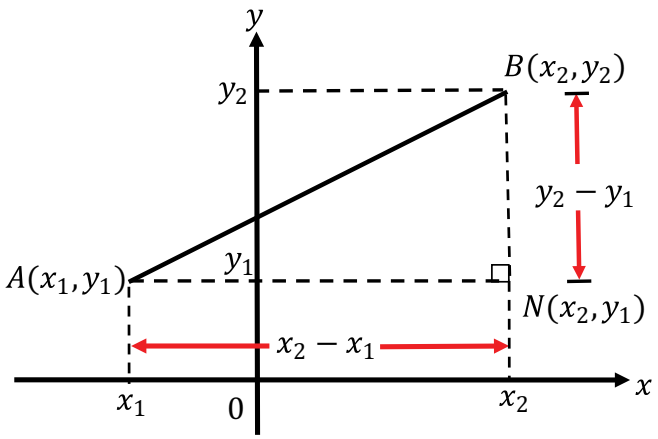
Episode 4.1 shows how a teacher posed questions to the class while explaining a property of quadratic equations.

Episode 4.1

Teacher : *So why cannot be, why cannot be 0? Why 'a' cannot be 0?* [Teacher writes "$a \neq 0$" on the whiteboard]
Teacher : *Anybody can tell me?*
Teacher : *Because the definition say 'a' cannot be 0. Why ah? 'a', 'b' and 'c' are constants. 0 is a constant what, it's a fixed number, but why cannot be 0?*
Student A : *Then it will be a linear equation.*
Teacher *Then it will be a linear equation. Okay. So what is linear?* [Teacher points to 'a' in $ax^2 + bx + c = 0$ on the whiteboard]
Teacher : *So if 'a' is 0 what will happen? $0 \times x^2$, what will happen to the x^2?*
Student B : *Zero.*
Teacher: *Becomes 0 right? So you will only be left with $bx + c = 0$. Okay, and Student A says this is call linear equation.*

Episode 4.2 shows how another teacher brought students' attention to derive the formula for the distance between two points (x_1, y_1) and (x_2, y_2) in a Cartesian plane by guiding the whole class to discover the formula based on the Pythagorean Theorem.

Episode 4.2



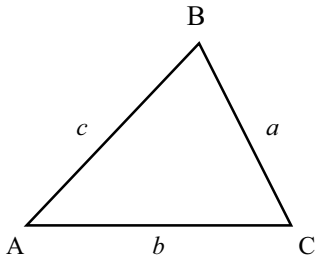
Teacher : *…How is this formula comes about? What type of triangle is this?*
Student A : *Right-angled triangle.*
Teacher : *Relationship. Pythagoras theorem.*
Teacher : *For the x-axis, and the y-axis. The distance between them. Okay, these are general letters, x and y. Numbers are given, y_2 - y_1, what length would you define? Length of BN or AN?*
Student B : *BN.*
Teacher : *BN is $y_2 - y_1$. Do you see, the deriving?*
Teacher : *How do I write AN?*
Student C : $x_2 - x_1$.
Teacher : *And to the left hand side (of the equation) you have the longest side, hypotenuse, that's what we need to find. Is this okay?*
[Teacher projects the length of line segment formula AB = $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ in the notes onto the screen]
Student D : *Square root.*
Teacher : *We want AB only so we put the square root on both sides.*

Episode 4.3 illustrates the task that one of the teachers assigned to students to discover the Cosine Rule using the Geometer Sketchpad (GSP). The task incorporated the use of an inductive approach to arrive at a conclusion based on observations.

Episode 4.3

The Cosine Rule

Work in pairs or a group of three and open GSP file "Cosine Rule". Vary the types of triangle below and record your observation in the table.



Types of Triangle	a^2	$b^2 + c^2 - 2bc\cos A$	b^2	$a^2 + c^2 - 2ac\cos B$	c^2	$a^2 + b^2 - 2ab\cos C$
Right-angled						
Acute-angled						
Obtuse-angled						
Equilateral						
Isosceles						
Scalene						

Conclusion of your observation:

How do mathematics teachers in Singapore secondary schools introduce concepts to students or engage students in constructing concepts, in general?

In the survey, participants (teachers) were asked to reflect on their own practices and indicate the frequency of the seven approaches demonstrated by the experienced and competent teachers in their lessons. The survey data are shown in Table 4.1.

Based on the percentage of responses for "frequently" and "mostly/always", it is evident that participants of the survey also draw on three key approaches when introducing concepts to students or engaging students in constructing concepts in the classrooms. They generally tend to explain concept/formula/property to the whole class while asking questions, guide the whole class to discover concept/formula/property and focus on telling the concept/formula/property directly to the class. While explaining concept/formula/property and asking questions to students along the way and guiding students through the discovery of concept/formula/property correspond to two of the key approaches used by the experienced and competent teachers as reported earlier, the third one - directly telling concept/formula/property to the class does not.

Table 4.1 Approaches for introducing concepts to students or engaging students in constructing concepts

Approach/Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
I explain the concept/formula/property to the whole class, asking students questions along the way.	1.2	15.6	61.6	21.6	83.2
I guide the whole class to discover the concept/formula/ property.	3.7	45.8	41.1	9.4	50.5
Students are asked to do a task to discover the concept/ formula/property.	12.9	59.2	24.8	3.1	27.9
Students view a video that explains the concept/ formula/property.	24.7	56.0	17.6	1.7	19.3
I focus on telling the concept/formula/property directly.	15.8	45.7	31.0	7.5	38.5
Students view video that tells the concept/formula/ property directly.	33.8	54.4	11.4	0.4	11.8
I explain the concept/formula/property, without asking students any questions.	41.8	44.2	12.8	1.2	14.0

Note: $N = 677$

From the data illustrated in Figure 4.1 and Table 4.1, it is apparent that although the general practices demonstrated by experienced and competent teachers when introducing concepts to students or engaging students in constructing concepts are also employed by participants of the survey, the rank order of the approaches differ. There appears preference for direct dissemination of knowledge over "discovering" knowledge by participants of the survey which contrasts with practices of experienced and competent teachers.

What are the implications for mathematics teachers in Singapore secondary schools?

- Mathematics concepts are best understood as a connected network of concepts. Learning should involve active processing and linking new inputs to the student's existing cognitive structure. Students should see connections among concepts rather than treating them as isolated ideas. To promote conceptual understanding, teachers should ask concept questions in lessons during the question-and-answer interactions or discussions. Teachers should ask more conceptually-oriented questions and fewer recall questions. The concept questions are about meaning, mode of representations, examples, non-examples and connections.
- The opportunity for students to engage in constructing concepts depends significantly on the nature of interaction generated in the classroom discourse. Therefore, teachers need to make critical decisions regarding the balance between developing conceptual understanding and skill fluency in their lessons.

How do you introduce concepts to students or engage students in constructing concepts in your mathematics lesson?



My notes

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QUESTION

5

How do mathematics teachers in Singapore secondary schools engage students in developing proficiencies in mathematics skills?



Question 5

How do mathematics teachers in Singapore secondary schools engage students in developing proficiencies in mathematics skills?

What does it mean for students to develop proficiencies in mathematics skills?

Mathematics skills refer to numerical calculation, algebraic manipulation, spatial visualisation, data analysis, measurement, use of mathematical tools, and estimation (Ministry of Education, 2012, p. 15). A student has developed proficiency in a mathematics skill if he/she has gained a strong grasp of the relevant mathematical language to communicate ideas and can apply the skill to a wide range of non-routine mathematical problems. In developing such proficiencies, teachers should identify clear learning goals and provide sufficient opportunities for the practice of these skills (Bloom, 1968). These skills should also be taught with an understanding of the underlying mathematical principles and not merely as procedures (Ministry of Education, 2012, p. 15). In other words, developing of such procedural fluency should promote not just instrumental understanding but also relational understanding (Skemp, 1987, p.166), i.e. students should not just be taught the 'how' of the procedural skills but also the 'why' of such procedural skills.

Our study of mathematics lessons enacted by 30 experienced and competent teachers in Singapore secondary schools revealed that they engage students in developing proficiencies in mathematics skills through two main approaches:

- Delivering well-structured lessons that include single or multiple cycle(s) of Development-Consolidation (Lee, 2009, p. 341)
- Adopting "I do, We do, You do" through effective use of worked example(s) and class practice

How do experienced and competent mathematics teachers in Singapore secondary schools structure their mathematics lessons?

Analysis of 211 lessons showed that experienced and competent teachers in Singapore secondary schools engage students in developing proficiencies in mathematics skills by structuring their lessons in three main ways. As illustrated in Figure 5.1, 56.7% of them structure their lesson using the Introduction-Development-Consolidation-Conclusion cycle and 26.7% of them structure their lesson using the Introduction-Development-Consolidation-Conclusion cycle but with more than one cycle of Development-Consolidation present within a lesson. 26.7% of the teachers were observed to have only one Introduction at the beginning and one Conclusion at the end of the entire chapter, with mainly Development and Consolidation cycles present in most lessons.

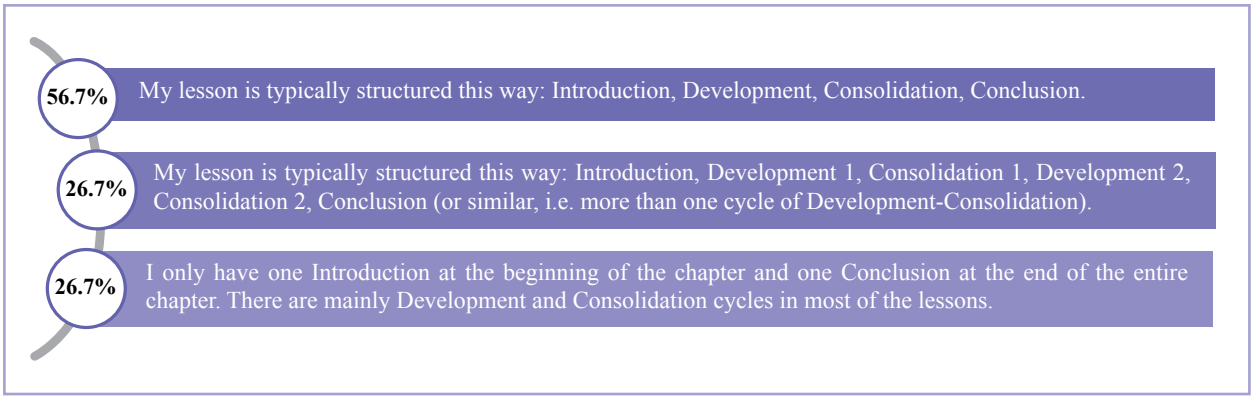


Figure 5.1 Cycles of Development-Consolidation enacted by experienced and competent teachers in developing proficiencies in mathematics skills

The three key structures present in the lessons taught by the experienced and competent teachers are:

- Introduction-Development-Consolidation-Conclusion cycle within one lesson
- Introduction-Development-Consolidation-Conclusion cycle, with more than one cycle of Development-Consolidation within one lesson
- One Introduction at the beginning and one Conclusion at the end of the entire chapter, with mainly Development-Consolidation cycles in most lessons

The following three episodes illustrate how three experienced and competent teachers engage students in developing proficiencies in mathematics skills by structuring their lessons in these three key ways.

Episode 5.1 shows how a teacher structured a lesson on Pythagoras Theorem through the use of a single Development-Consolidation cycle.

Episode 5.1

Phase	Objective	Activity
I	Set the stage for learning	1. Teacher recaps conditions for Pythagoras Theorem that were taught previously. 2. Teacher informs students of the lesson objectives (i.e. to prove that a particular triangle is a right-angled triangle).
D	Introduce new body of knowledge and convey expectations	1. Teacher explains the converse of Pythagoras Theorem. 2. Teacher explains the solutions to three worked examples.
CS	Check for understanding and review	1. Teacher assigns one practice question for students to attempt. 2. Teacher selects some students to share their answers on the whiteboard and uses the students' answers to explain the solution to the practice question.
CC	Bring lesson to closure	1. Teacher recaps the main ideas that were taught in the lesson and assigns homework on Pythagoras Theorem.

Note. I: Introduction, D: Development, CS: Consolidation, CC: Conclusion

Episode 5.2 shows how another teacher structured a lesson on Trigonometry using multiple cycles of Development-Consolidation.

Episode 5.2

Phase	Objective	Activity
I	Set the stage for learning	1. Teacher assigns students with entry card on Sine Rule. 2. Teacher explains solutions to the entry card.
D	Engage students with new body of knowledge	1. Students work with their peers to discover the Cosine Rule using Edmodo (an online learning platform) by themselves.
CS	Check for understanding	1. Teacher assigns three practice questions for students to attempt.
D	Provide teacher input on new body of knowledge	1. Teacher checks answers for the discovery of Cosine Rule, and explains the concept of Cosine Rule.
CS	Review student learning	1. Teacher checks and explains solutions to the three practice questions that students attempted.
CC	Bring lesson to closure	1. Teacher assigns practice questions as homework, and briefly recaps the concept of Cosine Rule.

Note. I: Introduction, D: Development, CS: Consolidation, CC: Conclusion

Episode 5.3 illustrates how one of the teachers structured the first three lessons on Pythagoras Theorem. Over the course of these lessons, it is observed that the teacher adopted mainly Development-Consolidation cycles.

Episode 5.3

Lesson 1		
Phase	**Objective**	**Activity**
I	Set the stage for learning	1. Teacher introduces lesson objectives.
D	Engage students with new body of knowledge and convey expectations	1. Teacher introduces new mathematical terms. 2. Teacher gets students to form groups and guides them to use manipulatives to discover the Pythagoras Theorem. 3. Teacher explains the solution to one worked example, with inputs from the students.
CS	Check for understanding and review	1. Students attempt two practice questions. 2. Teacher selects some students to share their answers on the whiteboard and uses the students' answers to explain the solutions to the practice questions.
D	Convey expectations	1. Teacher explains the solution to one worked example, with inputs from the students.
CS	Check for understanding	1. Students attempt two practice questions.
Lesson 2		
Phase	**Objective**	**Activity**
CS	Check for understanding and review	1. Teacher assigns two practice questions for students to attempt on their mini whiteboards. 2. Teacher explains solutions to the practice questions, with inputs from the students.
D	Introduce new body of knowledge	1. Teacher assigns students a question to attempt to discover the converse of Pythagoras Theorem. 2. Teacher shows one of the students' work and uses it to explain the solution to the question.
Lesson 3		
Phase	**Objective**	**Activity**
D	Convey expectations	1. Teacher explains the solution to one worked example, with inputs from the students.
CC	Closure	1. Teacher assigns homework on converse of Pythagoras Theorem.
Teacher went on to a new topic for the rest of Lesson 3.		

Note. I: Introduction, D: Development, CS: Consolidation, CC: Conclusion

How do mathematics teachers in Singapore secondary schools structure their mathematics lessons, in general?

In the survey, participants (teachers) were asked to reflect on their own practices and indicate the frequency of the three lesson structures, like those demonstrated by the experienced and competent teachers, in their lessons. The survey data are shown in Table 5.1.

Table 5.1 Cycles of Development-Consolidation enacted by teachers in developing proficiencies in mathematics skills

Lesson Structure/Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
My lesson is typically structured this way: Introduction, Development, Consolidation, Conclusion.	5.6	24.4	47.7	22.3	70.0
My lesson is typically structured this way: Introduction, Development 1, Consolidation 1, Development 2, Consolidation 2, Conclusion (or similar, i.e. more than one cycle of Development-Consolidation).	6.9	32.2	44.3	16.6	60.9
I only have one Introduction at the beginning of the topic and one Conclusion at the end of the entire topic. There are mainly Development and Consolidation cycles in most of the lessons.	14.5	33.1	38.7	13.7	52.4

Note: $N = 677$

Based on the percentage of responses for "frequently" and "mostly/always", it is evident that the participants of the survey also use these three lesson structures to engage students in developing proficiencies in mathematics skills. A higher proportion of the participants use single Development-Consolidation cycle within one lesson and multiple Development-Consolidation cycles within one lesson.

From the data illustrated in Figure 5.1 and Table 5.1, it is apparent that the lesson structures adopted by the experienced and competent teachers when developing students' proficiencies in mathematics skills are also used by participants of the survey. It is encouraging and interesting to note that a relatively higher proportion of participants of the survey adopted lesson structures that consist of Introduction, Development, Consolidation and Conclusion in a lesson.

In other words, mathematics teachers in Singapore secondary schools, including those experienced and competent mathematics teachers, promote relational understanding when engaging students in developing proficiencies in mathematics skills.

How do experienced and competent mathematics teachers in Singapore secondary schools use the worked example and class practice nexus in their lesson?

Analysis of 211 lessons showed that experienced and competent mathematics teachers in Singapore secondary schools engage students in developing proficiencies in mathematics skills by using worked example(s) and class practice through four approaches (Figure 5.2).

