Technical Report M2011-3
October 2011
Mathematics and Mathematics Education
National Institute of Education
Singapore

How Euler could have used Information & Communication Technology

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October 19, 2011

Abstract

This paper proposes a potential meeting point of the history of mathematics and information and communication technology (ICT), where the seemingly ill-matched duo collaborate to develop an interesting learner-centred lesson on undergraduate calculus. In addition, we present a novel proof of the famous product formula for sine which does not rely on theorems of uniform convergence or complex analysis.

1 Introduction

By now, the use of Information and Communication Technology (ICT) in mathematics classroom has become more of a norm than an exception. Because of their versatility in giving users visual representation of information, computers and graphing calculators are heavily harnessed in teaching pre-calculus and calculus concepts in high schools and universities all over the world; particularly, the graphing of functions (see [20, 19]). Specifically, the role of visualization in calculus was studied by David Tall in [18] with particular emphasis on appropriate choice of *complicated* examples using graphing facilities. The use of ICT in teaching and learning has become a characteristic milestone for mathematics education in the 21st century. Naturally, there is no lack of formal studies at all levels, investigating the feasibility, potentialities and risks involved when ICT is integrated into classroom teaching (see [7, 6, 17, 22, 14]).

Apart from their prowess in graphic display, computers and graphing calculators possess the important facility of handling tedious calculations – an inevitable challenge everyone (students and mathematicians alike) had to face during times of their non-existence. Exploiting the computational power of computers, mathematicians can now either formulate or refute conjectures based on empirical evidence churned out from exoteric 'number-crunching' that, otherwise, would have been impossible to perform. Recently, this aspect of harnessing computing power to look out for patterns, structures and even proofs in different areas of mathematics has been exploited and popularized by J.M. Borwein and D.H. Bailey under the banner of *Experimental Mathematics* (see, for instance, [4, 2, 1, 3]).

The main drive of this paper is to bring out this potentiality of ICT in teaching mathematics at an undergraduate (or even pre-university) course in Calculus. By thrusting the audience into a hypothetical scenario of how the famous Swiss mathematician, Leonhard Euler (1707–1783), might have exploited the present-day graphing calculators, we hope to create a deep impression in the reader of how meaningful mathematical learning can take place with the aid of modern-day computing facilities. While advocating the potentialities of computers in mathematics discovery, we hope that the reader can appreciate the *eureka* moments in the lives of mathematicians and be reminded of the indispensable role of rigorous proofs in mathematics. We drive home the importance of mathematical rigor by presenting an elementary proof of the infinite product formula for sine *without* appeal to uniform convergence (c.f. the remarks made on p. 520 of [21] about Euler's unjustified interchange of limits).

We organize our exposition as follows. In Section 2, we take advantage of a well-known historical anecdote of how Leonhard Euler solved the then-longstanding Basel's problem in 1735, taking *by faith* the famous infinite-product representation of the sine function. In our hypothetical scenario painted in Section 3, we picture how Euler chanced upon an access to modern-day graphing calculators which he quickly exploited to give him further confidence in the conjecture he formulated, i.e.,

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right). \tag{1}$$

In the process of obtaining a visual assurance for the above conjecture, we walk the reader (in the eyes of Euler) through the use of TI-nSPIRE – a modern graphing calculator. This leads us to Section 4, where an elementary proof of Equation (1), which Euler failed to supply during his time, is presented. This proof requires only minimum background knowledge of definite integration. We then conclude our paper in Section 5 with some pedagogical justification for our design and implementation of this historically-inspired ICT-based lesson.

2 The Basel problem and Leonard Euler

First posed by Pietro Mengoli in 1644, the *Basel problem* is a famous problem in mathematical analysis and number theory which demanded its solver to produce the exact value (in closed form) of the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^2} := \lim_{n\to\infty} \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right),$$

which is the summation of reciprocals of all perfect squares. Of course, a proof of its correctness was expected as part of its solution. This problem was later popularized by Jakob Bernoulli in around 1689 and survived the attacks of many leading mathematicians of the day, notably the Bernoulli family. According to historical records, Euler announced to the mathematical community in 1735 that the exact sum is $\frac{\pi^2}{6}$ (see [10]), though his arguments then were based on algebraic manipulations not entirely justified. Later in 1741, Euler did manage to produce a rigorous proof of his solution to the Basel problem (see [11]), albeit via a completely different method.

