Chromatic Roots of a Ring of Four Cliques

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Abstract

For any positive integers \(a, b, c, d\), let \(R_{a,b,c,d}\) be the graph obtained from the complete graphs \(K_a, K_b, K_c, K_d\) by adding edges joining every vertex in \(K_a\) and \(K_c\) to every vertex in \(K_b\) and \(K_d\). This paper shows that for arbitrary positive integers \(a, b, c, d\), every root of the chromatic polynomial of \(R_{a,b,c,d}\) is either a real number or a complex number with its real part equal to \((a + b + c + d - 1)/2\).

Keywords: graph, chromatic polynomial, chromatic root, ring of cliques

1 Introduction

A ring of cliques is a graph whose vertex set is the disjoint union of cliques, arranged in a cyclic order, such that the vertices of each clique are joined to all the vertices in the two neighbouring cliques. If the cliques have size \(a_1, a_2, \ldots, a_n\) then we denote this graph by \(R_{a_1,a_2,\ldots,a_n}\). Figure 1 shows the graph \(R_{2,2,3,3}\).

Graphs with this structure have occurred several times previously in the study of chromatic polynomials and their roots. In particular, in proving that there are non-chordal graphs with integer chromatic roots, Read [6] considered the graphs in this family with \(a_1 = 1\) (and he also used slightly different notation). Rings of cliques cropped up again recently in a preliminary investigation of the algebraic properties of chromatic roots (Cameron [1]) and in the course

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of this investigation, the chromatic roots of many of these graphs were computed. When the chromatic roots of the ring-of-clique graphs with *exactly four cliques* and a fixed number of vertices were plotted, an intriguing pattern was observed — all the complex chromatic roots lie on a single vertical line. Figure 2 shows the union of the chromatic roots of the 12-vertex graphs of the form $R_{a,b,c,d}$.

Faced with such a striking empirically-observed pattern, we were led to explain it theoretically. This appears to require a surprisingly intricate argument, but eventually we obtain the following result:

**Theorem 1** For arbitrary non-negative integers $a$, $b$, $c$ and $d$ the chromatic roots of $R_{a,b,c,d}$ are either real, or complex with real part equal to $(a + b + c + d - 1)/2$.

The overall structure of the argument is as follows: the chromatic polynomial $P(R_{a,b,c,d}, \lambda)$ is first expressed as the product of linear factors and a factor $Q_{a,b,c,d}(\lambda)$. It then suffices to show that the complex roots of $Q_{a,b,c,d}(\lambda)$ all lie on the vertical line $\Re(\lambda) = (a + b + c + d - 1)/2$ in the complex $\lambda$-plane. Next the polynomial $F_{a,p,q,n}(z)$ is defined to be $Q_{a,b,c,d}(z + (a + b + c + d - 1)/2)$ thus translating the vertical line supposed to contain the roots to the imaginary axis and also reparameterizing the problem (in a somewhat counterintuitive way). Then $F_{a,p,q,n}$ is shown to be an *even* polynomial and we define a fourth polynomial $W_{a,p,q,n}$ by $W_{a,p,q,n}(z^2) = F_{a,p,q,n}(z)$. The proof is completed by demonstrating that $W_{a,p,q,n}$ is real-rooted using polynomial interleaving techniques, and therefore $F_{a,p,q,n}$ has only real or pure imaginary roots as required.
Figure 2: Chromatic roots of the graphs $R_{a,b,c,d}$ where $a + b + c + d = 12$.

2 Basics

For any graph $G$ and any positive integer $\lambda$, let $P(G, \lambda)$ be the number of mappings $\phi$ from $V(G)$ to $\{1, 2, \cdots, \lambda\}$ such that $\phi(u) \neq \phi(v)$ for every two adjacent vertices $u$ and $v$ in $G$. It is well-known that $P(G, \lambda)$ is a polynomial in $\lambda$, called the chromatic polynomial of $G$.

The chromatic polynomial of a graph $G$ has the following properties (see, for instance, [3, 5, 7, 8]), which we will apply later.

Proposition 1   Let $G$ be a simple graph of order at least 2.

(i) If $u$ and $v$ are two non-adjacent vertices in $G$, then

$$P(G, \lambda) = P(G + uv, \lambda) + P(G/uv, \lambda),$$

where $G + uv$ is the graph obtained from $G$ by adding the edge joining $u$ and $v$, and $G/uv$ is the graph obtained from $G$ by identifying $u$ and $v$ and removing all parallel edges but one.