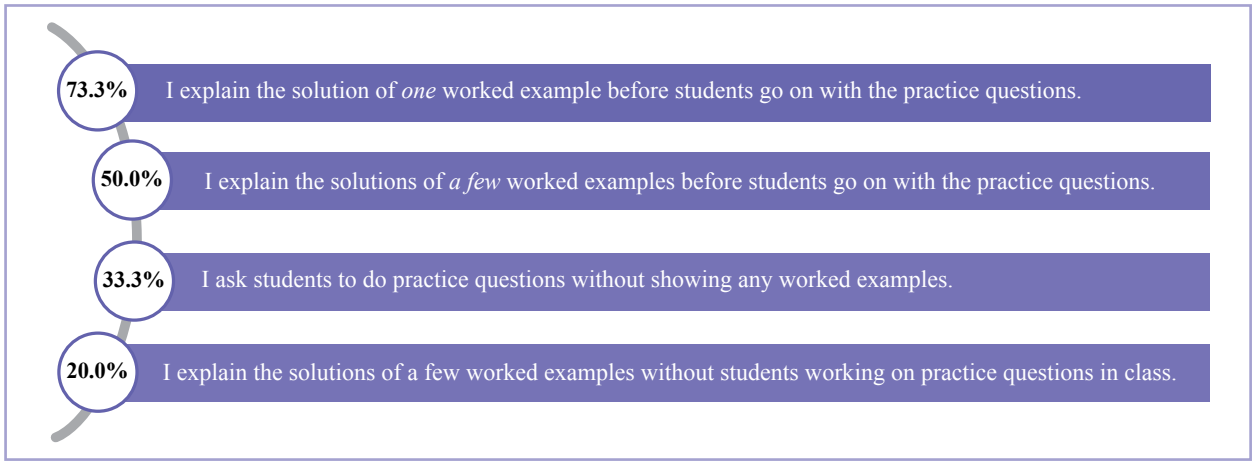


Figure 5.2 Approaches adopted by experienced and competent teachers in developing proficiencies in mathematics skills

Three key approaches, as observed from Figure 5.2, used by the experienced and competent teachers are:

- Explaining the solution of *one* worked example before assigning students with practice questions
- Explaining the solutions of *a few* worked examples before assigning students with practice questions
- Assigning students with practice questions without showing any worked example

The following three episodes illustrate how three experienced and competent teachers engage students in developing proficiencies in mathematics skills through the worked example and class practice nexus using these key approaches. In Episodes 5.4 and 5.5, the teachers were explicit in conveying their expectations on how to solve a worked example before assigning students with practice questions, while in Episode 5.6, the teacher assigned a task without showing worked examples as the task serves to bridge the concepts taught in previous lessons and their real-life applications.

Episode 5.4 shows how a teacher highlighted the importance of presenting solutions accurately for a mathematical problem in Vectors when explaining a worked example, with an emphasis on "I do".

Episode 5.4

Student A : *Teacher, the 0 must put the curl?*
Teacher : *With the curl. Without the curl, what does it mean?*
Students : *0.*
Teacher : *What is the difference between 0 and 0? The number 0 and the number zero with the curl below means one is a number, the other one is a? Vector. This is a vector, vector plus vector plus vector gives me?*
Student B : *Vector.*
Teacher : *Give me what? Vector. Not a scalar number. So, if you don't put a curl below, then you are not saying it is a vector. Please take note. You can be, you will be marked wrong, not you can be, you will be absolutely (be) marked wrong without the... You must put, that's what I'm trying to tell you. You hear what I'm saying? If not, you are not saying that it is a vector. You must. Do I need to write down? You what? Must.*

Episode 5.5 shows how another teacher brought students' attention to two types of Sine Rule formulas in Trigonometry (i.e. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ and $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$) to guide them in selecting and using the appropriate problem solving method for a worked example, with an emphasis on "We do".

Episode 5.5

Teacher : *Now, they are both talking about the same thing but there are instances where I would prefer this formula over this formula. Okay when do I use this, and when do I use the other one, the reciprocal relationship? Okay let's take a look. As long as I have an equation.*
[Teacher circles $\frac{\sin A}{a} = \frac{\sin B}{b}$ and $\frac{a}{\sin A} = \frac{b}{\sin B}$ on the whiteboard]

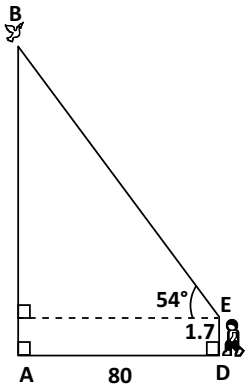
Teacher : *I just take any 2 will suffice, equation. For this case, $\frac{\sin A}{a} = \frac{\sin B}{b}$, my take is this. If I want to find the angle, from this relationship, from this equation, I can straightaway make sin A the subject right? I just have to multiply by 'a', then I get sin A = $\frac{\sin B}{b}$ × a. So if I want to find an unknown angle, I will tend to use this because algebraically, it's advantageous to me. Okay? Otherwise, the formula is the same thing.*

Teacher : *Now, as for this other one, $\frac{a}{\sin A} = \frac{b}{\sin B}$, if I want to find an unknown side, finding 'a', making small 'a' the subject will be very easy for me. It will be $\frac{b}{\sin B}$ × sin A. Okay, so this one is good for finding unknown sides.*

Episode 5.6 illustrates the task that one of the teachers assigned to students to attempt in class after showing a short video clip of real-life applications of trigonometric ratios but without showing a worked example, with an emphasis on "You do".

Episode 5.6

Problem Involving the Use of Trigonometric Ratios



In the diagram, B shows a bird flying above a point A on the horizontal ground AD. The distance of AD is 80 m. A boy's eye level, E, is 1.7 m above the ground. His line of sight to the bird makes an angle of 54° with the horizontal. Find the height of the bird above the ground.

How do mathematics teachers in Singapore secondary schools use the worked example and class practice nexus, in general?

In the survey, participants (teachers) were asked to reflect on their own practices and indicate the frequency of the four approaches demonstrated by the experienced and competent teachers, in their lessons. The survey data are shown in Table 5.2.

Table 5.2 Approaches adopted for developing proficiencies in mathematics skills

Approach/Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
I explain the solution of *one* worked example before students go on with the practice questions.	11.8	30.0	31.6	26.6	58.2
I explain the solutions of *a few* worked examples before students go on with the practice questions.	5.2	26.1	43.7	25.0	68.7
I ask students to do practice questions without showing any worked examples.	77.7	19.6	1.9	0.8	2.7
I explain the solutions of a few worked examples without students working on practice questions in class.	71.5	21.0	5.8	1.7	7.5

Note: $N = 677$

Based on the percentage of responses for "frequently" and "mostly/always", it is evident that the participants of the survey use two key approaches that relate to the worked example and class practice nexus for engaging students in developing proficiencies in mathematics skills. They generally tend to explain the solution of *one* or *a few* worked example(s) before assigning students with practice questions.

From the data illustrated in Figure 5.2 and Table 5.2, it is apparent that the general practices demonstrated by experienced and competent teachers that relate to the worked example and class practice nexus to engage students in developing proficiencies in mathematics skills are also used by participants of the survey. In particular, both groups of teachers tend to adopt "I do, We do, You do" through effective use of worked example(s) and class practice. However, it seems that relatively more experienced and competent teachers tend to assign class practice without explaining any worked examples and explain the solutions of a few worked examples without assigning students with class practice. This could be contributed by the strong Development-Consolidation cycles which convey clear expectations on the skills and knowledge required to develop proficiencies in mathematics skills for the particular mathematical unit.

What are the implications for mathematics teachers in Singapore secondary schools?

- Planning for a well-structured lesson that follows a logical teaching sequence allows for not only more effective teaching, but also ensures that students receive quality instruction. It is important that teachers plan for lessons that encompass clearly defined learning goals, allow sufficient room for development and consolidation of mathematics concepts and skills. This would help to students to develop procedural fluency with relational understanding.
- Checking on student learning creates awareness on whether the Development-Consolidation cycle has achieved its intended objectives through the explicit and effective use of worked example and class practice ("I do, We Do, You do"). Teachers should also ensure that students are given sufficient opportunities to apply on their own what they have learned, and provide remediation, such as additional practice or time for learning, whenever necessary.

How are you engaging your students in developing proficiencies in mathematics skills in your mathematics lesson?



My notes

Contributors
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QUESTION

6

How do mathematics teachers in Singapore secondary schools emphasise mathematical processes?



Question 6

How do mathematics teachers in Singapore secondary schools emphasise mathematical processes?

What does it mean for students to develop their use and understanding of mathematical processes?

Mathematical processes encompass three main areas below:

- **Mathematical Reasoning and Communication:** A student has displayed mathematical reasoning and communication if he/she is able to build coherent arguments by integrating mathematical language to express mathematical ideas logically (Ministry of Education, 2012). Teachers should provide opportunities for students to engage with a wide range of mathematical problems that involve processes such as inductive and deductive reasoning, explaining or justifying a mathematical solution/statement, and making inferences.
- **Thinking Skills and Heuristics:** According to curriculum documents, thinking skills refer to "skills that can be used in a thinking process", for example, "classifying, comparing, analysing parts and whole, identifying patterns and relationships, induction, deduction, generalising, and spatial visualisation" (Ministry of Education, 2012, p. 17). In contrast, heuristics are often used during problem solving attempts where the solver engages in "using a representation (e.g., drawing a diagram, tabulating), making a guess (e.g., trial and error, guess and check, making a supposition), walking through the process (e.g., acting it out, working backwards) or changing the problem (e.g., simplifying the problem, considering special cases)" so that he/she can navigate through the given information and conditions in the problem (Ministry of Education, 2012, p. 17). Teachers should encourage the discussion of thinking skills and heuristics when presenting students with various mathematical problems.
- **Mathematical Applications in Real World:** Applying mathematics in the real world involves students drawing connections between school mathematics and the real world. In other words, students solve problems situated in real-world contexts, applying what they learnt in class. Teachers should expose students to a wide range of problems, such as open-ended and real-world problems (Ministry of Education, 2012).

Our study of mathematics lessons enacted by 30 experienced and competent mathematics teachers in Singapore secondary schools revealed that they emphasise mathematical processes through three main approaches:

- Developing students' mathematical reasoning and communication skills
- Developing students' thinking skills and heuristics
- Making connections between mathematics in the classroom and real-world contexts

How do experienced and competent mathematics teachers in Singapore secondary schools develop students' mathematical reasoning and communication skills?

An analysis of 211 lessons showed that experienced and competent mathematics teachers in Singapore secondary schools emphasise students' mathematical reasoning and communication through two main approaches. As illustrated in Figure 6.1, 66.7% of the experienced and competent teachers encourage their students to explain and justify their working or their peers' working and 26.7% of them help students formulate mathematical relationships between quantities in a logical manner.

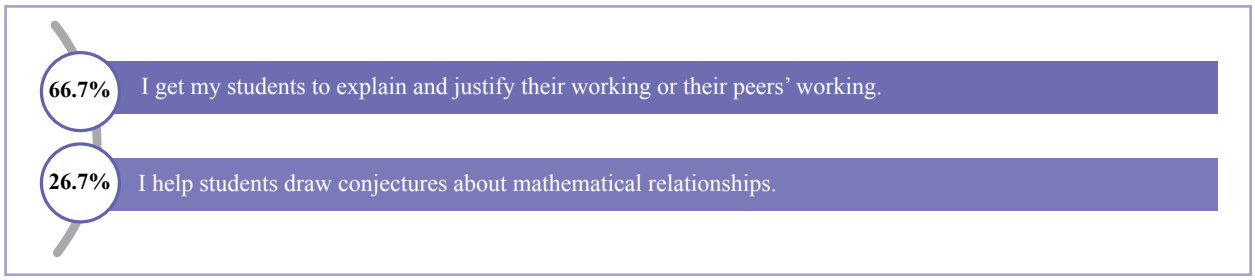


Figure 6.1 Approaches adopted by experienced and competent teachers when developing mathematical reasoning and communication

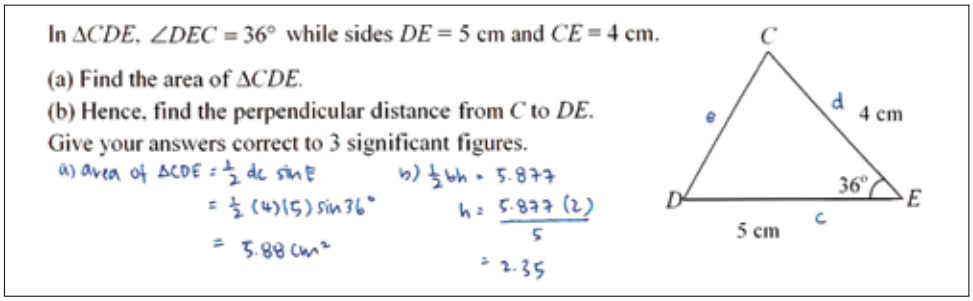
The two main approaches used by the experienced and competent teachers to develop mathematical reasoning and communication are:

- Getting students to explain and justify their working or their peers' working
- Helping students formulate mathematical relationships between quantities in a logical manner

The following two episodes illustrate how two experienced and competent teachers develop students' mathematical reasoning and communication using these two ways. Episode 6.1 shows how a teacher set aside time within her lesson for students to identify areas of improvement in their peer's solutions. She emphasised the need for "clarity" in communicating the solution process so that others can understand the logical argument made.

Episode 6.1

Student's work projected onto a screen



Teacher : *Okay, comments, come. People, you comment. Can I give you one minute, okay, discuss with your partner. Is there anything you want to keep or anything that you want to refine? Ask your partner. Comment on the solution. You can use the checklist to check if you want to. Okay you can check against the checklist I have just shown you earlier. I'm going to ask you for opinions. I'm sure everybody has more or less the same numerical answer. Okay, it's all the little final things but clarity of the solution.*

Teacher : *Okay, 5-4-3-2 and 1. Okay I'm going to ask you for some comments okay, anything okay. So... Student A, can you comment on the solution?*

Student A : *Part (b)...*

Teacher : *Part (b), what about part (b)?*

Student A : *The final answer needs units.*

Teacher : *Needs units. So... What are you going to say? What, what would you say?*

Student A : *Put "cm".*

Teacher : *Put "cm", just put like that, here, "cm".*
[Teacher inserts "*cm*" in the final answer]

Teacher : *Okay, is there anything else?*

Student B : *She should either define 'h' or write that perpendicular...*

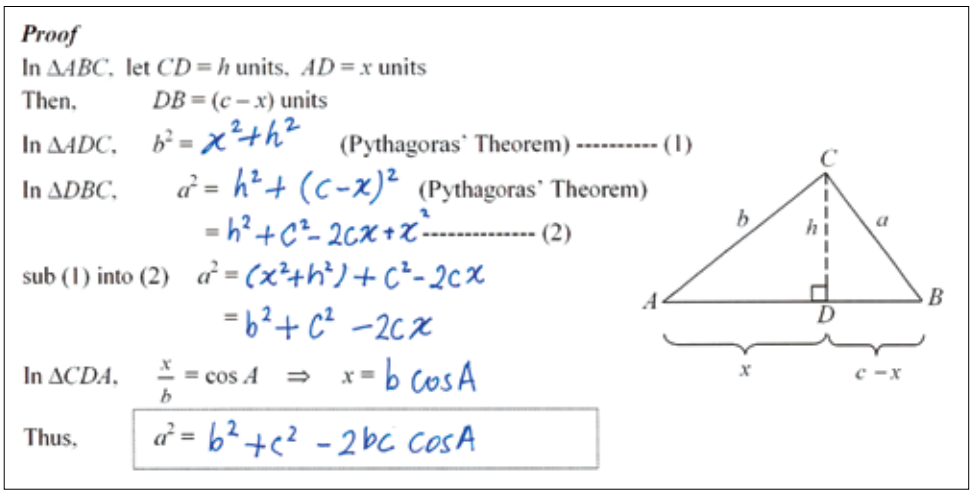
Teacher : *Ah, so there's a new unknown right? Okay you must introduce, define your new unknown, so... Let the perpendicular distance from C, whenever you introduce something that's not in the question, please define it. Clarity, clarity.*
[Teacher inserts "Let the perpendicular distance from C to DE be *h* cm" into the student's solutions]

Teacher : *Okay so, basically yeah I think you have made very good, sharp comments on that. That's very good.*

Episode 6.2 shows how another teacher provided students with an opportunity to work through a mathematical proof to derive the Cosine Rule for an acute-angled triangle (ΔABC). To promote fluency in mathematical reasoning and communication, she first examined how her students worked out the proof before challenging them to think about how their proof would work to show that the Cosine Rule still applies for an obtuse-angled triangle. Throughout the episode, the teacher emphasised the importance of establishing new mathematical relationships between quantities based on logical deduction using a prior piece of knowledge (Pythagoras' Theorem).

Episode 6.2

Student's work projected onto a screen



[The teacher explained the special case for Cosine Rule, where the included angle is 90°)]

Teacher : *Now so far this is our preliminary observation right? It works for specific values but does it work for all values? That's why we go into the proof. Let's take a look at your proof.*
[Teacher questions for generalisability of their mathematical observations and sets the stage for proof.]

[Teacher projects student's solutions onto the screen]

Teacher : *So I go through this proof with you okay? So* b^2 *is actually* $x^2 + h^2$*, Pythagoras Theorem. And therefore from triangle DBC, I have the following relationship right? So* $a^2 = h^2 + (c - x)^2$*, when you have expanded* $(c - x)^2$ *you get* $h^2 + c^2 - 2cx + x^2$*. So you substitute Equation 1 into Equation 2, and that helps you to replace your* $x^2 + h^2$*, so therefore* $a^2 = b^2 + c^2 - 2cx$*. And in triangle CDA, so you will have* $\frac{x}{b}$ *as your cos A, that implies that* $x = b \cos A$*. So now if I were to call this equation 3, and 4...*

[Teacher labels equation $a = b^2 + c^2 - 2cx$ as Equation 3, and $x = b \cos A$ as Equation 4]

Teacher : *I think some parts are missing. So if I were to substitute Equation 4 into 3...*

[Teacher inserts "Substitute (4) into (3)" into the student's solutions]

Teacher : *I have* $a^2 = b^2 + c^2 - 2bc \cos A$*. Okay what's the purpose? The purpose is to find a relationship between the sides of the triangle and the angle okay? So which I have done once. And this is for an acute-angled triangle.*
[Teacher challenges students to think about how their proof would work to show that the Cosine Rule still applies for obtuse-angled triangles.]

How do mathematics teachers in Singapore secondary schools develop students' mathematical reasoning and communication, in general?

In the survey, participating teachers were asked to reflect on their own practices and indicate the frequency in which they encourage students to explain and justify their or their peers' working, and help students formulate mathematical relationships between quantities in a logical manner (illustrated in Episodes 6.1 and 6.2). The survey data are shown in Table 6.1.

