

**Proof of  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$  without the use of area of sector**

Lu Pien Cheng  
 National Institute of Education, Singapore  
 Nanyang Technological University

Eng Guan Tay  
 National Institute of Education, Singapore  
 Nanyang Technological University

Tuo Yeong Lee  
 NUS High School of Mathematics and Science  
 Singapore

**Abstract**

The derivation for the result  $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$  is usually shown as coming from the inequality  $\cos \theta < \sin \theta / \theta < 1 / \cos \theta$ , which is itself derived from the area of a sector. In this short article, we provide a proof for the limit without the use of the area of a sector.

**1. Introduction**

One of the approaches to develop the area formula for circles is to arrange the sectors of a circle into a parallelogram such that as the number of sectors gets larger, the figure becomes close to a rectangle (Van de Walle, Karp, & Bay-Williams, 2010). This approach is a common and sufficient method to determine the circle's area. However, from our experience working with teachers, we observed that it is hard for teachers to accept 'the area of a rectangle as the area of a circle (see Figure 1 below).

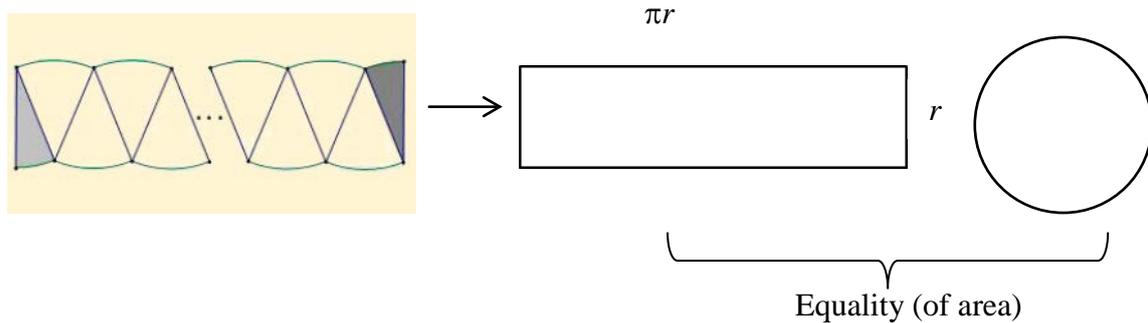


Figure 1. Area of circles using N-gon method.

In our paper (Cheng, Tay & Lee, 2012), we attempted to provide a conceptually and visually more acceptable proof for the teachers to accept the above equality.

In the proof, we assume that the limit as  $\theta$  approaches 0 of  $\frac{\sin \theta}{\theta}$  is 1. This assumption may give rise to some concerns that the derivation of this limit comes

from  $\cos\theta < \sin\theta/\theta < 1/\cos\theta$ , which is derived from the area of a sector, resulting in a circular argument (For example, see Richman, 1993).

**2. Proof for the result  $\lim_{\theta \rightarrow 0} \frac{\sin\theta}{\theta} = 1$  that does not require the area of circle**

Although Stewart (2003, p. 170) gives a proof of  $\lim_{\theta \rightarrow 0} \frac{\sin\theta}{\theta} = 1$  that uses the arc length, the rigour is questionable because he appeals to ‘sight’ when he claims that the arc length is shorter than the bounding line segments. The proof can be completed by referring to Schaumberger’s (1984) verification that “the length of the inscribed arc” (p. 144) or arc length, is shorter than the bounding line segments, “chord” (p. 144). The result  $\lim_{\theta \rightarrow 0} \frac{\sin\theta}{\theta} = 1$  has also been proven by Rose (1991) who used “arc length of a curve equals the least upper bound of the lengths of polygonal paths inscribed in the arc” (p.140) with the equation  $y = \sqrt{1-x^2}$ .

In this section, we provide another proof for the result  $\lim_{\theta \rightarrow 0} \frac{\sin\theta}{\theta} = 1$  without the use of the area of a sector.

Consider a curve given by the equation  $y = \sqrt{1-x^2}$  for  $0 \leq x \leq 1$  (see figure 2).