(ii) If $u$ is a vertex in $G$ which is adjacent to all other vertices in $G$, then

$$P(G, \lambda) = \lambda P(G - u, \lambda - 1),$$
where $G - u$ is the graph obtained from $G$ by removing $u$.

If $a = 0$, $R_{a,b,c,d}$ is a chordal graph and its chromatic polynomial is

$$P(R_{0,b,c,d}, \lambda) = \frac{(\lambda)_{b+c}(\lambda)_{c+d}}{(\lambda)_c},$$

(3)

and if $a \geq 1$ and $c \geq 1$, then applying Proposition 1 repeatedly yields that

$$P(R_{a,b,c,d}, \lambda) = \lambda P(R_{a-1,b,c,d}, \lambda - 1) + c\lambda P(R_{a-1,b,c-1,d}, \lambda - 1).$$

(4)

For a non-negative integer $a$ and real numbers $b, c$ and $d$, define a polynomial $Q_{a,b,c,d}(z)$ in $z$ as follows: $Q_{0,b,c,d}(z) = 1$ and for $a \geq 1$,

$$Q_{a,b,c,d}(z) = (z - b - c)(z - c - d)Q_{a-1,b,c,d}(z - 1) + c(z - a - c + 1)Q_{a-1,b,c-1,d}(z - 1).$$

(5)

It is clear that $Q_{a,b,c,d}(z)$ is a polynomial of order $2a$ in $z$.

**Proposition 2** Let $a, b, c$ and $d$ be any non-negative integers. Then

$$P(R_{a,b,c,d}, \lambda) = \frac{(\lambda)_{b+c}(\lambda)_{c+d}}{(\lambda)_{a+c}}Q_{a,b,c,d}(\lambda).$$

(6)

**Proof.** If $a = 0$, then (6) follows from (3) and the definition of $Q_{a,b,c,d}(\lambda)$. Now assume that $a \geq 1$. By (4) and induction, we have

$$P(R_{a,b,c,d}, \lambda) = \lambda P(R_{a-1,b,c,d}, \lambda - 1) + c\lambda P(R_{a-1,b,c-1,d}, \lambda - 1)$$

$$= \lambda \frac{(\lambda - 1)_{b+c}(\lambda - 1)_{c+d}}{(\lambda - 1)_{a+c-1}}Q_{a-1,b,c,d}(\lambda - 1)$$

$$+ c\lambda \frac{(\lambda - 1)_{b+c-1}(\lambda - 1)_{c+d-1}}{(\lambda - 1)_{a+c-2}}Q_{a-1,b,c-1,d}(\lambda - 1)$$

$$= \frac{(\lambda)_{b+c}(\lambda)_{c+d}}{(\lambda)_{a+c}}[(\lambda - b - c)(\lambda - c - d)Q_{a-1,b,c,d}(\lambda - 1)$$

$$+ c(\lambda - a - c + 1)Q_{a-1,b,c-1,d}(\lambda - 1)].$$

(7)

The result then follows. \hfill \Box

Assume that $\binom{r}{0} = 1$ and $\binom{r}{r} = x(x - 1) \cdots (x - r + 1)/r!$ for any positive integer $r$ and any complex number $x$.

**Proposition 3** For any non-negative integer $a$ and real numbers $b, c$ and $d$,

$$Q_{a,b,c,d}(\lambda) = a! \sum_{i=0}^{a} i!(a - i)! \binom{c}{i} \binom{\lambda - b - c}{a - i} \binom{\lambda - c - d}{a - i} \binom{\lambda - a - c + i}{i}. $$

(8)
Proof. It is trivial if \( a = 0 \) as \( Q_{0,b,c,d}(z) = 1 \). Now assume that \( a \geq 1 \). By (5) and induction,

\[
Q_{a,b,c,d}(\lambda) = (\lambda - b - c)(\lambda - c - d)Q_{a-1,b,c,d}(\lambda - 1) + c(\lambda - a - c + 1)Q_{a-1,b,c-1,d}(\lambda - 1)
\]

\[
= (\lambda - b - c)(\lambda - c - d)(a - 1)! \sum_{i=0}^{a-1} \binom{c}{i} \left( \frac{\lambda - b - c}{a - i - 1} \right) \left( \frac{\lambda - c - d}{a - i - 1} \right) \left( \frac{\lambda - a - c + i}{i} \right)
\]

\[
+ c(\lambda - a - c + 1)(a - 1)! \sum_{i=0}^{a-1} \binom{c}{i} \left( \frac{\lambda - b - c}{a - i - 1} \right) \left( \frac{\lambda - a - c + i + 1}{i + 1} \right)
\]