Table 6.1 Approaches for emphasising mathematical reasoning and communication

Approach/Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
I get my students to explain and justify their working or their peers' working.	6.5	45.1	40.2	8.2	48.4
I help students draw conjectures about mathematical relationships.	8.3	45.9	39.0	6.8	45.8

Note: $N = 677$

From Table 6.1, almost half of the participating teachers in the survey develop students' mathematical reasoning and communication frequently through encouraging students to explain and justify their working or their peers' working, and helping students draw conjectures about mathematical relationships. The latter often includes requesting students to make preliminary mathematical observations culminating to informal deductive statements or conjectures before launching into formal proofs pegged to the syllabus.

It is apparent from the data that practices demonstrated by experienced and competent teachers when developing students' mathematical reasoning and communication (Figure 6.1) were also employed by the participating teachers in the large-scale survey (Table 6.1). Comparatively, a relatively higher proportion of experienced and competent teachers tend to encourage students to explain and justify their working or their peers' working, whilst a relatively higher proportion of participants of the survey tend to help students draw conjectures about mathematical relationships.

How do experienced and competent mathematics teachers in Singapore secondary schools develop students' thinking skills and heuristics?

It was discovered after a further analysis of 211 lessons that some experienced and competent mathematics teachers in Singapore secondary schools develop students' thinking skills and heuristics through two main approaches. As illustrated in Figure 6.2, only 10.0% and 3.3% of experienced mathematics teachers in secondary schools teach problem-solving heuristics and thinking skills explicitly to their students respectively. This implies heuristics and thinking skills could have been taught implicitly during lessons or, perhaps not addressed at all. Nonetheless, video evidence points to instances of heuristics and thinking skills being discussed implicitly in mathematics lessons; without labelling these processes. Based on an informal conversation with one of the participating teachers, there was an assumption that problem-solving heuristics and thinking skills 'have already been taught explicitly in primary mathematics classrooms'. Hence, some teachers could have expected that students have already internalised such processes at secondary levels.

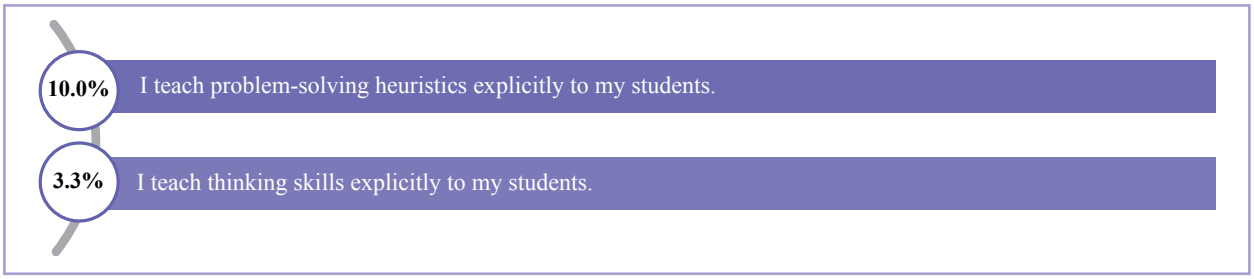


Figure 6.2 Approaches adopted by experienced and competent teachers when developing students' thinking skills and heuristics

Episodes 6.3 and 6.4 illustrate how an experienced and competent teacher weaved in thinking skills and heuristics as part of the process to derive the formula for arc length in a Year 10 (aged 16) class. In Episode 6.3, through use of thoughtful prompts and questions, the teacher guided students to apply the thinking skill "analysing parts and whole" (implicitly) in order to highlight arcs that are part of the circumference of a circle. She also used circular fraction discs in a subsequent instance of "analysing parts and whole" to guide students in visualising the relationship between the (arc) angle and the arc length mediated by proportional reasoning involving the size of the (arc) angle with respect to 360° (one full turn of a circle).

Episode 6.3

Teacher : *Anyone holding onto a circle that's been divided into two? Nobody? Just me? Okay here. We have a circle that is divided into two. Can you see that? Let's take a look. One full circle is the circumference right?*

Student : *Yeah.*

Teacher : *Okay so, Student A, if I were to cut it into two, how many arcs do I have here? How many arcs?*

Student A : *Two.*

Teacher : *Two. Okay I have two arcs right, one, and then the other one. Okay, so two arcs ah. Okay let's fill up this one first. Hold on, another thing, what is the angle here? You all should know right? Student B, what is the angle here? What is the angle of a semi-circle?*

Student B : *180.*

Teacher : *180, alright. Can we fill up as we discuss? So, it's 180 °, so this is* $\frac{180}{360}$ *and it means, there's half. Correct?* $\frac{180}{360}$ *right? And then what do you notice about the length of the arc over the circumference? How many as compared to a full circle?*

Student C : *Half.*

Teacher : *Half right? You can see that this arc is actually half of a full circle. So we say this is half.*

Episode 6.4 describes a later part of the same lesson as described in Episode 6.3. In this episode, the teacher drew the attention of the students to the fact that expressing the angle subtended by the arc as a fraction of 360° is instrumental in deriving the formula for length of the arc. The teacher made use of a table of values to scaffold students' thinking in "analysing patterns and relationships". The teacher then helped the students focus on the key variable (i.e. the angle subtended by the arc) when generalising through an inductive approach to derive the formula for arc length.

Episode 6.4

Activity 2: To find the arc length in the circle

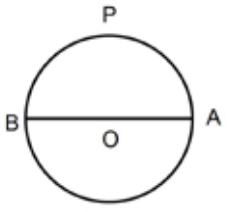


Figure 1

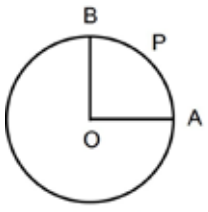


Figure 2

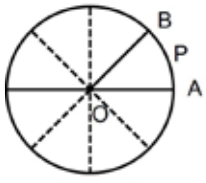


Figure 3

	(1) ∠AOB	(2) $\frac{\angle AOB}{360^\circ}$	(3) $\frac{\text{length of arc APB}}{\text{circumference}}$	(4) Length of the arc APB
Figure 1	180°	$\frac{180^\circ}{360^\circ}=\frac{1}{2}$	$\frac{1}{2}$	
Figure 2	90°	$\frac{90^\circ}{360^\circ}=\frac{1}{4}$	$\frac{1}{4}$	
Figure 3	45°	$\frac{45^\circ}{360^\circ}=\frac{1}{8}$	$\frac{1}{8}$	
Figure 4	60°	$\frac{1}{6}$	$\frac{1}{6}$	
Figure 5	x°	$\frac{x^\circ}{360^\circ}$	$\frac{x}{360}$	

Teacher : *Can you tell me what can you see about Column 2 and 3?*
Student A : *They are the same.*
Teacher : *They are the same. Well, can you specify? Student A, same as in what? So what can you conclude?*
Student A : *The numbers.*
Teacher : *The numbers. Okay, you realised the numbers are the same. Okay, look at the heading, I want you to look at the heading, what does it mean to you? What does the two numbers tell you? Look at the Column 2 ah, Column 2 says the angle over the total. The angle of one slice is over the total circle. Column 3 says the length of the arc as part of the whole circle. Can you see that? Okay, so how, what can you conclude? Yes, Student B, anything, anything, it's okay. What is it? So if I want to find the arc length, what can I do?*
[Teacher prompted for students to analyse patterns and relationships.]
Student B : *The circumference of the circle.*
Teacher : *Okay, very good. If I want to find the arc length, I need to know the circumference of the circle. Okay, what else, what else, Student C, what else? What else about the arc length that you know? It's always what?*
Student C : *Fraction.*
Teacher : *A fraction. Yes, fraction of what? Very good. A fraction of the?*
Student C : *Whole number.*
Teacher : *Fraction of the whole number? What do you mean by whole number?*
Student C : *360.*
Teacher : *A fraction of 360. Okay, alright, so the arc length is always, just now, Student C say arc length, a fraction of the circumference. Is that what you say Student C?*
[Teacher prompted students to generalise a key variable determining all arc lengths.]
Student C : *Yes.*
Teacher : *Okay, the arc length is a fraction of the circumference. And then what else? Student C said the angle of the arc is also a fraction of, he used the word, he says, 360. A fraction of 360 °, correct or not? Correct or not? The angle. Okay, 2 things here ah. This arc is always a fraction of the circumference of a circle, correct? Okay, at the same time, the angle, this angle here, is what? Also a fraction of the whole circle, which is how much?*
Student C : *360.*

How do mathematics teachers in Singapore secondary schools develop students' thinking skills and heuristics in general?

In the survey, participating teachers were asked to reflect on their own practices and indicate their frequency of teaching problem-solving heuristics and thinking skills explicitly to students (see Table 6.2). It is evident that almost half the participating teachers in the survey taught problem solving heuristics explicitly in their classrooms frequently or most of the time. In addition, more than half of the participating teachers taught thinking skills explicitly to their students frequently or always.

Table 6.2 Approaches for emphasising thinking skills and heuristics

Approach/Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
I teach problem-solving heuristics explicitly to my students.	9.5	47.6	36.5	6.4	42.9
I teach thinking skills explicitly to my students.	4.7	42.1	44.0	9.2	53.2

Note: $N = 677$

Compared to the video-recorded data of the 30 experienced and competent teachers, a higher proportion of the 677 participating teachers in the self-reported survey claimed to teach problem-solving heuristics and thinking skills explicitly in their mathematics lessons.

How do experienced and competent mathematics teachers in Singapore secondary schools make connections between mathematics in the classroom and real-world context?

Another analysis of 211 lessons showed that experienced and competent mathematics teachers in Singapore secondary schools emphasise connections between mathematics in the classroom and real-world contexts through three approaches. As illustrated in Figure 6.3, 46.7% of the experienced and competent teachers use real-world examples in their lessons, 43.3% of them draw connections between various mathematics topics and 6.7% of the teachers get students to discuss assumptions made during real-world contexts.

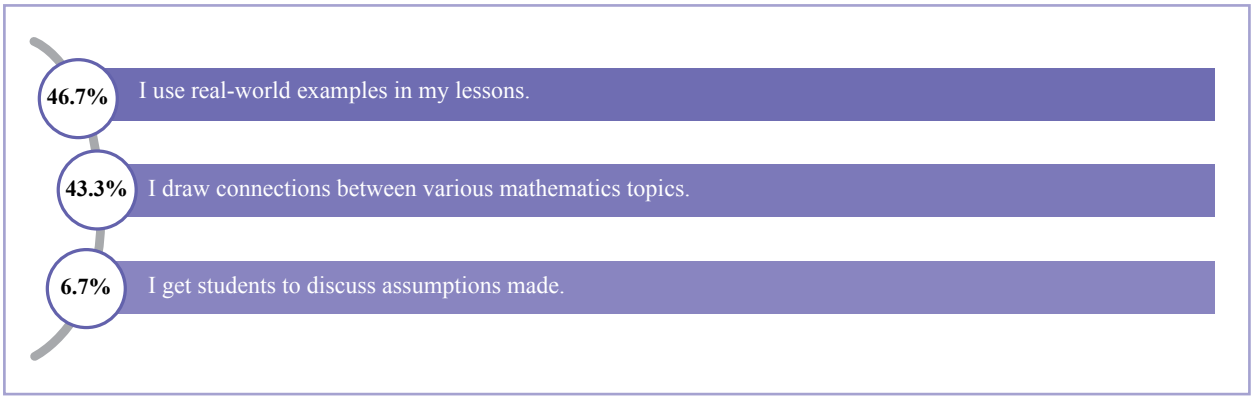


Figure 6.3 Approaches adopted by experienced and competent teachers when emphasising connections within school mathematics as well as connections between school mathematics and the real world

The following three episodes illustrate how two experienced and competent teachers drew connections between school mathematics and the real world. Episode 6.5 shows how a teacher encouraged students to think about the best way to calculate the area of the Bermuda Triangle given four formulas. The teacher used real-world examples in her lesson because she had wanted her high-progressing students to be discerning about the different types of real-world situations in which each formula can be applied and the conditions of use.

Episode 6.5

Formulae projected on the screen by the teacher

Formulae

$A = \frac{1}{2} \times \text{base} \times \text{height}$

$A = \frac{1}{2}\, ac \sin B$

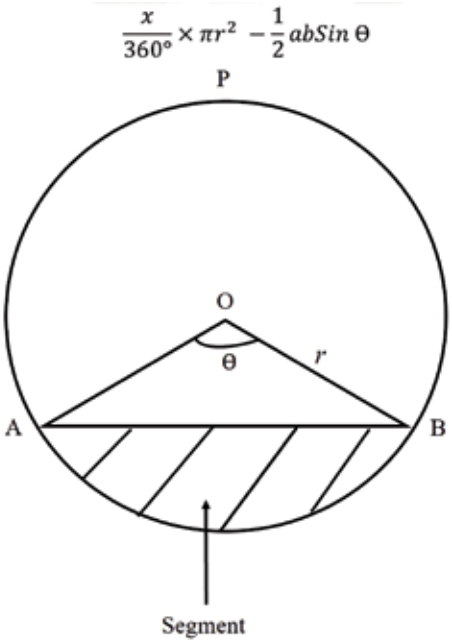
$A = \sqrt{S(S-a)(S-b)(S-c)}$

Shoelace formula (coordinate geometry)

Teacher : *Okay, Student A, can you offer us one opinion?*
Student A : *The shoelace (formula).*
Teacher : *Okay why?*
Student A : *Because the triangle may not be like an exact triangle.*
Teacher : *Not an exact triangle. Err, what do you mean by exact triangle?*
Student A : *Because the coordinates of the points on the...*
Teacher : *So Student A chose the shoelace formula because she said the triangle may not be exact. Okay, wait ah, can I reverse the question. Why don't you use this? Why don't you use "*$\frac{1}{2} \times base \times height$*"? Why can't we use this? So there are two ways of saying. One way is that we have this, we have a reason. Or some of us go by elimination. So why didn't you choose this?*
Student B : *Because it's difficult to find.*
Teacher : *Find what?*
Student B : *The base and the height.*
Teacher : *Why?*
Student B : *Because it's hard to find like where the right angle...*
Teacher : *Okay it's very hard to find the right angle, and knowing that, what is this? That's the sea right. Nobody would want to go, go canoeing. I don't think you want to do canoeing here okay, just to find the height. Student C, which one do you use?*
Student C : *Shoelace.*
Teacher : *Okay, why do you use the shoelace formula?*
Student C : *You can use the longitude and latitude of the place.*
Teacher : *Okay she said something interesting. You can use the longitude and latitude of the place, aha, and use this thing and then to find the area. Okay. I'm quite surprised, I mean I'm getting very good answers. And then, come, another person. Student D, what do you think? What would you use?*
Student D : *The* $A = \frac{1}{2}\, ac \sin B$.
Teacher : $\frac{1}{2}\, ac \sin B$. *So, you're going to use this one? So how are you going to find the angle?*
Student D : *Use a device that can measure the angle.*
Teacher : *Now comparatively, if I were to just look at these two formulas (i.e.* $A = \frac{1}{2}\, ac \sin B$ *and* $A = \sqrt{S(S-a)(S-b)(S-c)}$ *). If the distance is very very far, very long, okay, if I were to, if this distance and if this distance is very long, if I were to make a mistake in this angle, by let's say 1° off, what would happen to here? There would be an error isn't it? So 1°, if it's very short, it won't be very significant. But if it's very very far away, you will see that the area becomes more and more pronounced. So usually whenever possible, we, I would try to stick to, the finding of distances because distance, if I'm off by a metre, it would not, the error will not be as pronounced as the error caused by the angle...*

Episode 6.6 shows the same teacher drawing connections within mathematics learnt across levels and deepening understanding about the formulae for area of triangle and conditions of use. The first formula, $\frac{1}{2} \times \text{base} \times \text{height}$, was taught at Year 5 (aged 11) in the primary school whilst the next formula, $\frac{1}{2} ab \sin C$, is taught at Year 9 (aged 15) in the secondary school.

Episode 6.6



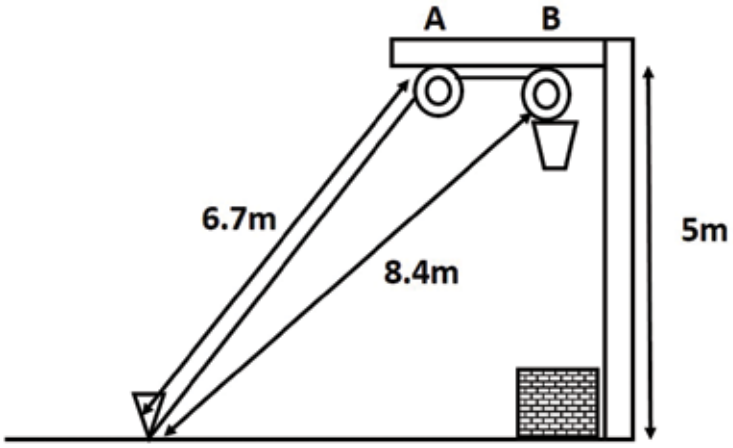
Teacher : *Let me ask Student A, can you tell us how do we find area of segment?*
Student A : *Find the area of the triangle.*
Teacher : *Find the area of triangle. Why do we find the area of triangle? Why? Because originally what is the shape? What is the shape? The whole thing here, does it look like the other one? What is this?*
Student A : *Sector.*
Teacher : *It's a sector. Originally it was a sector. But what separated it? What caused the segment?*
Student A : *The triangle.*
Teacher : *The triangle. Okay, there's a sector, and then there's a triangle inside. That gives us a, when there's a triangle inside, the difference between the sector and the triangle gives us the? Segment. Okay, and then according to Student B, the area of sector minus area of triangle. Student C, are you familiar with this area of triangle? You were absent yesterday. Aha, you are. Okay can you tell us a little bit more about this formula? Why do we use this formula? What's another area of triangle you know? Common one, you have learned in primary school.*
Student C : $\frac{1}{2} \times base \times height$.
Teacher : *Why do you think I can't use* $\frac{1}{2} \times base \times height$ *here?*
Student C : *Because it's not a right-angled triangle.*
Teacher : *Student D, do you agree with what Student C said? We use* $\frac{1}{2} ab \sin C$ *here because it's not a right-angled triangle.*
Student D : *No.*
Teacher : *No? So I can use* $\frac{1}{2} \times base \times height$ *to find the area of this triangle. Can, how? Tell me. Find what? It is quite possible if you do what? What is missing there? Yes, Student D, can we use* $\frac{1}{2} \times base \times height$ *or not? What do you think? Possible right? Provided I have what? I have the height and the?*
Student D : *The base.*
Teacher : *So if I don't have the height and the base, Student D, is there another area of triangle I can use, which is what?*
Student D : $\frac{1}{2} ab \sin C$.