For details of the history of the Basel problem and the biography of Euler, the reader is encouraged to consult the excellent works of L. Debnath [8, 9] and V.S. Varadarajan [21], all written as tricentennial tributes to Leonhard Euler.

3 The hypothetical scenario

In this section, we consider a hypothetical scenario in which Euler somehow gained access to the modern-day graphing calculator. While cautioning the reader *not* to take this scenario as actual historical fact, we *do* encourage a healthy amount of imagination on the part of the reader in order to appreciate the difference ICT can make in creating a new learning experience. Let us proceed in Euler's *first person.* We have to begin somewhere in 1730.

Figure 1: Leonhard Euler (1707 - 1783)

For some time, I, Leonhard Euler of Basel, have been contemplating the idea that the sine-function (written below with my newly invented symbol for function[1])

$$f(x) = \sin x$$

behaves in many ways like a polynomial. Well, a polynomial is nothing but an expression in an indeterminate x of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0, \quad a_i \in \mathbb{R}.$$

Thanks to René Descartes (1596 – 1650) and his *Cartesian philosophy*[2] some 80 years ago, one can now conveniently visualize the variation of a given function in the Cartesian plane. Any section of $f(x) = \sin x$ can be graphed without having my quill-pen leave the surface of the paper in the midst of the sketch. Furthermore, the sine-function f has all its

[1] The function symbol $f(x)$ first appeared in print around 1734.

[2] In 1723 Euler completed his Master's degree in philosophy with a scholastic comparison between the philosophies of René Descartes and Isaac Newton.

derivatives, just like the polynomials. Of course, the periodicity of f is a distinctive feature which distinguishes it from any polynomial.

Nevertheless, polynomials can be used to approximate any function that possesses all derivatives; in particular, the sine-function. In fact, I had earlier worked out the infinite series expansion for the sine-function, i.e.,

$$\sin x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k-1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots \tag{2}$$

relying on a formula (discovered by myself), which I learnt later was first derived by Taylor and later popularized by Maclaurin (through private communication with James Stirling). From the above formula, the sine-function can be regarded as a 'polynomial of infinite degree'. Sketching a succession of partial sums of this series certainly illustrates this point. But such an enterprise will grow monstrously tedious, if not impossible by hand, as the number of summands increases indefinitely. Yes, even for me, despite having an extraordinary gift for handling complex mental calculations. Alas, *if only* one could have in quiet possession a calculating machine that can relieve this heavy burden of mundane calculations!

Just then, something caught my attention. Whatever is this? It must be the cataract[3] in my left eye playing tricks on me. Lately, such visual ailments have worsened. Pondering over its origin, I begin to study the object's structure. This gadget has several buttons, marked with numerals and mathematical symbols, and even has a glass window through which one might seem to peer into the future. It bears the label 'TI-*n*spire'[4]. Judging from its appearance, I have every belief that its design is intended for calculations of all kinds. Is this a dream or, should I say, God answering my fervent prayers? [Accidentally, hitting the ON button...] Wow, it seems I have just set the machine rolling.

Fiddling with this gadget for a while, it is not difficult to use the various functions on the *calculator* (I shall use this nomenclature for the moment) that results in the plotting of the graph of $y = \sin(x)$ (as shown below) as the values of x runs from -2π to 2π.

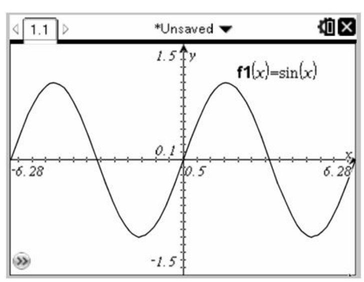

Figure 2: Graph of $y = \sin x$ on $[-2\pi, 2\pi]$

[3]R. Calinger in [5] suggests that Euler's left eye turned blind later because of cataract. Readers might have noticed Euler's eye defect from the 1753 portrait of Euler in Figure 1.