\[
= (a - 1)! \sum_{i=0}^{a-1} i!(a - i) \binom{c}{i} \left( \frac{\lambda - b - c}{a - i} \right) \left( \frac{\lambda - c - d}{a - i} \right) \left( \frac{\lambda - a - c + i}{i} \right)
\]

\[
+ (a - 1)! \sum_{i=0}^{a-1} i!(a - i - 1)!(i + 1) \left( \frac{c}{i + 1} \right) \left( \frac{\lambda - b - c}{a - i - 1} \right) \left( \frac{\lambda - c - d}{a - i - 1} \right) \left( \frac{\lambda - a - c + i + 1}{i + 1} \right)
\]

\[
= (a - 1)! \sum_{i=0}^{a-1} i!(a - i) \binom{c}{i} \left( \frac{\lambda - b - c}{a - i} \right) \left( \frac{\lambda - c - d}{a - i} \right) \left( \frac{\lambda - a - c + i}{i} \right)
\]

\[
+ (a - 1)! \sum_{i=1}^{a} (i - 1)!(a - i) \binom{c}{i} \left( \frac{\lambda - b - c}{a - i} \right) \left( \frac{\lambda - c - d}{a - i} \right) \left( \frac{\lambda - a - c + i}{i} \right)
\]

\[
= a! \sum_{i=0}^{a} i!(a - i) \binom{c}{i} \left( \frac{\lambda - b - c}{a - i} \right) \left( \frac{\lambda - c - d}{a - i} \right) \left( \frac{\lambda - a - c + i}{i} \right). \tag{9}
\]

The result then follows. \(\square\)

For any non-negative integer \( a \) and real numbers \( p, q, n \), define

\[
F_{a,p,q,n}(z) = a! \sum_{i=0}^{a} i!(a - i) \left( \frac{a + p + q - 1}{i} \right) \left( \frac{z + n + i - 1}{i} \right) \left( \frac{z - p}{a - i} \right) \left( \frac{z - q}{a - i} \right). \tag{10}
\]

Then (8) and (10) implies that \( Q_{a,b,c,d}(z + (a + b + c + d - 1)/2) = F_{a,p,q,n}(z) \), where

\[
\begin{align*}
p &= (b + c - a - d + 1)/2 \\
q &= (c + d - a - b + 1)/2 \\
n &= (b + d - a - c + 1)/2.
\end{align*}
\tag{11}
\]

In the next section, we shall show that \( F_{a,p,q,n}(z) \) is an even polynomial in \( z \), and the polynomial obtained from \( F_{a,p,q,n}(z) \) by replacing \( z^2 \) by \( z \) (i.e., \( W_{a,p,q,n}(z) \) defined on Page 9) has only real roots for an arbitrary positive integer \( a \) and arbitrary real numbers \( p, q, n \) satisfying the condition that \( p + q, p + n \) and \( q + n \) are all non-negative (see Proposition 10). This result implies that every root of \( F_{a,p,q,n}(z) \) is either a real number or a complex number with its real part equal to 0 if \( a \) is a positive integer and \( p + q, p + n \) and \( q + n \) are all non-negative real
numbers. For arbitrary positive integers \(a, b, c, d\), if \(a \leq \min\{b, c, d\}\) and \(p, q\) and \(n\) are given in (11), then \(p + q = c - a + 1 > 0\), \(p + n = b - a + 1 > 0\) and \(q + n = d - a + 1 > 0\). Since \(Q_{a,b,c,d}(z + (a + b + c + d - 1)/2) = F_{a,p,q,n}(z)\), where \(p, q\) and \(n\) are given in (11), the following result is obtained.

**Proposition 4** For arbitrary positive integers \(a, b, c, d\), if \(a \leq \min\{b, c, d\}\), then every root of \(Q_{a,b,c,d}(z)\) is either a real number or a complex number with its real part equal to \((a + b + c + d - 1)/2\). Therefore, for arbitrary non-negative integers \(a, b, c\) and \(d\), every root of \(P(R_{a,b,c,d}(\lambda))\) is either a real number or a complex number with its real part equal to \((a + b + c + d - 1)/2\).

### 3 The polynomial \(F_{a,p,q,n}(z)\)

From the definition of \(F_{a,p,q,n}(z)\), we have \(F_{0,p,q,n}(z) = 1\) and \(F_{1,p,q,n}(z) = z^2 + pq + pn + qn\). We shall show that \(F_{a,p,q,n}(z)\) has a recursive expression in terms of \(F_{a-1,p,q,n}(z)\) and \(F_{a-2,p,q,n}(z)\). We first prove two properties of \(F_{a,p,q,n}(z)\).