Episode 6.7 illustrates how another teacher guided students to make assumptions for a mathematical problem situated in a real-world context.

Episode 6.7

Task 3

The cooks required water to cook up a feast for the celebration. Dan Bi came up with a plan to help them retrieve the water from the well by constructing a pulley system. The rope ran through two pulleys that are 5m above the ground as shown in the diagram. Find the distance between the two pulleys.



Teacher : *Task number 3 is a little bit difficult. When I talk about the distance between the two pulleys I am actually talking about this distance here. Now you might have to do some assumptions. That means what have you assumed in this picture? So you might have to simplify the picture a bit and draw on your own. And then after that what is the assumption in this case.*
[Students attempt Task 3]

Teacher : *When you simplify the picture, what happens? How can I make it to the one that I want? First of all Pythagoras' Theorem, what do I need?*

Student A : *Right-angled triangle.*

Teacher : *Ah, right-angled triangle. Then you have to adjust, how can I adjust the picture such that I get a right-angled triangle? Then from there, you simplify, so what is the assumption that you have made?*

Student B : *Ohh, I know how to do it already.*
[Students attempt on Task 3]

Teacher : *Now before you do this, what is the simplified version now? What is the thing that you have assumed? Because if you follow the diagram, you realised that, does it form, really a triangle? Not really. So what is the assumption that you have made here? You basically assume that it actually forms two triangles. So that's why Student A came up with this diagram that she has used to help her to solve the distance between the two pulleys.*

How do mathematics teachers in Singapore secondary schools make connections between mathematics in the classroom and real-world context, in general?

In the survey, participating teachers were asked to reflect on their own practices and indicate the frequency of the 3 approaches discussed above. The survey data is shown in Table 6.3.

Table 6.3 Approaches for emphasising mathematical applications in real world

Approach/Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
I use real-world examples in my lessons.	0.4	33.4	54.2	12.0	66.2
I draw connections between various mathematics topics.	0.7	21.9	57.8	19.6	77.4
I get students to discuss assumptions made.	14.0	54.8	26.7	4.5	31.2

Note: $N = 677$

Based on the percentage of responses for "frequently" and "mostly/always", it is evident that the participants of the survey also employ the same three approaches to emphasise connections between school mathematics and real-world contexts. They generally tend to use real-world examples in their lessons, draw connections between various mathematics topics (at times across levels), and discuss how assumptions can be formulated using problems situated in real-world contexts.

Table 6.3 shows that participating teachers in the survey reported frequent episodes of using real-world examples in their mathematics lessons. In addition, frequent attempts were also made by these teachers to highlight connections between various mathematics topics. However, discussion of assumptions associated to the real-world context appeared to be less frequently done by these teachers. This is observed across the whole sample, including experienced and competent teachers.

What are the implications for mathematics teachers in Singapore secondary schools?

- Teachers should continue to provide students with opportunities to engage in mathematical processes such as reasoning and communication, discussion or use of heuristics and thinking skills, and making connections between mathematics taught in the classroom and the real world.
- Making assumption is an important aspect of formulating a real-world problem into a mathematical problem. It also sensitises students to real world interpretations of mathematical solutions. Teachers should provide students with more opportunities to articulate assumptions made with respect to real-world problems discussed in the classroom.

How are you emphasising mathematical processes in your mathematics lesson?



My notes

Contributors
Ng Kit Ee Dawn • Liyana Safii

References

Ministry of Education. (2012). *O-Level mathematics teaching and learning syllabus*. Singapore: Ministry of Education, Curriculum Planning and Development Division.

QUESTION

7

How do mathematics teachers in Singapore secondary schools facilitate the development of metacognitive strategies amongst their students?



Question 7

How do mathematics teachers in Singapore secondary schools facilitate the development of metacognitive strategies amongst their students?

What does it mean for students to develop metacognition?

Flavell (1976), who first coined the term metacognition, defined metacognition as "one's knowledge concerning one's own cognitive processes and products or anything related to them… Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective" (p. 232). According to curriculum documents (Ministry of Education, 2012), metacognition involves a person reflecting on his/her thinking. In other words, metacognition refers to the "awareness of, and the ability to control one's thinking processes, in particular the selection and use of problem-solving strategies. It includes monitoring of one's own thinking, and self-regulation of learning" (p. 17). In other words, metacognition involves three aspects – awareness, monitoring and regulating (Lee, Ng, & Yeo, 2019). Teachers can foster students' "metacognitive awareness and strategies", and knowledge of "when and how to use the strategies" by providing "opportunities to solve non-routine and open-ended problems, to discuss their solutions, to think aloud and reflect on what they are doing, and to keep track of how things are going and make changes when necessary" (Ministry of Education, 2012, p. 17).

How do experienced and competent mathematics teachers in Singapore secondary schools facilitate the development of metacognitive strategies amongst their students?

Analysis of mathematics lessons enacted by 30 experienced and competent mathematics teachers in Singapore secondary schools revealed that they facilitate the development of metacognitive strategies amongst their students through four approaches. As illustrated in Figure 7.1, data from 211 lessons showed that 43.3% of the teachers get their students to compare different ways of solving a problem, 40.0% of them get their students to reflect on their learning, and 30.0% of the teachers get their students to check for reasonableness of their answers after solving a problem.

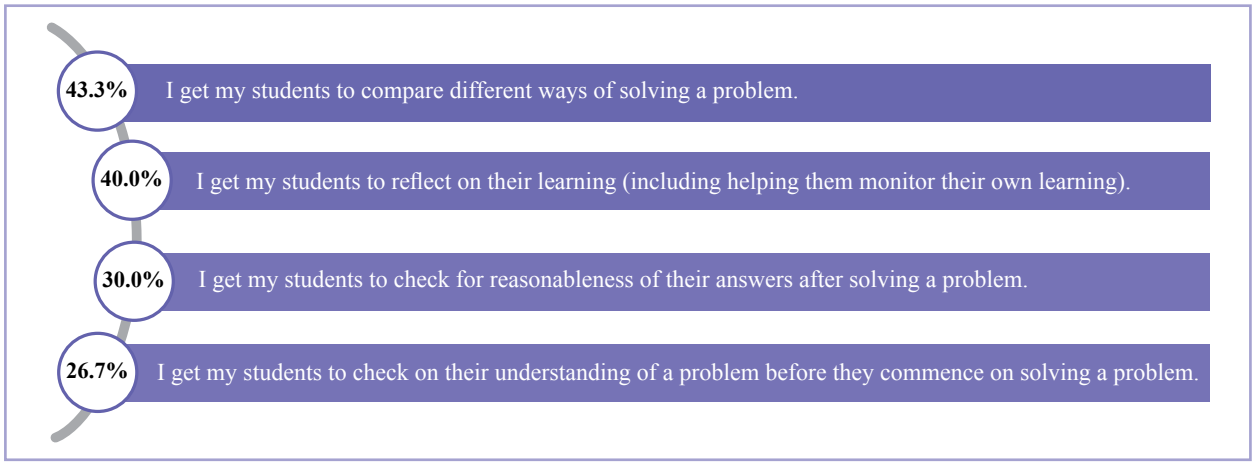


Figure 7.1 Approaches adopted by experienced and competent teaches when facilitating the development of metacognitive strategies

The following three episodes illustrate how three experienced and competent teachers facilitated the development of metacognitive strategies amongst their students using the following approaches:

- Encouraging students to compare different ways of solving a problem
- Providing opportunities for students to reflect on their learning, including helping them to monitor their learning
- Getting students to check for reasonableness of their answers after solving a problem

The first approach builds the student's repertoire of solving problems from which he or she can then decide which to choose from. The second teaches the student how to recall relevant cognitive processes and objects so that he or she is aware of them and can choose when to use them. The third approach reminds the student to monitor his or her solution, i.e. cognitive product, to assess its viability and applicability.

Episode 7.1 shows how a teacher encouraged her students to compare two different types of solution steps in order to highlight the equivalent statements. At the same time, students' awareness of their understanding was heightened.

Episode 7.1

Solutions written on the class whiteboard by the teacher

Teacher's solution	**Student A's solution**
$k^2 = 4(k+3)$	$4(k+3) = k^2$
$k^2 = 4k+12$	$4k+12 = k^2$
$k^2 - 4k - 12 = 0\ (shown)$	$-k^2 + 4k + 12 = 0$

Teacher : *You all see, help him to see why his answer is right or wrong. Nothing wrong what?* $k^2 = 4(k+3)$, *nothing wrong. This and this* (referring to the first steps in both solutions), *isn't it the same?*

Student B : *The zero should be at the other side.*

Teacher : *So this is the part where he is confused. It's okay, let's help him. So he has a negative but there's no negative there* (in the teacher's solution). *So what can we do?*

Student C : *I know how to do it, but I don't know how to explain.*
[Students discuss among themselves]

Teacher : *So actually this is an equation. When there is an equation, I can actually multiply both sides by negative, I multiply by negative, I multiply by negative.*
[Teacher writes "multiply by – " on both the left and right hand side of the equation in Student A's solution]

Teacher : *So then this whole sign will change, and 0 is still 0. Actually it's the same.*
[Teacher writes $k^2 - 4k - 12 = 0\ (shown)$]

Episode 7.2 shows how another teacher provided opportunities for students to reflect on their own learning, using prompts to help them recall prior knowledge that has been purposefully selected in view of what was to be learnt in that particular lesson. She wanted to understand what each student remembered so that she could subsequently plan an approach to move each student forward from his/her entry point.

Episode 7.2

Teacher : *Now, what I want to do on your whiteboard, can you please draw a circle for me, small or big, up to you. I want you to try and label parts of the circle. Let's see whether you can recall anything about circle. Why? Because this whole chapter is about the circle. So you have to know, the parts of the circle. Alright, so can you draw, not too small. Okay, because later when you put up I need to see. Okay, not too big and not too small. Alright, label the parts of the circle. Anything that comes into your head now. No discussion.*

Student A : *What kind of circle?*

Teacher : *What do you mean what kind of circle? How many types of circle? Anything, anything, anything okay. Anything that you want to tell me about circle that you remember.*

Student B : *Teacher, I can't remember anything ah.*

Teacher : *You can't remember anything at all? Okay, then just draw the circle. If you can remember some formulas associated with the circle, also write it down. Okay, anything, not to worry. I'm just trying to check on your prior knowledge, what you remember. And how much you remember. Okay, perhaps you can write down any formula that you remember, with regards to circle, anywhere by the side. Ah, anything, just put it down, anywhere on the whiteboard.*

Teacher : *Okay, done, done, done? I don't want to spend so much time on it. Okay, alright, let's see whether you are able to get all the information from the back of your head. Remember I told you about information stored at the back of your brain? Okay, let's see whether you are able to bring it to the front. Okay, okay, looks like this row is ready. Can I have your whiteboard please?*
[Teacher walks around and checks students' mini whiteboards]

Teacher : *Okay. Most of you remembered radius, most of you remembered diameter. I do see the word circumference for some of you. And I do see a few formulas, there on the whiteboard.*

Episode 7.3 shows how another teacher taught students how to check for reasonableness of their answers after solving problems relating to quadratic equations.

Episode 7.3

Teacher : *In the exam right, in the exam, do you have answers at the back to check or not?*

Students : *No.*

Teacher : *No right? Obviously not right? Then how do you check your own answers in the exam?*

Student A : *Backtrack.*

Teacher : *Backtrack. So you check step-by-step?*

Student B : *Substitute.*

Student C : *You sub the answer and...*

Teacher : *Okay, wait, everybody, listen to what Student C say. This is very important. Okay please say it loudly to the whole class.*

Student C : *You sub the solution of the answer right...*

Teacher : *Substitute the answer that you obtain...*

Student C : *Into the question.*

Teacher : *Into the original question. Substitute and make sure what? Make sure it satisfies the equation. Okay, so you substitute into the left-hand side, substitute into the right-hand side and make sure they are equal. Then you know, if it satisfies the equation, then you know the solution that you obtained is correct. Okay the answers that you obtained are correct. Understand or not? Because in the exam, you cannot turn at the back and check you know, right? So this is a way to check to help yourself. If you check step-by-step, you backtrack and check step-by-step, it's very difficult to find your own errors. Very difficult.*

How do mathematics teachers in Singapore secondary schools facilitate the development of metacognitive strategies amongst their students, in general?

In the survey, participating teachers were asked to reflect on their own practices and indicate the frequency of the four approaches, demonstrated by the experienced and competent teachers, in their lessons. The survey data are shown in Table 7.1.

Table 7.1 Approaches for facilitating development of metacognitive strategies in students

Approach/Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
I get my students to compare different ways of solving a problem.	4.3	44.0	41.8	9.9	51.7
I get my students to reflect on their learning (including helping them monitor their own learning).	8.1	45.9	39.7	6.3	46.0
I get my students to check for reasonableness of their answers after solving a problem.	3.4	27.5	52.9	16.2	69.1
I get my students to check on their understanding of a problem before they commence on solving a problem.	2.5	31.3	54.5	11.7	66.2

Note: $N = 677$

Based on the percentage of responses for "frequently" and "mostly/always", it is evident that participants of the survey also use three key approaches when facilitating the development of metacognitive strategies amongst students. They generally tend to get their students to check for reasonableness of their answers after solving a problem, check on their understanding of a problem prior to solving a problem and compare different ways of solving a problem. Two of these approaches correspond to those used by the experienced and competent teachers reported above. Getting students to check on their understanding of a problem prior to solving one appears to be a distinctive key approach adopted by the participants of the survey.

From the data illustrated in Figure 7.1 and Table 7.1, it is apparent that the general practices demonstrated by experienced and competent teachers when facilitating the development of metacognitive strategies amongst their students are also used by participants of the survey. It is interesting and encouraging to observe that a relatively higher proportion of participants of the survey claimed to use the same approaches when facilitating the development of metacognitive strategies amongst their students as compared to experienced and competent teachers.

What are the implications for mathematics teachers in Singapore secondary schools?

- Encouraging students to compare and contrast different approaches when solving a problem will help students better regulate their problem-solving processes, so that they can appreciate flexibility in problem solving and become more effective in getting themselves 'unstuck' during problem solving.
- Providing opportunities for students to reflect on their learning, including helping them to monitor their own learning, is crucial as it helps students to develop metacognition in the process of learning mathematics. Developing such metacognitive disposition will help students to be more aware of their knowledge and the possible learning gaps – crucial attributes of self-directed learners.
- Getting students to check for reasonableness of their answers after solving a problem helps students to direct themselves to their thinking processes where gaps or misconceptions could have occurred. This enables students to calibrate their thinking processes through self-regulation of their learning.

How are you facilitating the development of metacognitive strategies amongst your students in your mathematics lesson?



My notes

Contributors

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QUESTION

8

How do mathematics teachers in Singapore secondary schools imbue desired attitudes for the learning of mathematics amongst their students?



Question 8

How do mathematics teachers in Singapore secondary schools imbue desired attitudes for the learning of mathematics amongst their students?

What does it mean for students to develop desired attitudes for the learning of mathematics?

The study focused on four types of desired attitudes specified in the Singapore School Mathematics Curriculum Framework (Ministry of Education, 2012, p. 17): interest, appreciation, confidence and perseverance. We investigated whether Singapore mathematics teachers imbued these types of attitudes among their students and how they did it. A student has developed desired attitudes for the learning of mathematics if he/she shows interest and enjoyment in learning mathematics, appreciation of the beauty and power of mathematics, confidence in using mathematics and perseverance in solving mathematical problems.

How do experienced and competent mathematics teachers in Singapore secondary schools imbue desired attitudes for the learning of mathematics amongst their students?

The study of mathematics lessons enacted by 30 experienced and competent teachers in Singapore secondary schools revealed that they imbued desired attitudes through five approaches. As illustrated in Figure 8.1, the analysis of 211 lessons showed that 66.7% of them focused on building students' confidence, 50.0% of them encouraged their classes to persevere, and 36.7% of them helped their students appreciate the relevance of mathematics. Only six of them (20.0%) tried to make lessons interesting, of which four of them (13.3%) told non-mathematics-related jokes or stories, while three of them (10.0%) used mathematics-related resources (including a teacher who did both).