[4]Those who are curious about the looks of this machine may like to visit `http://education.ti.com/calculators/products/US/Nspire-Family/`.

Unimaginable! Let me quickly exploit this machine to sketch the first few partial sums of the infinite series (2). Presto! I immediately obtain the following graphs.

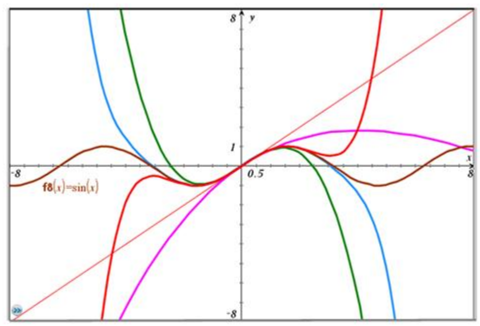

Figure 3: Graphs of partial sums in Taylor series of the sine function

Inspecting the graph of each member of the family, i.e.,

$$P_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{(2k-1)!} x^{2k-1}, \quad n = 1, 2, \ldots$$

it is easy to see that none of these have any zeros in common with $f(x) = \sin x$; this is a little unsatisfactory. But one cannot ask for more since we generate these polynomials by *recursively adding* a polynomial of higher degree to the existing partial sum. Perhaps, a better way to get polynomials whose zeros agree with the sine-function is to consider a scheme that *recursively multiplies* suitable linear factors to certain existing finite products.

The first zero of the sine-function is $\alpha_0 = 0$. So, the simplest polynomial that has this zero is

$$Q_0(x) = x.$$

The next two zeros of sine are $\alpha_1 = \pi$ and $\alpha_{-1} = -\pi$. If we require the curve of a cubic polynomial to pass through the coordinates $(\alpha_k, 0)$ for $k = -1, 0, 1$, then a simple choice would be

$$(x + \pi)x(x - \pi).$$

However, the knowledge that $\lim_{x\to 0} \frac{\sin x}{x} = 1$ indicates that this choice is inappropriate since

$$\lim_{x\to 0} \frac{(x+\pi)x(x-\pi)}{x} = \lim_{x\to 0}(x+\pi)(x-\pi) = -\pi^2.$$

But this imperfection can be easily restored via a division of $Q_1(x)$ by precisely the constant of $-\pi^2$ so as to obtain

$$Q_1(x) = \frac{-1}{\pi^2}(x+\pi)x(x-\pi) = \left(1 + \frac{x}{\pi}\right) x \left(1 + \frac{x}{\pi}\right) = x\left(1 - \frac{x^2}{\pi^2}\right).$$

Catching on the pattern, one naturally forms

$$Q_n(x) = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right),$$

which surely has roots $\alpha_k = \pm k\pi,\ k = 1, 2, \ldots, n$ as desired. Notice that

$$\lim_{x\to 0}\frac{Q_n(x)}{x} = \lim_{x\to 0}\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right) = 1,$$

confirming that my choice of these polynomials, $Q_n(x)$, is good.

All of what I have obtained looks good so far. But there is still an nudging guilt that is eating into my mathematical conscience – the following conjecture is ill-justified:

Conjecture 3.1. *For any value of $x \in \mathbb{R}$, it holds that*

$$\sin x = \lim_{n\to\infty} Q_n(x),$$

where $Q_n(x) = x\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2\pi^2}\right)$.

To be honest, I am not even totally sure if it is correct. Now that I have this wonderful graphing machine with me, why not make use of it to *check the credibility* of my conjecture?

One good thing about this TI-*n*spire (CAS) calculator is that it has a built-in "slider" functionality that allows users to vary the size of parameters easily – that is useful for my investigative work. Let me make use of the slider to control the value of n in the expression $Q_n(x) = x\prod_{k=1}^{n}\left(1-\frac{x^2}{k^2\pi^2}\right)$. Life is now made so easy through the exploit of this miraculous machine: the first two sketches of $y = Q_n(x)$ for $n = 10$ and $n = 10^2$ are already very promising (see Figure 4).