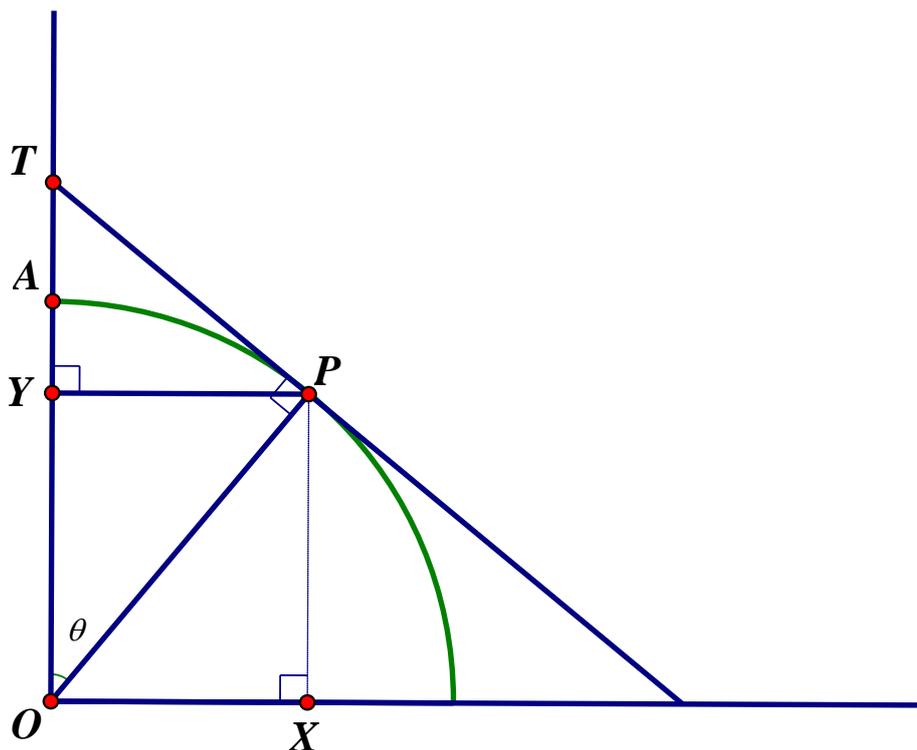


Figure 2. A curve given by the equation  $y = \sqrt{1-x^2}$  for  $0 \leq x \leq 1$ .

This curve will intercept the y-axis at a point A whose coordinate is (0, 1). Let O be the origin (0, 0) and P be a point on the curve such that OP makes an acute angle  $\theta$  (rad) with the y-axis. Then P will have coordinates  $(\sin\theta, \cos\theta)$ . Let  $(\sin\theta, 0)$

and  $(0, \cos\theta)$  be labeled X and Y respectively. T lies on the y-axis such that PT is perpendicular to OP.

At A,  $\frac{dy}{dx} = 0$ , and at P,  $\frac{dy}{dx} = -\tan\theta$  since  $\angle TPY = \theta$ . Observe that  $\left|\frac{dy}{dx}\right|$  is increasing in the interval  $[0, 1]$ . Thus, we have (note that  $\theta$  is constant):

$$\int_0^{\sin\theta} \sqrt{1+(0)^2} dx < \int_0^{\sin\theta} \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx < \int_0^{\sin\theta} \sqrt{1+(\tan\theta)^2} dx$$

$$\sin\theta < \theta < \int_0^{\sin\theta} \sec\theta dx \quad (\text{since } \int_0^{\sin\theta} \sqrt{1+\left(\frac{dy}{dx}\right)^2} dx \text{ is the length of the arc PA, which is } \theta)$$

$$\sin\theta < \theta < \sin\theta \sec\theta$$

$$\sin\theta < \theta < \tan\theta.$$

From  $\sin\theta < \theta$ , we have  $\frac{\sin\theta}{\theta} < 1$ . From  $\theta < \tan\theta$ , we have  $\cos\theta < \frac{\sin\theta}{\theta}$ . Thus we

$$\text{have } \cos\theta < \frac{\sin\theta}{\theta} < 1.$$

Taking limits as  $\theta$  tends to 0, the Squeeze Theorem gives us  $\lim_{\theta \rightarrow 0} \frac{\sin\theta}{\theta} = 1$ .

### 3. Concluding remarks

We hope the proof provided in this paper helps school teachers reinforce their own understanding that the area of circle can indeed be perceived as area of rectangle. The proof in this paper uses concepts of analysis and trigonometry and may also help undergraduate students apply The Squeeze Theorem and trigonometry to prove what they would normally see in their elementary school textbook as “area of circle equals to area of rectangle”. Applying undergraduate level mathematics into everyday life, common or familiar situations helps undergraduate students reflect, recognise and use connections among mathematical ideas and foster the view of mathematical learning as continuous – from elementary to undergraduate level mathematics.

### REFERENCES

- [1] Cheng, L. P., Tay, E. G., & Lee, T. Y. (2012). On area of circle. *Mathematics Teacher*, 108(8), 564 – 565.
- [2] David A. R. (1991). The Differentiability of  $\sin x$ . *The College Mathematics Journal*, 22(2), 139 – 142.
- [3] Richman, F. (1993). A circular argument. *The College Mathematics Journal*, 24(2), 160 – 162.
- [4] Schaumberger, N. (1984). The derivatives of  $\sin x$  and  $\cos x$ . *The College Mathematics Journal*, 15(2), 143 – 145.
- [5] Stewart, J. (2003). *Calculus* (5th ed.). Thomson: Brooks/Cole.
- [6] Van de Walle, J. A, Karp, K. S., & Bay-Williams, J. M., (2010). *Elementary and middle school mathematics: Teaching developmentally* (7<sup>th</sup> ed.). Boston: Allyn & Bacon

Lu Pien Cheng  
National Institute of Education, Singapore  
1 Nanyang Walk  
Singapore 637616  
lupien.cheng@nie.edu.sg

Eng Guan Tay  
National Institute of Education, Singapore  
1 Nanyang Walk  
Singapore 637616  
engguan.tay@nie.edu.sg

Tuo Yeong Lee  
NUS High School of Mathematics and Science  
20 Clementi Avenue 1  
Singapore 129957  
nhsleety@nus.edu.sg