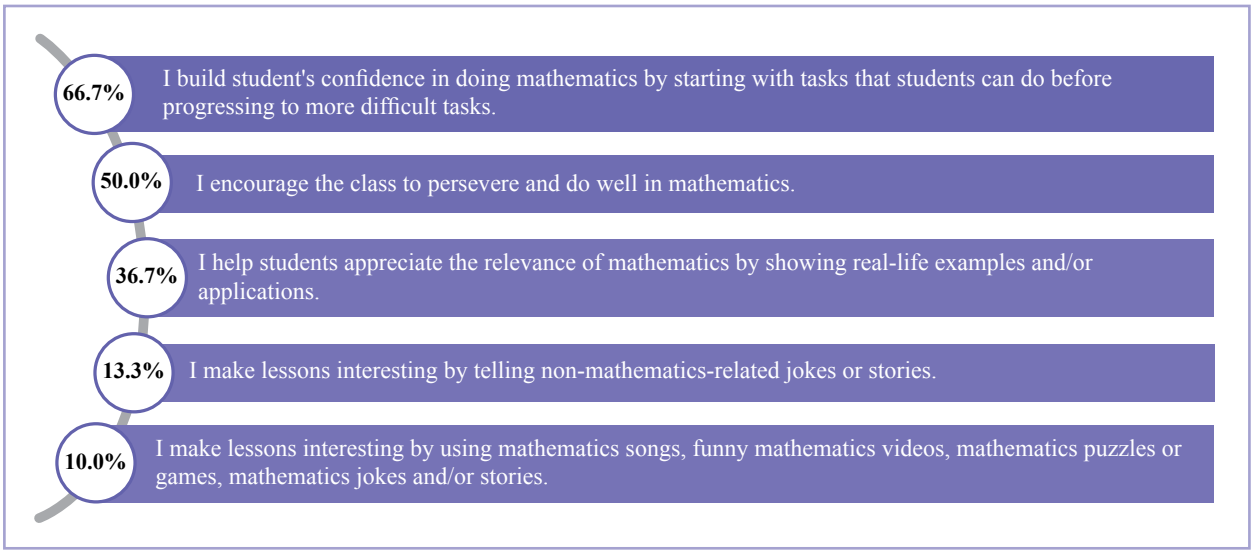
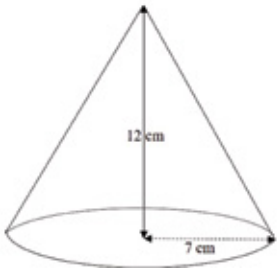
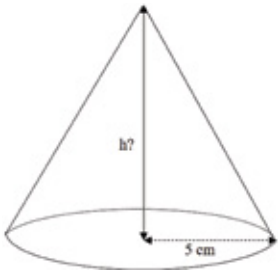
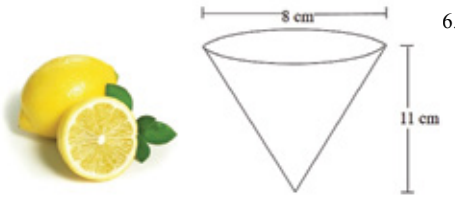


Figure 8.1 Approaches adopted by experienced and competent teachers when imbuing desired attitudes for the learning of mathematics amongst students

The following five episodes illustrate how five experienced and competent teachers imbued desired attitudes amongst their students.

Episode 8.1 shows how a teacher built Normal Technical students' confidence by starting with tasks that they could do before progressing to more difficult tasks.

Episode 8.1

Level of Difficulty	Sample Questions
Level 1	 1. A cone has a base radius of 7 cm and a height of 12cm. Calculate the volume of the cone. [to 3 s.f.] Basic question: Given base radius and height, use formula to find volume of cone.
Level 2	 4. The volume of a cone is 225cm^3. The base radius of the cone is 5 cm. Calculate h, the height of the cone. [to 3 s.f.] Reverse question: Given base radius and volume, use formula to find height of cone.
Level 3	 6. <Optional> you have 37 850 cm^3 of lemonade to sell. Each customer uses one paper cup shown in the diagram. How many cups do you need? Ans: 206 *[Hint: find the vol of a paper cup first]* Application question

Episode 8.2 shows how another teacher encouraged the class to persevere when solving a mathematics problem.

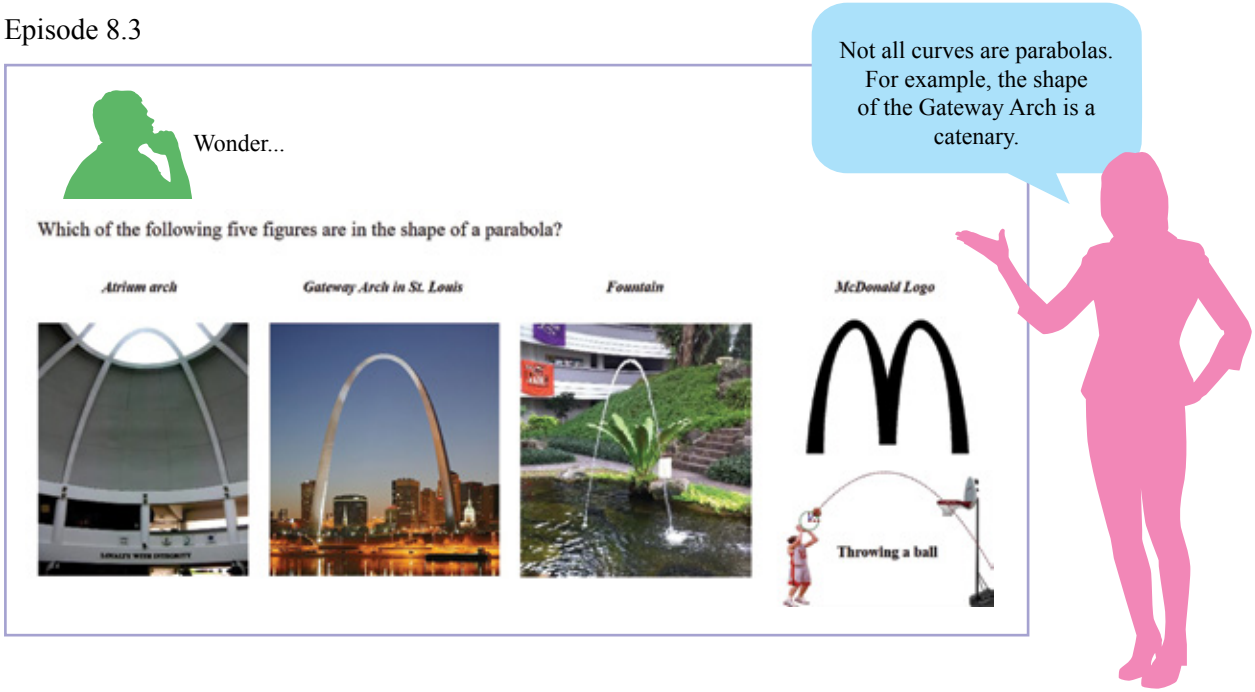
Episode 8.2



This is what I want you to do. Some of you may not be able to do a full question, but as you actually pluck out the information from the question, put it inside the diagram, and write down the information, and write down the name of the property. Ok, this question during O-level, is going to be worth 6 to 8 marks. Did I say everyone must score full marks? What is my aim? You must earn some marks. Nobody is going to give up on this question. Nobody is going to get zero.

Episode 8.3 shows how another teacher used real-life examples and applications to help Integrated Programme students appreciate the relevance of mathematics.

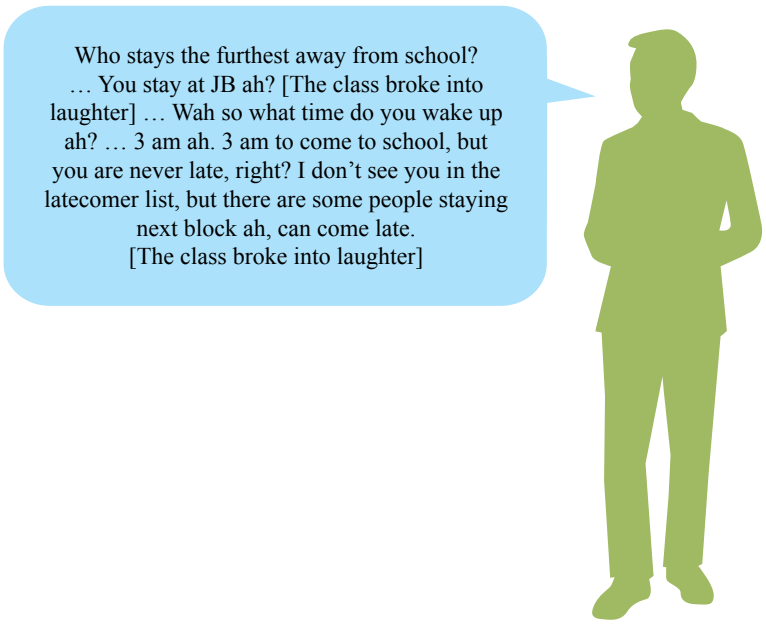
Episode 8.3



While trying to link mathematics to real life, the teacher was also concerned that her students (who were in the Integrated Programme) might just model any curve in the real world with a quadratic function, so she highlighted that some of these curves are not parabolas but catenaries (the equation of a catenary is a hyperbolic cosine function).

Episode 8.4 shows how a teacher made lessons interesting by telling non-mathematics related jokes or stories.

Episode 8.4



You may not find what the teacher said funny because situational jokes and slapstick humour, when narrated, are no longer hilarious. However, the two focus students for the lesson, who were interviewed separately after the lesson, said that this part (the teacher's jokes) was the high of the lesson.

Episode 8.5 shows how the teacher made lessons interesting by using a funny mathematics-related video.

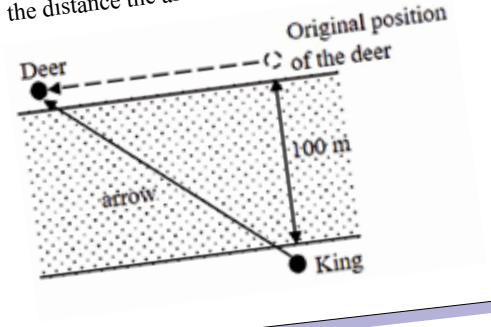
Episode 8.5

The teacher showed snippets of a Korean comedy (with English subtitles) about a girl called Dan Bi who somehow travelled back in time to ancient Korea and helped the king solve a mathematics problem using Pythagoras' theorem. The whole class found the drama hilarious because of the situational jokes and slapstick humour, but there was really nothing much about learning the application of the theorem. However, the teacher continued the storyline in the drama by giving her students three problems to solve in class using Pythagoras' theorem.

Task 1:

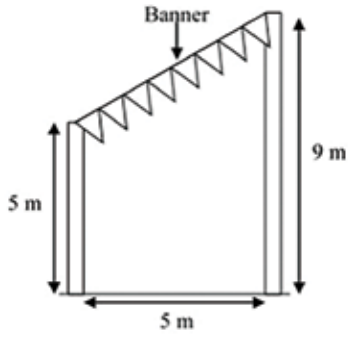
During one hunting trip, the king saw a deer on the opposite shore of a river that is 100 m wide. The deer ran along the shore at a speed of 20 m/s. After 8 seconds, the arrow hit the deer.
The king wanted his hunting trip to be recorded in history, hence he needed the following information. Dan Bi would need to help him to calculate

(a) the distance the deer travelled,
(b) the distance the arrow travelled.



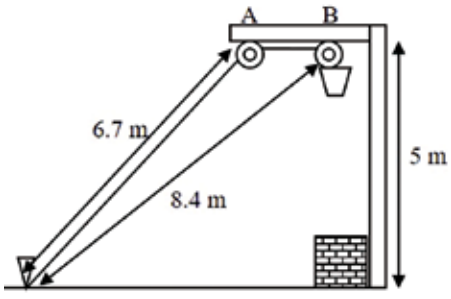
Task 2:

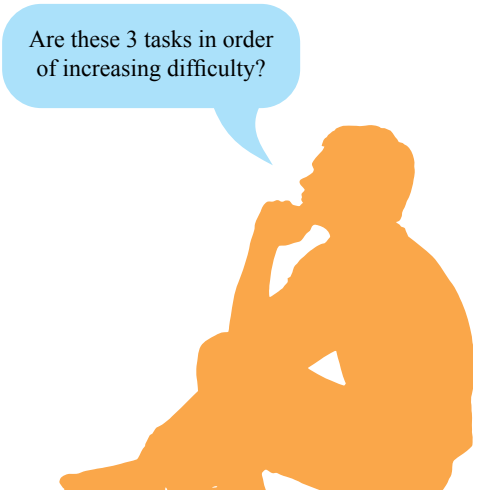
To celebrate the king's victory in hunting the deer, the palace was having a celebration. The guards were going to put up a decorative banner between two poles. One of the vertical poles was 5 m tall and the other was 9 m tall, and both of them were 5 m apart. Dan Bi was assigned to assist the guards with the amount of materials used to make the banner. Calculate the minimum length of the banner required.



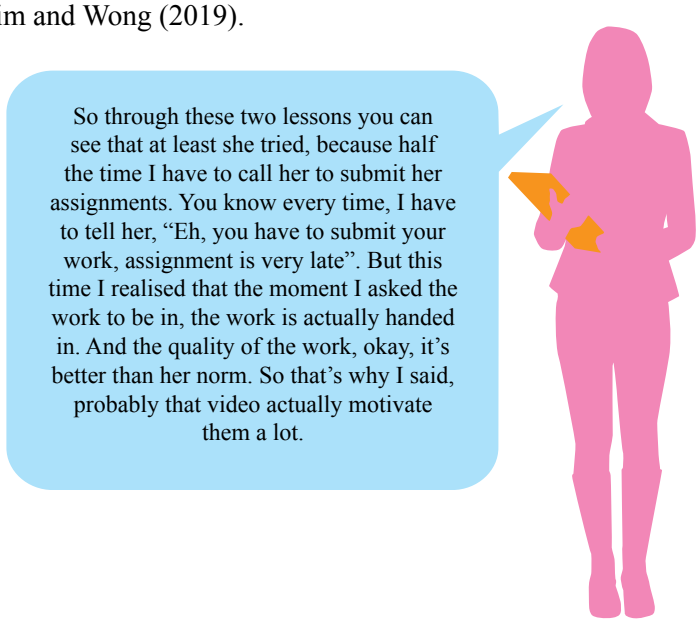
Task 3:

The cooks required water to cook up a feast for the celebration. Dan Bi came up with a plan to help them retrieve the water from the well by constructing a pulley system. The rope ran through two pulleys that were 5 m above the ground as shown in the diagram. Find the distance between the two pulleys.



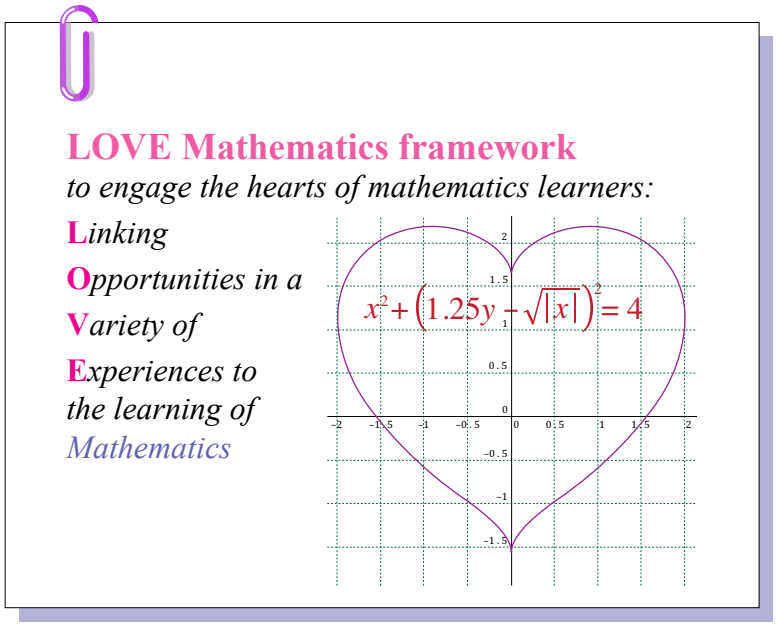


At the end of the lesson, the teacher gave the class homework, which consisted of a few textbook questions that were not as interesting as the three tasks given in class. However, a student, who seldom handed in homework on time, unexpectedly handed in this set of homework punctually the following day. It seems that the student was somehow motivated by the Korean drama and the solving of the three tasks in class to solve her routine homework questions promptly. These findings have been published in Yeo (2018) and Yeo, Choy, Lim and Wong (2019).



Yeo (2018) has proposed a theoretical framework on how to engage mathematics learners in their hearts.

- Variety: Use different resources to interest different students (since people have different tastes)
- Opportunity: Do it often enough, at least once in most lessons (as it is not possible to make every part of every lesson interesting)
- Linkage: Link the resources to the learning of mathematics (not just to make students laugh)



How do mathematics teachers in Singapore secondary schools imbue desired attitudes for the learning of mathematics amongst their students, in general?

In the survey, participants (teachers) were asked to reflect on their own practices and indicate the frequency of the five approaches demonstrated by the experienced and competent teachers, in their lessons. The survey data are shown in Table 8.1.

Table 8.1 Approaches for imbuing desired attitudes for the learning of mathematics amongst students

Approach/Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
I build students' confidence in doing mathematics by starting with tasks that students can do before progressing to more difficult tasks.	0.0	2.7	40.8	56.5	97.3
I encourage the class to persevere and do well in mathematics.	0.0	1.5	29.5	69.0	98.5
I help students appreciate the relevance of mathematics by showing real-life examples and/or applications.	0.6	29.4	50.5	19.5	70.0
I make lessons interesting by telling non-mathematics-related jokes or stories.	12.3	43.6	25.0	19.1	44.1
I make lessons interesting by using mathematics songs, funny mathematics videos, mathematics puzzles or games, mathematics jokes and/or stories.	14.3	51.6	24.4	9.7	34.1

Note: $N = 677$

Based on the percentage of responses for "frequently" and "mostly/always", it is evident that participants of the survey use three key approaches in imbuing desired attitudes for the learning of mathematics amongst their students. They generally tend to encourage the class to persevere, build students' confidence by starting with tasks that students can do before progressing to more difficult tasks, and help students appreciate the relevance of mathematics by showing real-life examples and/or applications. More than half of the survey participants (53.3%) also indicated making lessons interesting, of which 44.1% told non-mathematics-related jokes or stories, while 34.1% used mathematics-related resources. Interestingly, more teachers tried to make lessons interesting by telling non-mathematics-related jokes/stories than by using mathematics-related resources.

What are the implications for mathematics teachers in Singapore secondary schools?

- Many teachers in Singapore are already building students' confidence by starting with tasks that students can do before progressing to more difficult tasks, encouraging their classes to persevere, and helping students appreciate the relevance of mathematics by showing real-life examples and/or applications. So, teachers should continue to do so.
- Teachers should also consider making lessons interesting. Although telling non-mathematics-related jokes/stories may make students laugh, using interesting mathematics-related resources can also motivate students to learn mathematics at the same time.