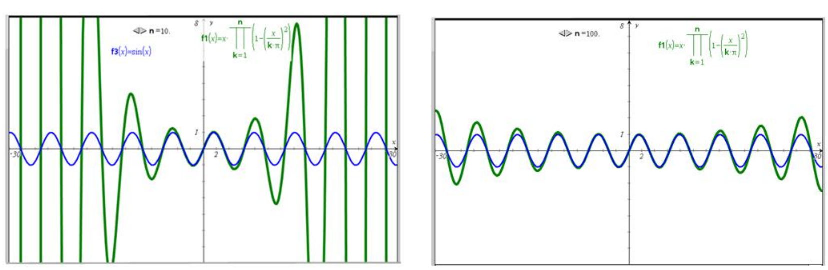

Figure 4: Sketches of $y = Q_n(x)$ for $n = 10,\ 10^2$

The blue curve is the sine curve, while the green curves are $y = Q_n(x)$ for the various values of n. By the time one reaches $n = 10^3$, the curve of $y = Q_n(x)$ almost completely overlaps with that of $y = \sin x$ (see Figure 5).

Extremely reassuring! Clearly, these three sketches provide strong empirical evidence for the convergence of $Q_n(x)$ to $\sin x$, thus confirming my conjecture!

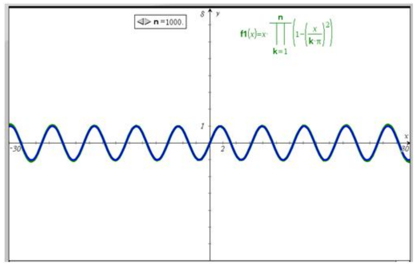

Figure 5: Sketch of $y = Q_n(x)$ for $n = 10^3$

Rigor aside, my sixth sense tells me something nontrivial is hiding under this identity. So far, I have two different representations for the sine function (namely, the infinite summation and the infinite product). Surely nothing should stop me from equating them:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} x^{2k-1} = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right).$$

Comparing the coefficients of x^3 on both sides of the equation, I have

$$-\frac{1}{3!} = -\frac{1}{\pi^2} - \frac{1}{2^2\pi^2} - \frac{1}{3^2\pi^2} - \dots$$

This then implies that

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}.$$

Isn't this precisely the solution to the famous Basel problem everyone is seeking after? To be sure, let me enslave the mystery machine to verify the 10^5-th partial sum of $\sum_{k=1}^{\infty} \frac{1}{k^2}$ against $\frac{\pi^2}{6} \approx 1.64493406685$ (see Figure 6). With this added confidence, I am now ready to announce this remarkable result to the Academy!

The rest of the story is, as they say, history. Eventually, Euler announced his solution to the Basel problem in 1735.

	A	B	C
1	n	nth*partial_sum	
2	99990	1.6449240659	
3	99991	1.644924066	
4	99992	1.6449240661	
5	99993	1.6449240662	
6	99994	1.6449240663	
7	99995	1.6449240664	
8	99996	1.6449240665	
9	99997	1.6449240666	
10	99998	1.6449240667	
11	99999	1.6449240668	
12	100000	1.6449240669	
13	...		

Figure 6: Tabulation of $\sum_{k=1}^{n} \frac{1}{k^2}$ for $n = 99990, \dots, 100000$

4 Proof of the infinite product identity

Euler himself was aware that his original solution of the Basel problem (which involved the audacious generalization of the Newton's formula for sums of powers of the roots of a polynomial to the case when the polynomial was replaced by a power series) were open to objections. He spent a good ten years filling in the gap, appealing crucially to the famous product formula for sine, i.e., Equation (1). The proof that Euler supplied, according to Varadarajan (p. 519 of [21]), involved the following equation:

$$\frac{\sin x}{x} = \lim_{n\to\infty} \frac{\left(1+\frac{ix}{n}\right)^n - \left(1-\frac{ix}{n}\right)^n}{2ix} = \lim_{n\to\infty} \prod_{k=1}^{2n+1} \left(1 - \frac{x^2}{n^2}\frac{1+\cos\frac{2k\pi}{n}}{1-\cos\frac{2k\pi}{n}}\right)$$

At this juncture, Euler made a passage to limit termwise by using the clever estimate of $\left|\frac{1+\cos\frac{2k\pi}{n}}{1-\cos\frac{2k\pi}{n}}\right| \leq C\frac{x^2}{k^2\pi^2}$ which, by present day standards, can be justified by *uniform convergence*. The keen reader should also be informed that Equation (1) can be alternatively established using complex analysis, e.g., see Section 8.4 of [9], p. 248.

In this section, we deliver what we promised in our introduction: a new elementary proof for Equation (1) which does not depend on theorems involving uniform convergence or complex variables. In contrast, our method is far simpler and easily accessible to undergraduate students in a first year calculus course. Our result depends on two technical lemmata. The first is a well-known identity, whose proof we give a reference:

Lemma 4.1 (Infinite partial fractions representation of $\cot t$)**.**
Let $t \in \mathbb{R}$ which is not an integral multiple of π. Then the following identity holds:

$$\cot t \equiv \frac{1}{t} + \sum_{k=1}^{\infty}\left(\frac{1}{t+k\pi} + \frac{1}{t-k\pi}\right). \tag{3}$$

Proof. For an elementary proof (also independent of uniform convergence and complex analysis results) of this, see a recent work of the authors [16]. □

Lemma 4.2. *Let r be a positive integer. Then the following holds:*

$$\ln 2 + \sum_{k=1}^{r-1} \ln\left(\frac{r^2}{k^2} - 1\right) + \sum_{k=r+1}^{\infty} \ln\left(1 - \frac{r^2}{k^2}\right) = 0. \tag{4}$$

Proof. The first two terms of Equation 4 simplify to

$$\begin{aligned}
\ln 2 + \sum_{k=1}^{r-1} \ln\left(\frac{r^2}{k^2} - 1\right) &= \ln 2 + \sum_{k=1}^{r-1}\Big(-\ln k + \ln(r-k) - \ln k + \ln(r+k)\Big) \\
&= \ln 2 - \sum_{k=1}^{r-1} \ln k + \sum_{k=1}^{r-1} \ln k - \sum_{k=1}^{r-1} \ln k + \sum_{k=r+1}^{2r-1} \ln k \\
&= \ln\left(\frac{(2r)!}{(r!)^2}\right),
\end{aligned}$$

while the third term is equivalent to

$$
\begin{aligned}
& \sum_{k=r+1}^{\infty} \ln\left(\frac{k^2-r^2}{k^2}\right) \\
= \quad & \sum_{k=r+1}^{\infty} \Big(\ln(k+r) - \ln k + \ln(k-r) - \ln k \Big) \\
= \quad & \sum_{k=r+1}^{\infty} \sum_{s=1}^{r} \Big(\ln(k+s) - \ln(k+s-1) - \ln(k-s+1) + \ln(k-s) \Big) \\
= \quad & \sum_{s=1}^{r} \lim_{n\to\infty} \sum_{k=r+1}^{n} \Big(\ln(k+s) - \ln(k+s-1) - \ln(k-s+1) + \ln(k-s) \Big) \\
= \quad & \lim_{n\to\infty} \sum_{s=1}^{r} \Big(\ln(n+s) - \ln(r+s) - \ln(n-s+1) + \ln(r-s+1) \Big) \\
= \quad & \sum_{s=1}^{r} \Big(-\ln(r+s) + \ln(r-s+1) \Big) \\
= \quad & -\sum_{s=r+1}^{2r} \ln s + \sum_{s=1}^{r} \ln s \\
= \quad & \ln\left(r! \div \left(\frac{(2r)!}{r!}\right) \right) \\
= \quad & \ln\left(\frac{(r!)^2}{(2r)!}\right).
\end{aligned}
$$