How are you imbuing desired attitudes for the learning of mathematics in your lesson?



My notes

Contributors
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QUESTION

9

How do mathematics teachers in Singapore secondary schools use instructional materials to help students make connections?



Question 9

How do mathematics teachers in Singapore secondary schools use instructional materials to help students make connections?

What is "making connections using instructional materials"?

In the context of the Singapore school mathematics curriculum, "*connections* refer to the ability to see and make linkages among mathematical ideas …" (Ministry of Education, 2012, p.15). These connections may be facilitated using instructional materials. Instructional materials refer to materials which teachers bring to class and use during instruction.

How do mathematics teachers in Singapore secondary schools use instructional materials to help students make connections?

The 30 experienced and competent teachers who participated in the study help students make connections using their instructional materials in a number of ways. Some of the ways are as follows:

- Connect across different modes of representations
- Develop from earlier concepts
- Reinforce conceptual links established earlier (Leong, Cheng, Toh, Kaur, & Toh, 2019)

9.1 Connect across different modes of representations

Some equations such as $x^2 - x - 3 = 0$*, cannot be easily solved by the factorisation method because* _____.

An alternative way is to solve these equations graphically.

By drawing the graph of $y = ax^2 + bx + c$*, the roots /* _____ *of* $ax^2 + bx + c = 0$ *are given by the* _____ *of the graph.*

The roots of the equation obtained by the graphical method are often approximated.

3 cases are possible:

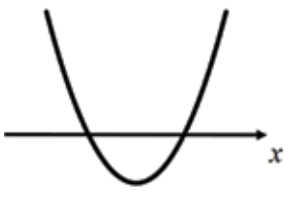
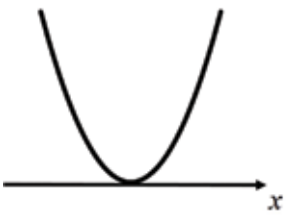
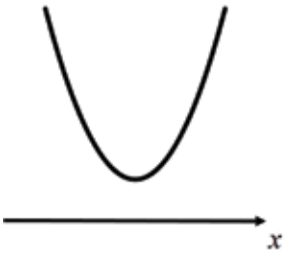
(i) Two real & distinct roots	(ii) Two real & equal roots	(iii) No real roots
		

Figure 9.1.1 An extract showing how an experienced and competent teacher connects across two modes of representations within the topic of Quadratic Equations

Figure 9.1.1 shows an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Quadratic Equations. From this extract, it can be seen that the teacher helps students make connections by using graphical representations to connect with algebraic representations of quadratic equations.

The extract from this teacher's instructional materials was used in the survey to find out if mathematics teachers in general connect across different modes of representations in their instructional materials. Participants of the survey were asked to indicate the frequency of their own practices. The data collated for this item are displayed in Table 9.1.1.

The sum of the percentages for "frequently" and "mostly/always" shows that 80.4% of the participants who responded make connections across different modes of representations in their instructional materials at least on a frequent basis.

Table 9.1.1 Connect across different modes of representations

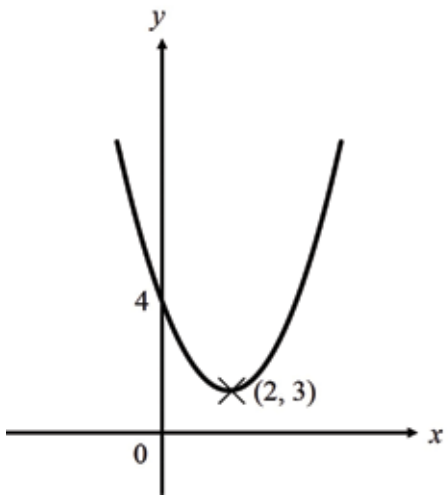
Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 9.1.1] connect different representations	0.0	19.6	51.9	28.5	80.4

To further illustrate this design principle, we present other extracts of instructional materials used by experienced and competent teachers who participated in the study.

Figure 9.1.2 is an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Quadratic Graphs and Inequalities. The task requires students to connect the graphical mode to the algebraic mode of representing quadratic equations.

Determine the equation of each of the following graphs.

(i)



(ii)

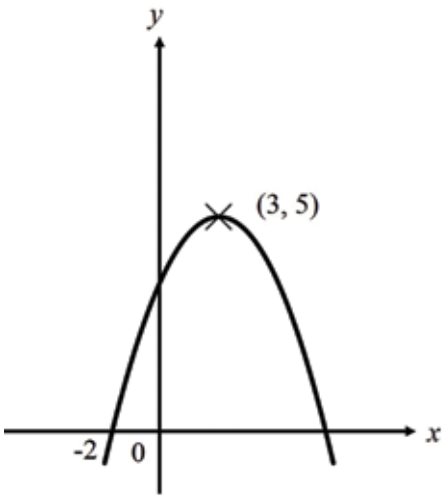
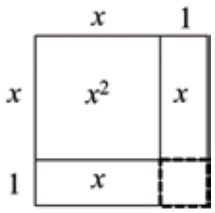


Figure 9.1.2 An extract showing how an experienced and competent teacher connects across two modes of representations within the topic of Quadratic Graphs and Inequalities.

Figure 9.1.3 is an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Quadratic Equations. This task requires students to connect the algebraic mode and the geometric mode of representing quadratic expressions of the form $x^2 + ax$.

Constructing New Knowledge

When we write $x^2 + bx + c$ in the form $(x - h)^2 - k^2$ where b, c, h and k are real numbers, we are **completing the square**. Study the examples shown and complete the table below.

Expression	Geometric Representation	Term to be Added	Algebraic Representation
$x^2 + 2x$		$1^2 = 1$	$x^2 + 2x$ $= x^2 + 2x + 1^2 - 1^2$ $= (x + 1)^2 - 1$
$x^2 + 4x$			
$x^2 + 6x$			
$x^2 + 7x$			

**** You can stop drawing the geometric representation once you have understood its connection to the algebraic representation.***

Figure 9.1.3 An extract showing how an experienced and competent teacher connects across two modes of representations within the topic of Quadratic Equations.

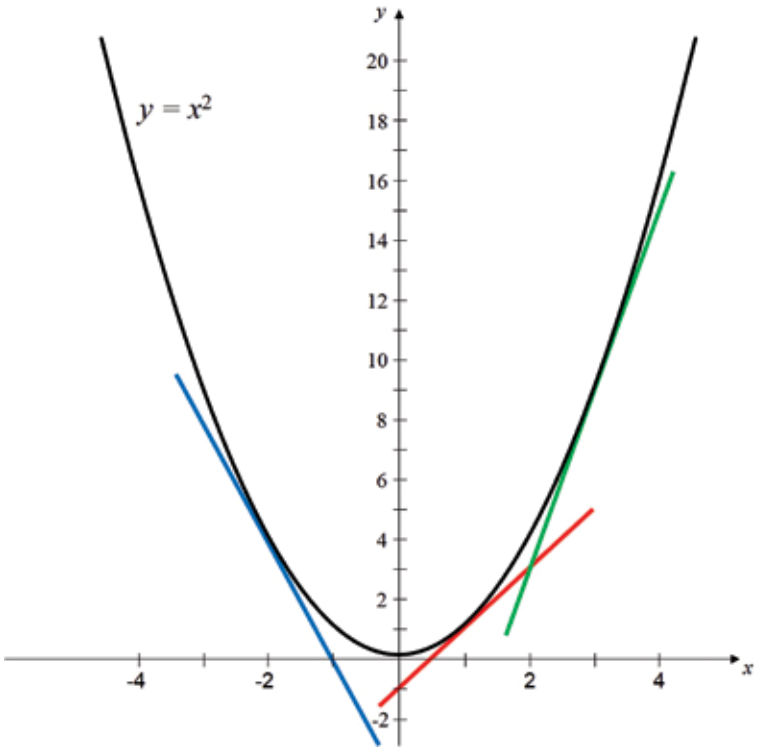
9.2 Develop from earlier concepts

Figure 9.2.1 shows an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Differentiation. From this extract, it can be seen that the teacher uses previously taught concept of calculating gradients of curves to link to differentiation.

Topic: **Derivative of x^n and simple rules of differentiation**

Objective: To find derivatives of x^n for any rational n.

1. Introduction



1) Gradient of the curve at point A = Gradient of tangent to curve at same point A.
2) Gradient of a curve is always changing.
   Using the graph of $y = x^2$,

x	**Gradient of tangent**

   Thus, for $y = x^2$, it has a gradient function _____.
3) The process of finding the gradient of a curve at any point is called ***differentiation***.

Figure 9.2.1 An extract showing how an experienced and competent teacher connects an earlier concept learnt (calculating gradients of curves) to find the derivative of x^n

The extract from this teacher's instructional materials was used in the survey to find out if mathematics teachers in general help their students make connections by developing from previous concepts. Participants of the survey were asked to indicate the frequency of their own practices. The data collated are displayed in Table 9.2.1.

The sum of the percentages for "frequently" and "mostly/always" shows that 87.1% of the participants who responded help their students make connections by developing from previous concepts in their instructional materials at least on a frequent basis.

Table 9.2.1 Develop from earlier concepts

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 9.2.1] draw on previously taught concept/s to develop new concepts	2.6	10.3	44.2	42.9	87.1

To further illustrate this design principle, we present other extracts of instructional materials used by experienced and competent teachers who participated in the study.

Figure 9.2.2 is an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Quadratic Graphs and Inequalities. The earlier concept taught was sketching of quadratic graphs. Here, the teacher connects this to the new concept involved in solving quadratic inequalities.

Example 3
Solve the inequality $2x^2 - 7x + 6 < 0$.

Solution:
Step 1: To use a graphical approach, we first obtain a sketch of the graph – note that for quadratic inequalities, we only look at the **shape** and the ***x*-intercepts**, so the factorised form of the quadratic function is the most useful one for this purpose.

Step 2: Since the graph we have sketched is $\boldsymbol{y} = 2x^2 - 7x + 6$, and the inequality we want to solve is $2x^2 - 7x + 6 < \mathbf{0}$, we are looking for the section of the sketch where the y-values are smaller than 0, i.e. below the x-axis.

Step 3: Having identified the portion of the sketch which represents the solution, write down the solution in the proper form.

Figure 9.2.2 An extract showing how an experienced and competent teacher connects an earlier concept learnt (sketching of quadratic graphs) to solve quadratic inequalities

Figure 9.2.3 is an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 4/5 (Year 10/11) Vectors. An earlier concept taught was the application of Pythagoras' Theorem to calculate the length of a line segment. Here, the teacher connects this to the new concept involved in calculating the magnitude of a vector.

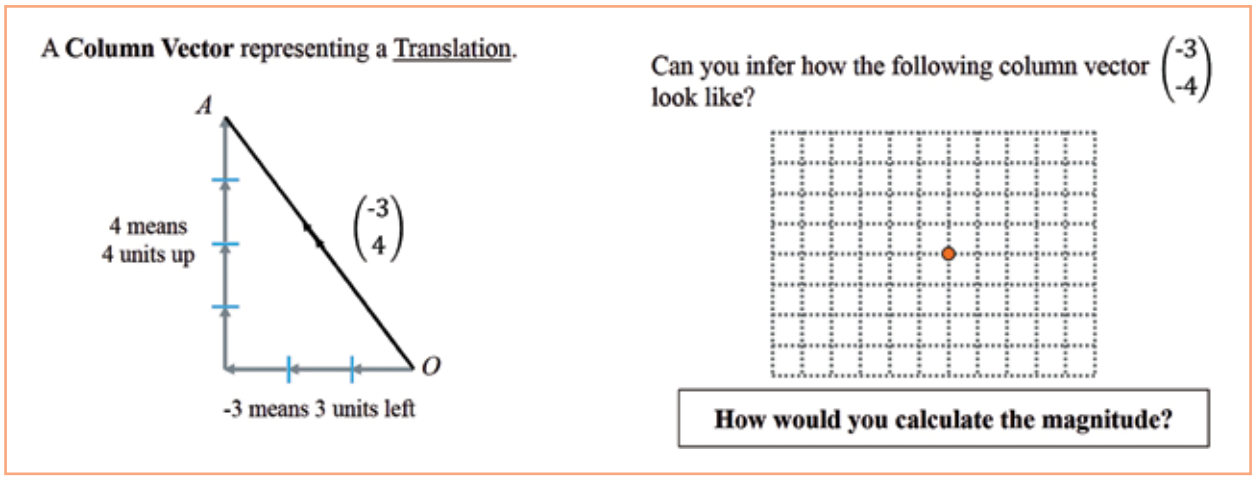


Figure 9.2.3 An extract showing how an experienced and competent teacher connects an earlier concept learnt in Trigonometry (Pythagoras Theorem) to find the magnitude of a vector in Geometry

9.3 Reinforce conceptual links established earlier

This refers to links made earlier by the teacher and now reinforced. The mathematical task in Figure 9.3.1 is an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Differentiation. From this extract, it can be seen that the teacher reinforces the conceptual link between gradient and differentiation by inserting a mathematical task in the instructional materials.

Calculate the gradient of the curve at the given point:

$$y = \sqrt{x}(2 - x), x = 9$$

Figure 9.3.1 An extract showing how an experienced and competent teacher reinforces conceptual link established earlier within the topic of Differentiation

This extract was used in the survey to find out if mathematics teachers in general help their students make connections by reinforcing conceptual links established earlier in their instructional materials. Participants of the survey were asked to indicate the frequency of their own practices. The data collated are displayed in Table 9.3.1.

The sum of the percentages for "frequently" and "mostly/always" shows that 92.9% of the teachers who responded make connections by reinforcing conceptual links in their instructional materials at least on a frequent basis.

Table 9.3.1 Reinforce conceptual links established earlier

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 9.3.1] reinforce conceptual links established earlier	0.0	7.1	45.5	47.4	92.9

To further illustrate this design principle, we present other extracts of instructional materials used by experienced and competent teachers who participated in the study.

Figure 9.3.2 is an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Quadratic Equations. The teacher earlier made links between different methods of solving quadratic equations. Here, the teacher uses this task to reinforce the conceptual links between two methods.

> Given two equations $(x + 1)^2 = 4$ and $x^2 + 2x - 3 = 0$, how are they related?
>
> Which equation is easier to solve and why?

Figure 9.3.2 An extract showing how an experienced and competent teacher reinforces conceptual link established earlier within the topic of Quadratic Equations

Figure 9.3.3 is an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 4/5 (Year 10/11) Vectors. The teacher earlier made implicit links between vectors and its negative counterparts. Here, the teacher uses this task to reinforce and formalise this link.

Refer to the diagram.

Write down the column vector for:

(a) $\overrightarrow{KL}$

(b) $\overrightarrow{MN}$

(c) $\overrightarrow{PQ}$

(d) $\overrightarrow{RS}$

(e) $\overrightarrow{QQ}$

How are the vectors above related?

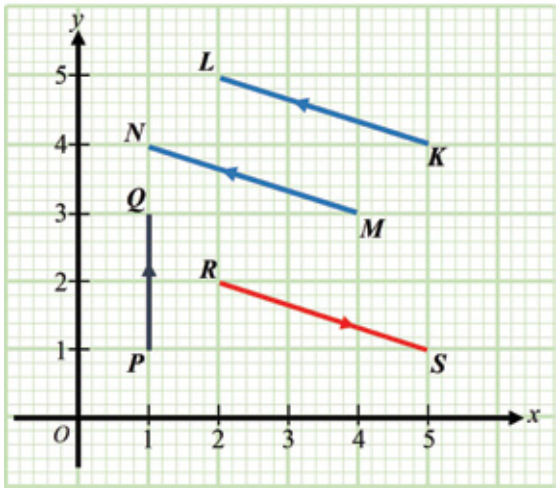


Figure 9.3.3 An extract showing how an experienced and competent teacher reinforces conceptual link established earlier within the topic of Vectors

What are the implications for mathematics teachers in Singapore secondary schools?

In designing instructional materials, teachers may consider good practices such as

- crafting mathematical tasks that provide students with opportunities to realise that mathematical concepts can be connected across different modes of representations;
- careful development of ideas in a connected manner rather than presenting each as an isolated one; and
- reinforcement of concepts that are developed as well as their links to past concepts.

So how do you make connections using your instructional materials?



My notes

Contributors

Leong Yew Hoong • Cheng Lu Pien • Toh Wei Yeng Karen

References

Leong, Y. H., Cheng, L. P., Toh, W. Y., Kaur, B., & Toh, T. L. (2019). Making things explicit using instructional materials: A case study of a Singapore teacher's practice. *Mathematics Education Research Journal, 31*(1), 47-66.

Ministry of Education, Singapore. (2012). *Ordinary-level mathematics teaching and learning syllabus*. Singapore: Author.

QUESTION

10

How do mathematics teachers in Singapore secondary schools use instructional materials to stimulate students' thinking?



Question 10

How do mathematics teachers in Singapore secondary schools use instructional materials to stimulate students' thinking?

What is an activity that "stimulates students' thinking"?

It refers to an activity that helps students go beyond the conventional mode of procedural operation so that they experience a challenge in their mental activity.

How do mathematics teachers in Singapore secondary schools use instructional materials to stimulate students' thinking?

The 30 experienced and competent teachers who participated in the study stimulate students' thinking using their instructional materials in a number of ways. Some of the ways are as follows:

- Include mathematical items that are deemed challenging to students
- Provide opportunities for students to make comparisons across methods

10.1 Include mathematical items that are deemed challenging to students

Figure 10.1.1 shows an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Differentiation. From this extract, it can be seen that the teacher stimulates students' thinking with a mathematical item that is challenging.

Challenging Problem

Given that $f(x) = \sqrt{1+\sqrt{x}}$ where $x \geq 0$, show that $f'(x) = \frac{1}{4\sqrt{x+x\sqrt{x}}}$.

Figure 10.1.1 An extract showing a challenging item within the topic of Differentiation

The extract from this teacher's instructional materials was used in the survey to find out if mathematics teachers in general stimulate students' thinking by including challenging items in their instructional materials. Participants of the survey were asked to indicate the frequency of their own practices. The data collated for this item are displayed in Table 10.1.1.