□

We are ready to establish the main result:

Theorem 4.3. *Let $x \in \mathbb{R}$. Then the following identity holds:*

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right).$$

Proof. Rearranging the identity (3) to

$$\cot t - \frac{1}{t} = \sum_{k=1}^{\infty} \frac{2t}{t^2 - k^2\pi^2}. \tag{5}$$

A direct integration of Equation (5) over $(0, x)$ for $0 < x < \pi$ yields:

$$\left[\ln\left(\frac{\sin t}{t}\right)\right]_0^x = \sum_{k=1}^{\infty} \left[\ln\left(k^2\pi^2 - t^2\right)\right]_0^x, \tag{6}$$

which results in

$$\sin x = x \prod_{k=1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right).$$

Note that the interchange of limits leading to Equation (6) can be achieved without appealing to any convergence theorems: One first compute a closed formula for the finite N-th partial sum $\sum_{k=1}^{N} \frac{2t}{t^2-k^2\pi^2}$, followed by direct integration and taking limits. Such a technique has already been advertised in the derivation of the infinite partial fractions representation of the cotangent function (see the proof of Corollary 4.2 of [16]).

To extend the above result to the whole real line, consider an arbitrary positive real number x. The Archimedean property of real numbers ensures that there is a positive integer r so large that $x \in (r\pi, (r+1)\pi)$. The proof commences by considering the following definite integral

$$I = \int_{r\pi}^{x} (\cot t - \frac{1}{t - r\pi} - \frac{1}{t + r\pi})\ dt. \tag{7}$$

On one hand, by virtue of Lemma 4.1, we can trade away the cotangent function with its partial fractions representation, resulting in

$$I = \int_{r\pi}^{x} \frac{1}{t}\ dt + \sum_{\substack{k=1 \\ k\neq r}}^{\infty} \int_{r\pi}^{x} (\frac{1}{t + k\pi} + \frac{1}{t - k\pi})\ dt. \tag{8}$$

Evaluating the definite integral (8) yields

$$\begin{aligned} &\ln x - \ln(r\pi) + \sum_{k=1}^{r-1} \ln\left(\frac{x^2 - k^2\pi^2}{(r^2 - k^2)\pi^2}\right) + \sum_{k=r+1}^{\infty} \ln\left(\frac{k^2\pi^2 - x^2}{(k^2 - r^2)\pi^2}\right) \\ =& \ln(\frac{x}{r\pi}) + \sum_{k=1}^{r-1} \ln\left(\frac{x^2 - k^2\pi^2}{k^2\pi^2}\right) - \sum_{k=1}^{r-1} \ln(\frac{r^2}{k^2} - 1) \\ &+ \sum_{k=r+1}^{\infty} \ln\left(\frac{k^2\pi^2 - x^2}{k^2\pi^2}\right) - \sum_{k=r+1}^{\infty} \ln(1 - \frac{r^2}{k^2}). \end{aligned}$$

On the other hand, one can evaluate (7) directly to obtain

$$\ln|\sin x| - \ln\left|x^2 - r^2\pi^2\right| + \ln(2r\pi)$$

so that equating these expressions, one has

$$\ln\left|\frac{\sin x}{x}\right| = \sum_{k=1}^{\infty} \ln\left|1 - \frac{x^2}{k^2\pi^2}\right| - \ln 2 - \sum_{k=1}^{r-1} \ln(\frac{r^2}{k^2} - 1) - \sum_{k=r+1}^{\infty} \ln(1 - \frac{r^2}{k^2}). \tag{9}$$

But by invoking Lemma 4, the above expression then simplifies to

$$\ln\left|\frac{\sin x}{x}\right| = \sum_{k=1}^{\infty} \ln\left|1 - \frac{x^2}{k^2\pi^2}\right|. \tag{10}$$

Finally, since our choice of r (such that $x \in (r\pi, (r+1)\pi)$) yields the following inequalities:

$$(-1)^r \left(\frac{\sin x}{x}\right) > 0, \quad (-1)^r \cdot \prod_{k=1}^{r} \left(1 - \frac{x^2}{k^2\pi^2}\right) > 0 \quad \text{and} \quad \prod_{k=r+1}^{\infty} \left(1 - \frac{x^2}{k^2\pi^2}\right) > 0,$$

this warrants a legitimate removal of the modulus signs in Equation (10), and the desired result follows immediately. □

5 Concluding remarks

The question of whether history of mathematics has any role to play in the teaching and learning of mathematics has recently attracted the attention of many mathematics educators. The most systematic, in-depth and large-scale study of this question was unarguably the ICMI study on History in Mathematics Education ([13]) carried out in 2000. A similar study was carried out in [15] with special emphasis on Singapore mathematics education.

In this concluding section, we appeal to the didactic framework suggested by [15] to support the methodology we adopt herein. This particular framework advocates a two-fold process: (1) *backward sourcing*, and (2) *forward implementing*. Roughly speaking, while planning to integrate the history of mathematics in a mathematics lesson, the teacher must begin by identifying the learning points in the topic where the learner is likely to experience an intellectual leap. By understanding the learning difficulties or potential confusion anticipated in such an experience, the teacher then explores the psycho-genetical mechanisms that are needed to help the learner transit smoothly over these identified learning points/difficulties. Once these psycho-genetical mechanisms are identified, the corresponding historical mechanisms are to be sought after. This is then followed by an understanding of what problems these specific historical mechanisms or processes were used to tackle. Having identified the salient historical mechanisms, the teacher then browse through the historical resources to look for relevant historical episodes where such identified mechanisms had been invoked. Collectively, what the teacher has just performed is backward sourcing. The lesson plan should then be developed to take advantage of these powerful historical moments with a primary focus of addressing the anticipated learning difficulties. This portion is known as forward implementation and must consists of meaningful activities or teaching moves that incorporate the historical element to achieve the specific learning objectives.

In the preceding sections, we demonstrate how a hypothetical situation of Leonhard Euler establishing his famous infinite product representation of the sine function *with* the help of an ICT tool such as TI-*n*spire. Here, the learning point we have identified is the topic of graphs and approximation by functions[5]. Learning difficulties concerning limiting processes and graphing techniques are anticipated. The backward sourcing process allows us to identify a historical moment in which a similar difficulty was experienced by a mathematician, i.e., Leonhard Euler. What was then taken by faith can now be verified by modern day technology. It is exactly at this juncture of blatent mismatch and conflict where we believe that history of mathematics collaborates seamlessly with information technology.

On one hand, it has long been identified that history of mathematics helps improve students' learning attitude, sustain interest and arouse excitement in mathematics (see [12, 15]). On the other hand, with fast-paced advancement and pervasive use of ICT in teaching and learning cutting

[5]This can be seen as a precursor to the Stone-Weierstrass approximation theorem.

across every grade level and subject, Mathematics educators have both recognized and harnessed the strengths of ICT tools to design increasingly inspiring and engaging mathematics lessons. Our novel pedagogical approach proposed herein fuses the old (history) and the new (ICT), and should warrant further investigation and development. Adopting this approach to teaching Calculus topics, for instance, not only helps students "visualise" an abstract theorem, but also allows students to see how that particular mathematical result was brewed in the minds of mathematicians. For example, the dynamic and interactive feature of the TI-*n*spire calculator helps deliver the abstract concepts of limits, convergence and functional approximation in a convincing way; learners see for themselves how $Q_n(x)$'s get closer and closer to $\sin(x)$. It is hoped by immersing learners in experiencing mathematics through the tangibles that they begin to see the relevance of justifying these observations rigorously. Mathematical proofs then become more meaningful when learners are sufficiently motivated.

There are many great episodes in the history of mathematics; several of which, we believe, can be suitably coupled with modern day ICT tools to help students gain insight as well as inspirations in their mathematics learning journey. Let us begin to excavate those precious gems of mathematics history and incorporate them with ICT to create a fresh platform for teaching and learning of mathematics.

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