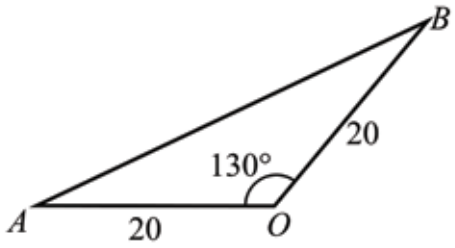
The sum of the percentages for "frequently" and "mostly/always" shows that 74.4% of the participants who responded stimulate students' thinking by including challenging items in their instructional materials at least on a frequent basis.

Table 10.1.1 Include mathematical items that are deemed challenging to students

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 10.1.1] include challenging tasks	0.0	25.6	51.9	22.5	74.4

Figure 10.1.2 is an extract from another experienced and competent teacher's instructional materials. The topic was Sec 3 (Year 9) Trigonometry Ratios. From this extract, it can be seen that the teacher stimulates students' thinking with a mathematical item that is challenging.

In the diagram, OA and OB represent two positions of the minute hand of a clock, $OA = OB = 20$ cm and $\angle AOB = 130°$.
Find
(a) the distance AB,
(b) the perpendicular distance from O to AB.



[*At this point, students were not yet taught cosine rule.*]

Figure 10.1.2 An extract showing a challenging item within the topic of Trigonometry Ratios

The extract from this teacher's instructional materials was used in the survey to find out if mathematics teachers in general stimulate students' thinking by including challenging items in their instructional materials. Participants of the survey were asked to indicate the frequency of their own practices. The data collated for this item are displayed in Table 10.1.2.

The sum of the percentages for "frequently" and "mostly/always" show that 48.7% of the teachers who responded stimulate students' thinking by including challenging items in their instructional materials at least on a frequent basis.

Table 10.1.2 Include mathematical items that are deemed challenging to students

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 10.1.2] include challenging tasks	7.4	43.9	38.7	10.0	48.7

Figure 10.1.3 is an extract from a third experienced and competent teacher's instructional materials. The topic was Sec 3 (Year 9) Quadratic Equations. From this extract, it can be seen that the teacher stimulates students' thinking with a mathematical item that is challenging.

Given that the roots of the equation $ax^2 + bx = 6$ are $\frac{2}{3}$ and $-\frac{3}{7}$, find the values of a and b.

Figure 10.1.3 An extract showing a challenging item within the topic of Quadratic Equations

The extract from this teacher's instructional materials was used in the survey to find out if mathematics teachers in general stimulate students' thinking by including challenging items in their instructional materials. Participants of the survey were asked to indicate the frequency of their own practices. The data collated for this item are displayed in Table 10.1.3.

The sum of the percentages for "frequently" and "mostly/always" show that 50.5% of the teachers who responded stimulate students' thinking by including challenging items in their instructional materials at least on a frequent basis.

Table 10.1.3 Include mathematical items that are deemed challenging to students

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 10.1.3] include challenging tasks	3.4	46.1	37.8	12.7	50.5

10.2 Provide opportunities for students to make comparisons across solution methods

This refers to providing opportunities for students to compare different methods used to solve mathematical items. Figure 10.2.1 is an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Quadratic Equations. From this extract, it can be seen that the teacher provides students with an opportunity to compare the different methods used to solve quadratic equations.

Task:
Examine each of the following questions. Discuss with your partner and decide which method ***you prefer*** to solve each of the following questions. Justify your choices.

No.	Question	Your Preferred Method	Reason(s) for Your Choice
1	Find the roots of the equation $2x^2 - 5 = 9x$.		
2	Solve $2x^2 + 9x - 15 = 0$, giving your answers correct to 2 decimal places where necessary.		
3	Solve the equation $2(x - 5)^2 = 100$.		

Figure 10.2.1 An extract showing an item that requires students to compare methods within the topic of Quadratic Equations

Participants of the survey were asked to indicate the frequency of their own practices based on this same extract. The data collated for this item are displayed in Table 10.2.1.

Table 10.2.1 Provide opportunities for students to make comparisons across solution methods

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 10.2.1] include tasks for students to compare solution methods	13.8	43.6	33.3	9.3	42.6

The sum of the percentages for "frequently" and "mostly/always" show that 42.6% of the teachers who responded stimulate students' thinking by inserting items that require comparison across methods in their instructional materials at least on a frequent basis.

To further illustrate this design principle, we present another extract of instructional material used by an experienced and competent teacher who participated in the study.

Figure 10.2.2 is an extract from the teacher's instructional materials. The topic was Sec 3 (Year 9) Quadratic Graphs and Inequalities. It can be seen that students are required to compare the algebraic solution for the inequality with the graphical solution they obtained earlier.

Example 10
Solve the inequality $2x^2 - 7x + 6 < 0$ using an algebraic method.

Solution:
This is the same problem as that of Example 3.
Compare and contrast the graphical method and the algebraic method.

[*For the reader's reference, "Example 3" is "Solve the inequality* $2x^2 - 7x + 6 < 0$ *(using a graphical method)".*]

Figure 10.2.2 An extract showing an item that requires students to compare methods within the topic of Quadratic Graphs and Inequalities

What are the implications for mathematics teachers in Singapore secondary schools?

In designing instructional materials, teachers may consider good practices such as including mathematical tasks

- that are challenging at suitable junctures so that students are given opportunities to grapple with mathematical conceptual knowledge that is beyond mere mechanical applications; and
- for students to compare different/alternative methods so as to encourage reasoning.

So how do you stimulate students thinking using your instructional materials?



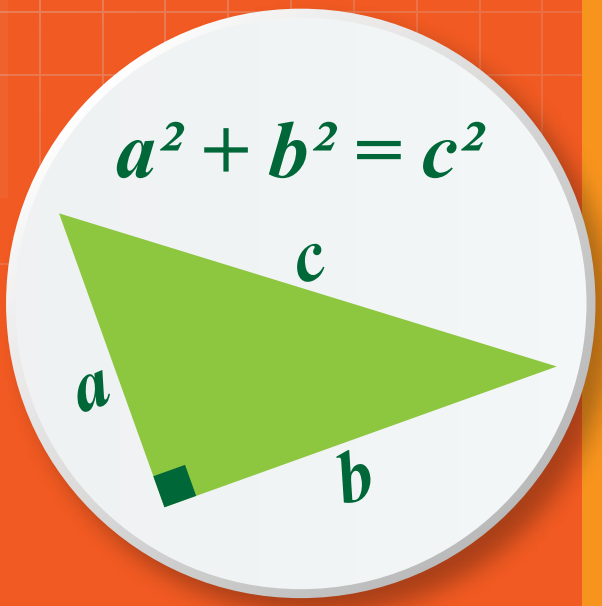
My notes

Contributors
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QUESTION

11

How do mathematics teachers in Singapore secondary schools use instructional materials to help students develop and apply formula?



Question 11

How do mathematics teachers in Singapore secondary schools use instructional materials to help students develop and apply formula?

What is a mathematical "formula"?

The term "formula" is not formally defined within the academic discipline of mathematics. The popular image of formula involves an application of an algebraic identity. We define it more broadly as steps to follow based on a prescribed procedure (Leong, Cheng, Toh, Kaur, & Toh, Accepted).

How do mathematics teachers in Singapore secondary schools use instructional materials to help students develop and apply formula?

The 30 experienced and competent teachers who participated in the study help students gain fluency in applying formula using their instructional materials in a number of ways. Some of the ways are as follows:

- Begin with easier items
- Use a variety of examples
- Develop from previous methods or ideas

11.1 Begin with easier items

Figure 11.1.1 shows an extract from one of the 30 experienced and competent teachers' instructional materials. The topic was Sec 3 (Year 9) Differentiation. From this extract, it can be seen that the teacher helps students gain fluency in applying formula by beginning with easier items. Notice x^3 is already of the form x^n and thus easiest to recognise the formula in order to apply. This is followed by other items that require rewriting into the x^n form or using a different representation such as graphical.

Formula 1: $\frac{d}{dx}[x^n] = nx^{n-1}$ where n is a constant.

Example 14.1.1

(a) $y = x^3$ ➔ $\frac{dy}{dx} =$

(b) $y = 5$ ➔ $\frac{dy}{dx} =$

(c) $y = \frac{1}{x} =$ ➔ $\frac{dy}{dx} =$

(d) $y = \sqrt{x}$ ➔ $\frac{dy}{dx} =$

Figure 11.1.1 An extract showing how an experienced and competent teacher begins with easier items within the topic of Differentiation

The extract from this teacher's instructional materials was used in the survey to find out if mathematics teachers in general help students gain procedural fluency by beginning with easier items in their instructional materials. Participants of the survey were asked to indicate the frequency of their own practices. The data collated for this item are displayed in Table 11.1.1.

The sum of the percentages for "frequently" and "mostly/always" shows that 96.8% of the teachers who responded help students gain procedural fluency by beginning with easier items in their instructional materials at least on a frequent basis.

Table 11.1.1 Begin with easier items

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 11.1.1] begin with easier items to build confidence	0.0	3.2	31.4	65.4	96.8

Figure 11.1.2 is an extract from another experienced and competent teacher's instructional materials. The topic was Sec 3 (Year 9) Trigonometry Ratios. From this extract, it can be seen that the teacher helps students gain fluency in applying formula by beginning with easier items.

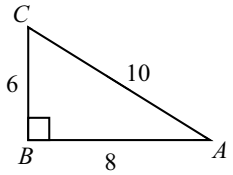
An easier way to remember

$\tan A = \frac{O}{A}$ **TOA** $\cos A = \frac{A}{H}$ **CAH** $\sin A = \frac{O}{H}$ **SOH**

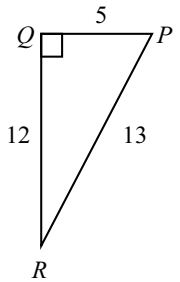
(*big leg aunty*)

Test Yourself!

1. For the right-angled triangle shown, find
   (a) sin A
   (b) cos A
   (c) tan A



2. For the right-angled triangle shown, find
   (a) sin P
   (b) cos P
   (c) tan R



3. For the right-angled triangle shown, find
   (a) sin K
   (b) cos J
   (c) tan K

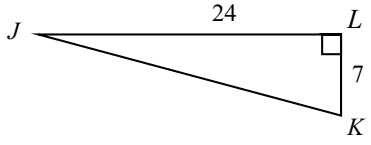


Figure 11.1.2 An extract showing how an experienced and competent teacher begins with easier items within the topic of Trigonometry Ratios

The extract from this teacher's instructional materials was also used in the survey to find out if mathematics teachers in general help students gain procedural fluency by beginning with easier items in their instructional materials. Participants of the survey were asked to indicate the frequency of their own practices. The data collated for this item are displayed in Table 11.1.2.

The sum of the percentages for "frequently" and "mostly/always" show that 96.1% of the teachers who responded help students gain procedural fluency by beginning with easier items in their instructional materials at least on a frequent basis.

Table 11.1.2 Begin with easier items

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 11.1.2] begin with easier items to build confidence	0.0	3.9	27.8	68.3	96.1

To further illustrate this design principle, we present another extract of instructional material used by an experienced and competent teacher who participated in the study.

Figure 11.1.3 is an extract from the teacher's instructional materials. The topic was Sec 1 (Year 7) Linear Equations. In this extract, it can be seen that the first item requires the least amount of manipulation, and then the degree of complication increases gradually with every item.

Worksheet on Simple Linear Equations in One Variable

(1) Solve the following equations.
(2) Check that the solution obtained is correct.

(a) $13x - 22 = 30$

(b) $2y = 6y + 11$

(c) $7x - 1 = 4x + 10$

(d) $-3 - w = -2w - 3$

(e) $4q + 3q - 5 = 1 - 3q + 3$

Figure 11.1.3 An extract showing how an experienced and competent teacher begins with easier items within the topic of Linear Equations

11.2 Use a variety of examples

This refers to the selection of items for students to gain procedural fluency. From the same extract, as in Figure 11.1.1, it could be also be observed that the experienced and competent teacher used a variety of examples in the exercise. Note that the exponents vary from positive integer to zero to negative number, and then to positive rational. Participants of the survey were asked to indicate the frequency of their similar practice based on this same extract. The data collated for this item are displayed in Table 11.2.1.

The sum of the percentages for "frequently" and "mostly/always" show that 96.8% of the participants who responded help students gain procedural fluency by using a variety of examples in their instructional materials at least on a frequent basis.

Table 11.2.1 Use a variety of examples

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 11.1.1] include a variety of examples to help students recognise cases of a given form	0.0	3.2	38.5	58.3	96.8

Similarly, the extract in Figure 11.1.2 was used in the survey to find out if mathematics teachers in general help their students gain procedural fluency by using a variety of examples in their instructional materials. There is variation in the orientation of triangles and also the reference angles. Participants of the survey were asked to indicate the frequency of their own practices. The data collated are displayed in Table 11.2.2.

The sum of the percentages for "frequently" and "mostly/always" show that 94.8% of the participants who responded help students gain procedural fluency by using a variety of examples in their instructional materials at least on a frequent basis.

Table 11.2.2 Use a variety of examples

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 11.1.2] include a variety of examples to help students recognise cases of a given form	0.4	4.8	37.0	57.8	94.8

A third extract from an experienced and competent teacher's instructional materials, as shown in Figure 11.2.1, was also used in the survey. The topic was Sec 3 (Year 9) Quadratic Equations. From this extract, it can be seen that this teacher helps students gain fluency in applying formula by using a variety of examples. Note that the set of quadratic equations produce solutions which illustrate all the cases in the nature of roots: two distinct real roots; two equal roots; and no real roots.

Self-Check:

Apply the quadratic formula to solve each of the following equations.

$2x^2 + 6x - 19 = 0$	$6x^2 - 3x - 7 = 0$
$6x^2 - 3x + 7 = 0$	$9x^2 - 24x + 16 = 0$

Write down your observations:

Figure 11.2.1 An extract showing how an experienced and competent teacher uses a variety of examples within the topic of Quadratic Equations

Participants of the survey were asked to indicate the frequency of their own practices. The data collated are displayed in Table 11.2.3.

The sum of percentages for "frequently" and "mostly / always" shows that 88.7% of the participants who responded help their students gain procedural fluency by using a variety of examples in their instructional materials at least on a frequent basis.

Table 11.2.3 Use a variety of examples

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 11.2.1] include a variety of examples to help students recognise cases of a given form	0.3	11.0	54.3	34.4	88.7

11.3 Develop from previous methods or ideas

Figure 11.3.1 is extracted from one of the 30 experienced and competent teachers' notes. The topic was Sec 3 (Year 9) Quadratic Equations. From this extract, it can be seen that the teacher uses this task in the instructional materials to develop the general formula from the previous method—Completing the Square.

The roots of the general quadratic equation $ax^2 + bx + c = 0$ can be obtained by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Let's try to derive this formula by applying the completing the square method.

Figure 11.3.1 An extract showing how an experienced and competent teacher develops a formula from previous method within the topic of Quadratic Equations

This extract was used in the survey to find out if mathematics teachers in general help students develop formulas from previous methods or ideas in their instructional materials. Participants of the survey were asked to indicate the frequency of their own practices. The data collated are displayed in Table 11.3.1.

The sum of percentages for "frequently" and "mostly/always" shows that 78.0% of the teachers who responded help students develop formulas from previous methods or ideas in their instructional materials at least on a frequent basis.

Table 11.3.1 Develop from previous methods or ideas

Survey Item	Percentage of Participants				
	Never/ Rarely	Sometimes	Frequently	Mostly/ Always	Frequently & Mostly/ Always
Your instructional materials like the extract [Figure 11.3.1] develop from previous methods or ideas	2.4	19.6	54.3	23.7	78.0

To further illustrate this design principle, we present another extract of instructional materials used by an experienced and competent teacher who participated in the study.

Figure 11.3.2 is an extract from the teacher's instructional materials. The topic was Sec 3 (Year 9) Trigonometry. From this extract, it can be seen that the teacher helps students develop another formula for the area of a triangle apart from the usual "$\frac{1}{2} \times base \times height$" formula.

The Area of a Triangle

We have learnt that the area of a triangle is given by the formula:

$$area\ of\ triangle = \frac{1}{2} \times base \times height$$

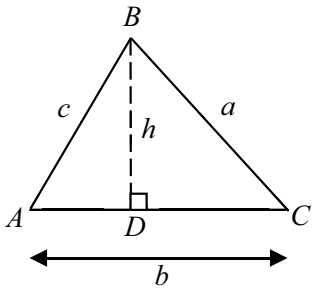
There is a need to find another formula for the area of a triangle as the height of the triangle may not be given all the time.

The diagram shows an acute-angled ΔABC.

$$area\ of\ \Delta ABC = \frac{1}{2}bh \text{ -------- (1)}$$

In the right-angled ΔBCD, $\sin C = \frac{h}{a} \Rightarrow h = a \sin C$ -------- (2)

Substitute (2) into (1), $area\ of\ \Delta ABC = \frac{1}{2}ab \sin C$



[*The proceeding section of the instructional materials is similar to the above for the case of obtuse Angle C*]

In general, Area of a $\Delta = \frac{1}{2}$ product of two sides × sine of the **included** angle.

Figure 11.3.2 An extract showing how an experienced and competent teacher develops a formula from a previous method within the topic of Trigonometry

What are the implications for mathematics teachers in Singapore secondary schools?

In designing instructional materials, teachers may consider good practices such as

- helping students acquire procedural fluency in applying formulas by arranging mathematical items in gradually-increasing levels of difficulty;
- selecting a variety of examples and items judiciously to help students gain familiarity with the different ways of applying the formula in an appropriate range of tasks; and
- developing formulas by building on previously-learnt methods or ideas to strengthen the links in students' learning.

So how do you help students develop and apply formula using your instructional materials?



My notes

Contributors

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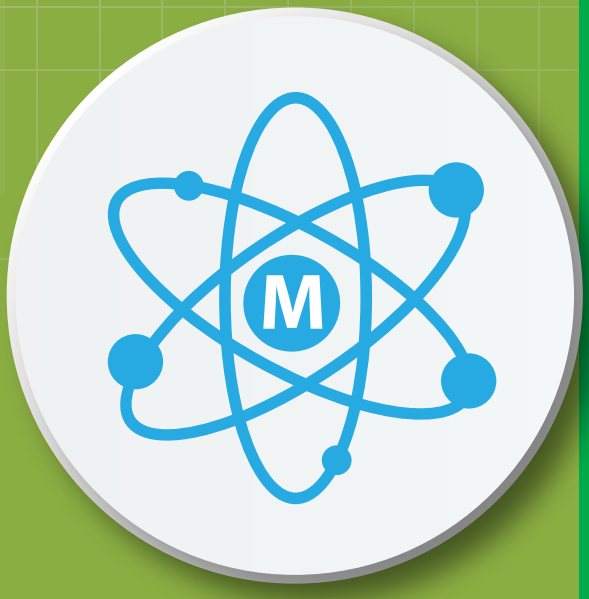
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QUESTION

12

How mathematically powerful are the classrooms in Singapore secondary schools?



Question 12

How mathematically powerful are the classrooms in Singapore secondary schools?

What is a mathematically powerful classroom?

According to Schoenfeld's (2014) Teaching for Robust Understanding of Mathematics (TRU Math) framework, the five dimensions of mathematically powerful classrooms are:

(1) The Mathematics
(2) Cognitive Demand
(3) Access to Mathematical Content
(4) Agency, Authority and Identity
(5) Uses of Assessment

Figure 12.1 provides a general top-level description of the TRU Math framework and Figure 12.2 shows the summary rubric that describes the kinds of classroom activities that will score high (level 3) or low (level 1) along each of the dimensions.

The Five Dimensions of Mathematically Powerful Classrooms	
The Mathematics	The extent to which the mathematics discussed is focused and coherent and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained.
Cognitive Demand	The extent to which classroom interactions create and maintain an environment of productive intellectual challenge conducive to students' mathematical development.
Access to Mathematical Content	The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed by the class.
Agency, Authority and Identity	The extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to their development of agency and authority resulting in positive identities as doers of mathematics.
Uses of Assessment	The extent to which the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings.

Figure 12.1 The five dimensions of mathematically powerful classrooms (Schoenfeld, 2014)

LEVEL	The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Authority and Identity	Uses of Assessment
	How accurate coherent, and well justified is the mathematical content?	*To what extent are students supported in grappling with and making sense of mathematical concepts?*	*To what extent does the teacher support access to the content of the lesson for all students?*	*To what extent are students the source of ideas and discussion of them? How are student contributions framed?*	*To what extent is students' mathematical thinking surfaced; to what extent does instruction build on student ideas when potentially valuable or address misunderstandings when they arise?*
1	Classroom activities are unfocused or skills-oriented, lacking opportunities for engagement in key practices such as reasoning and problem solving.	Classroom activities are structured so that students mostly apply memorized procedures and/ or work routine exercises.	There is differential access to or participation in the mathematical content, and no apparent efforts to address this issue.	The teacher initiates conversations. Students' speech turns are short (one sentence or less), and constrained by what the teach says or does.	Student reasoning is not actively surfaced or pursued. Teacher actions are limited to corrective feedback or encouragement.
2	Activities are primarily skills-oriented, with cursory connections between procedures, concepts and contexts (where appropriate) and minimals attention to key practices.	Classroom activities offer possibilities of conceptual richness or problem solving challenge, but teaching interactions tend to "scaffold away" the challenges, removing opportunities for productive struggle.	There is uneven access or participation but the teacher makes some efforts to provide mathematical access to wide range of students.	Students have a chance to explain some of their thinking, but "the student proposes, the teacher disposes": in class discussions, student ideas are not explored or built upon.	The teacher refers to student thinking, perhaps even to common mistakes, but specific students' ideas are not built on (when potentially valuable) or used to address challenges (when problematic).
3	Classroom activities support meaningful connections between procedures, concepts and contexts (where appropriate) and provide opportunities for engagement in key practices.	The teacher's hints or scaffolds support students in productive struggle in building understandings and engaging in mathematical practices.	The teacher actively supports and to some degree achieves broad and meaningful mathematical participation; **OR** what appear to be established participation structures result in such engagement.	Students explain their ideas and reasoning, The teacher may ascribe ownership for students' ideas in exposition, **AND/ OR** students respond to and build on each other's ideas.	The teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings.

Figure 12.2 The Teaching for Robust Understanding of Mathematics summary rubric (Schoenfeld, 2014)

How mathematically powerful are the classrooms in Singapore secondary schools?

To answer this question, we map the pedagogy of the teachers in Singapore mathematics classrooms onto that of mathematically powerfully classrooms advocated by Schoenfeld, based on the findings presented in the preceding questions in this book.

Dimension 1 – The Mathematics

This is the quality of the mathematics that gets discussed. According to Schoenfeld, if the mathematics is impoverished or if the mathematics is learnt procedurally, students cannot grow to be powerful thinkers and sensemakers. It is not just the mathematics and what the teacher presents, but what sense the students make of the mathematics.

From Question 4, we see that the three key approaches used by the 30 experienced and competent teachers in engaging students in learning mathematics are: (1) explaining the concept/formula/property to the whole class while asking questions along the way; (2) guiding the whole class to discover the concept/formula/property; and (3) asking students to do a task to discover the concept/formula/property. However, data from the survey reveals that, while participants (teachers) of the survey also draw on the first two approaches, there appears preference for direct dissemination of knowledge over "discovering" knowledge in contrast with practices of experienced and competent teachers.

From Question 6, we see that an approach that teachers adopt to develop students' mathematical reasoning and communication skills is helping students formulate mathematical relationships between quantities in a logical manner. However, video-recorded data of the 30 experienced and competent teachers shows only a quarter of the teachers while survey data shows slightly less than half of the participants (teachers) help students draw conjectures about mathematical relationships – an approach that requires students to make preliminary mathematical observations culminating to informal deductive statements or conjectures before launching into formal proofs. This is an interesting finding that perhaps needs further investigation.

From Question 9, we see that teachers also help students to make connections among mathematical ideas in their instructional materials in a number of ways, including: (1) connecting across different modes of representations; (2) developing from earlier concepts rather than presenting each as isolated ideas; and (3) reinforcing conceptual links established earlier.

Dimension 2 – Cognitive Demand

This is the extent students are engaged in productive struggle to make sense of mathematical concepts. According to Schoenfeld, if the mathematics is cut up into little bitesize pieces such that students do not get to do any real thinking, then they are not building their mathematical muscles; or if the mathematics is beyond their level and they are given challenges they cannot make sense of, they end up just memorising, then they are not doing the right kind of sensemaking. The real challenge for instruction with regard to this dimension, is how the teacher makes sure that the students are engaged in a way that they are actually doing some thinking and problem solving within the range of what they can do so they grow by virtue of having done that sensemaking, and doing that means that the teacher has to have arranged the contingencies of instruction so that the mathematics and the instructions meet the students where they are and support them to grow their mathematics.

From Question 6, we see that an approach teachers adopt is to develop students' thinking skills and heuristics through: (1) teaching problem solving heuristics explicitly; and (2) teaching thinking skills explicitly. While video-recorded data of the 30 experienced and competent teachers points to instances of heuristics and thinking skills being discussed implicitly in mathematics lessons without labelling the processes in problem solving, about half of the participants (teachers) in the survey claimed to teach these skills explicitly in their mathematics lessons.

From Question 7, we see that another approach teachers adopt is to facilitate the development of metacognitive strategies amongst their students through: (1) getting students to compare different ways of solving a problem; (2) providing opportunities for their students to reflect on their own learning, including helping them to monitor their learning; and (3) getting students to check for reasonableness of their answers after solving a problem.

From Question 10, we see that teachers also stimulate students' thinking using their instructional materials in a number of ways, including: (1) including mathematical items that are deemed challenging to students

at appropriate junctures so that students are given opportunities to grapple with mathematical conceptual knowledge that is beyond mere mechanical applications; and (2) providing opportunities for students to make comparisons across methods.

From Question 11, we see that teachers, to help students to gain fluency in applying formula, arrange mathematical items in gradually-increasing levels of difficulty. It is important to note that in these instances, the mathematics is not cut up into little bitesize pieces such that students do not get to do any real thinking, but is organised into manageable chucks that require appropriate levels of cognitive demands, helping students to make sense of what they are learning and progressively build their mathematical muscles.

Dimension 3 – Access to Mathematical Content

This is about enabling every single student in the classroom to engage powerfully with the mathematical content and to do sensemaking with the content. Different students come to different tasks with different strengths and different understandings. According to Schoenfeld, narrowband activities that can be approached in only one way limit the mathematical riches, the kinds of conversations that the class is engaged in, and the numbers of students engaged in the tasks. Tasks that enable all students access to mathematical content have low floors and high ceilings – they are easy to begin to understand, and students make progress on them in relatively unsophisticated ways, but also in very sophisticated ways, making mathematical connections along the way.

From Question 6, we see that teachers emphasise mathematical processes by developing students' mathematical reasoning and communication skills, getting them to explain and justify their or their peers' working. Teachers also adopt different approaches to make connections between mathematics in the classroom and real-world context, including: (1) using real-world examples in lessons; and (2) encouraging students to discuss assumptions made, though it is observed that the discussion of assumptions associated to the real-world context appears to be less frequently done.

From Question 11, we see that teachers also help students to gain fluency in applying formula using their instructional materials in a number of ways, including: (1) arranging mathematical items in gradually-increasing levels of difficulty; (2) using a variety of examples and items judiciously to help students gain familiarity to the different ways of applying the formula in an appropriate range of tasks; and (3) developing formulas by building on previously-learnt methods or ideas so as to strengthen the links in students' learning.

Dimension 4 – Agency, Authority and Identity

This is getting students to develop a sense of agency in mathematics, seeing themselves as people who can do mathematics, develop an identity as a mathematics person who can do it when is confronted with a mathematical challenge. According to Schoenfeld, students in a mathematically powerful classroom not only have the opportunity to do sensemaking, but to talk to each other about it, to make contributions to the classroom discourse, to have others build on their ideas so that mathematics class becomes a place where people sort out reasoning collectively and build on one another's ideas.

From Question 3, we see that teachers create opportunities for and orchestrate math talks in their classrooms, mostly drawing on the basic repertoire of teaching talk: rote, recitation, and instruction/exposition, and occasionally the two kinds: discussion, and dialogue. Students are mostly engaged in the basic repertoire of learning talk: narrate and explain; and are given occasional opportunities to develop their repertoire of learning talk, such as speculate, imagine, explore, analyse, evaluate, discuss, argue, justify, and question. These math talks lie in a continuum between univocal and dialogic, depending on the instructional goals.

From Question 8, we see that teachers also adopt various approaches to imbue desired attitudes, such as interest, appreciation, confidence and perseverance, for the learning of mathematics amongst their student, including: (1) building students' confidence by starting with tasks that they can do before progressing to more difficult tasks; (2) encouraging students to persevere when solving a mathematics problem; and (3) using real-life examples and applications to help students appreciate the relevance of mathematics.

Dimension 5 – Uses of Assessment

This is assessment done in the moment of instruction in the service of helping teacher and students understand what the students understand so that teacher can do something about it. It is to understand students' thinking well enough to anticipate what students can do, to provide students the opportunity to reveal what they know. Then instruction can meet them where they are and help them move forward and cognitive demands can be adjusted accordingly so that the students are doing sensemaking and building their own understanding.

From Question 2, we see that teachers adopt the following actions in their moments of instruction so as to assess students' understanding:

- walking around the class noting student work that teacher would draw on to provide the class feedback during whole class review;
- exchanging ideas with students on how to solve a problem;
- asking students open-ended questions and allowing them to build on each other's responses to develop concepts or clarify their understanding;
- building on students' responses rather than merely receiving them;
- providing students with probing guidance (open-ended questions about their thinking and why they are considering certain approaches) when they face difficulty with a mathematical task they are doing;
- providing feedback to individuals for in-class work and homework to serve as information and diagnosis so that students can correct their errors or improve;
- providing collective feedback to whole class for common mistakes and misconceptions related to in-class work and homework;

and students are also encouraged to adopt the following actions so as to self-assess and deepen their own understanding:

- asking questions when they do not understand;
- explaining how their solutions or how their answers are obtained;
- analysing why a procedure (that teacher has shown on the board) works or why a solution method makes sense;
- offering alternative method(s) or solution(s) to a problem teacher has shown on the board;
- asking questions (such as 'what if') to probe further or deepen understanding;
- defending and explaining to classmate(s) why their approach/method to solve a problem is better (more efficient or more elegant);
- critiquing one another's work presented on board/screen so as to improve their understanding of concepts or elegance in their presentation/solution;
- discussing features of a problem that teacher has shown on the board so as to select a particular procedure or solution method.

What are the implications for mathematics teachers in Singapore secondary schools?

The findings in the preceding section affirm that for all the five dimensions of mathematically powerful classrooms, advocated by Schoenfeld (2014), activities enacted by mathematics teachers in Singapore secondary schools are beyond level 1.

The challenge now remains for teachers to advance their instruction such that activities they enact are at level 3 for all the five dimensions.

To do so, teachers need to reflect on their instruction, adopt the TRU Math Conversation Guide (Baldinger & Louie, 2014), and start professional conversations about the "what and how" of level 3 activities with an aim to make every mathematics classroom a mathematically powerful one.

So how mathematically powerful is your classroom?



My notes

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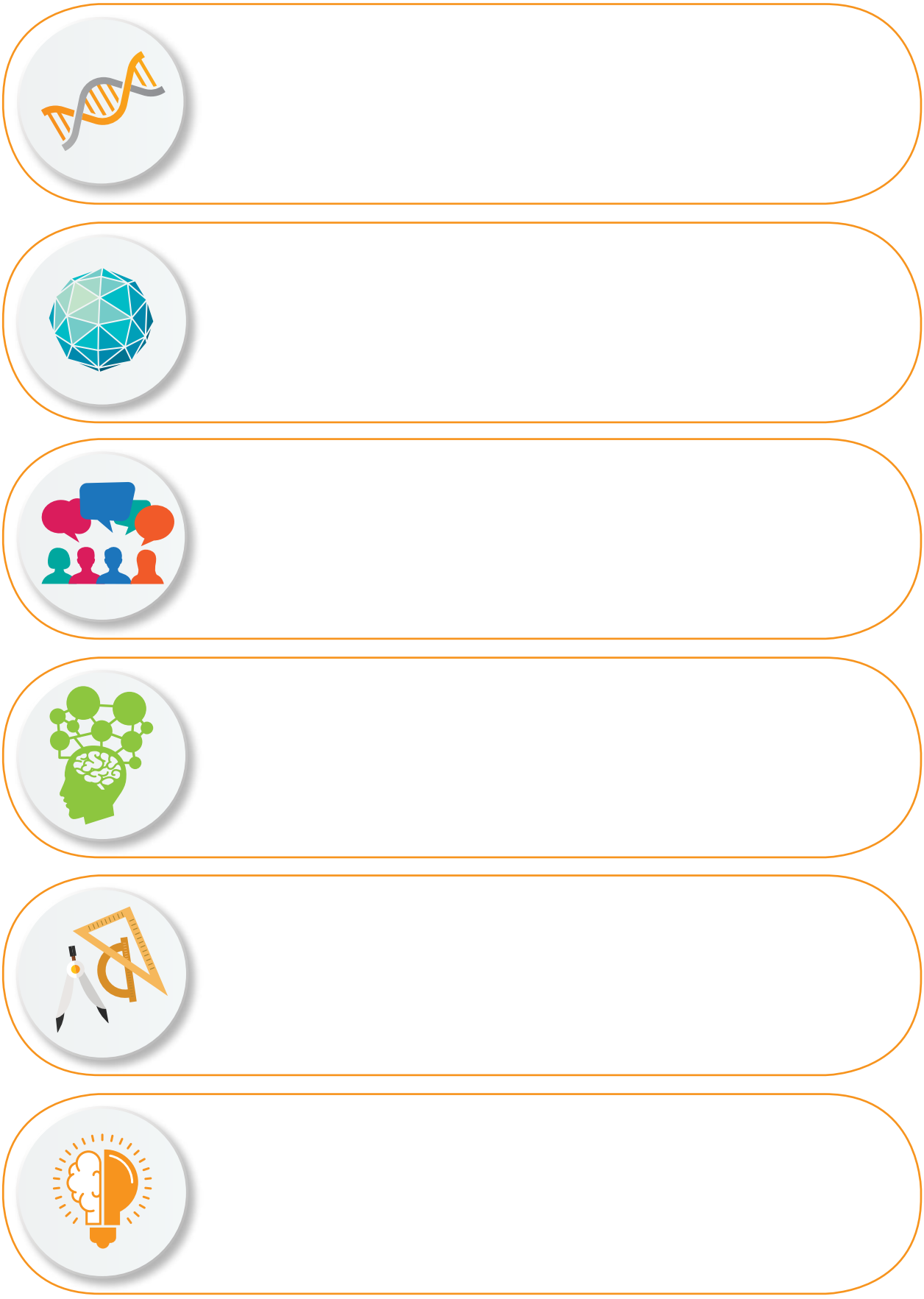
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My Summary



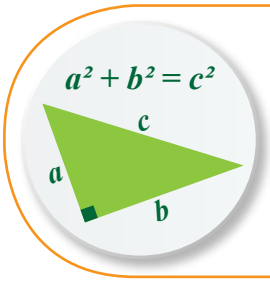
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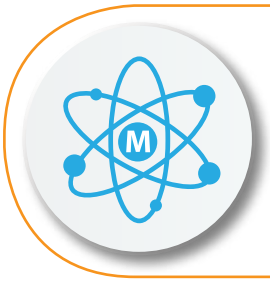












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