**Proposition 5** For any integer \(a \geq 1\) and arbitrary real numbers \(p, q, n\), if \(p + q = 0\), then

\[
F_{a,p,q,n}(z) = (z - p)(z - q)F_{a-1,p+1,q+1,n}(z). \quad (12)
\]

**Proof.** For \(a \geq 1\),

\[
F_{a,p,q,n}(z) = a! \sum_{i=0}^{a} \frac{i!(a - i)!}{i!} \binom{a - 1}{i} \binom{z + n + i - 1}{a - i} (z - p) (z - q) (a - i) = (z - p)(z - q)(a - 1)! \sum_{i=0}^{a-1} \frac{i!(a - 1 - i)!}{i!} \binom{a}{i} \binom{z + n + i - 1}{a - 1 - i} (z - p - 1) (z - q - 1) (a - 1 - i) = (z - p)(z - q) \times F_{a-1,p+1,q+1,n}(z). \quad (13)
\]

**Proposition 6** For any integer \(a \geq 1\) and arbitrary real numbers \(p, q, n\),

\[
F_{a,p+1,q,n}(z) - F_{a,p,q,n}(z) = a(a + n + q - 1)F_{a-1,p+1,q,n}(z). \quad (14)
\]

**Proof.** For \(a \geq 1\),

\[
F_{a,p+1,q,n}(z) - F_{a,p,q,n}(z) = a! \sum_{i=0}^{a} \frac{i!(a - i)!}{i!} \binom{z + n + i - 1}{a - i} (z - q) (a - i) = \]

\[
\left(\binom{a+p+q}{i} \binom{z-p-1}{a-i} - \binom{a+p+q-1}{i} \binom{z-p}{a-i}\right) =
\]
\[
a! \sum_{i=0}^{a} i!(a-i)! \binom{z-n+i-1}{i} \binom{z-q}{a-i} - \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1}\]
\[
= a! \sum_{i=1}^{a} i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1}\]
\[
+ a! \sum_{i=0}^{a-1} i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1}\]
\[
- a! \sum_{i=1}^{a-1} (i+1)!(a-i-1)! \binom{z+n+i}{i+1} \binom{z-q}{a-i-1} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1}\]
\[
- a! \sum_{i=0}^{a-1} i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1}\]
\[
= (n+q+a-1)a! \sum_{i=0}^{a-1} i!(a-i)\binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1}\]
\[
= a(a+n+q-1)F_{a-1,p+1,q,n}(z). \tag{15}\]
\]

Now we can prove that \(F_{a,p,q,n}(z)\) can be expressed in terms of \(F_{a-1,p,q,n}(z)\) and \(F_{a-2,p,q,n}(z)\).

**Proposition 7** Let \(p, q, n\) be arbitrary real numbers. For any integer \(a \geq 2\),
\[
F_{a,p,q,n}(z) = (z^2 + (a-1)(2p + 2q + 2n + 2a - 3) + pq + pm + qn)F_{a-1,p,q,n}(z)
- (a-1)(p + q + a)(q + n + a - 2)(p + n + a - 2)F_{a-2,p,q,n}(z). \tag{16}
\]

**Proof.** By the definition of \(F_{a,p,q,n}(z)\), we have \(F_{0,p,q,n}(z) = 1\), \(F_{1,p,q,n}(z) = z^2 + pq + pm + qn\) and
\[
F_{2,p,q,n}(z) = z^4 + (2q + 2pq + 1 + 2pm + 2p + 2qn + 2)z^2 + pq^2 + pq
+ qn + q^2n + p^2q^2 + p^2n^2 + p^2q + 4pqn + pm^2 + 2p^2qn
+ pn + 2pq^2n + 2pqn^2 + qn^2 + q^2n^2 + p^2n. \tag{17}
\]
Thus it can be verified that (16) holds when \(a = 2\).

Assume that (16) holds for every integer \(2 \leq a < k\), where \(k \geq 3\). Now consider the case that \(a = k\).

By the definition of \(F_{a,p,q,n}(z)\), \(F_{a,p,q,n}(z)\) is also a polynomial of order \(a\) in \(p\). Let \(q, n, z\) be any fixed real numbers. If (16) holds for all numbers \(p\) in the set \(-q + r : r = 0, 1, 2, \cdots\), then the result is proven.
By assumption on \(a\), (16) holds for \(F_{a-1,-q+1,q+1,n}(z)\) and thus

\[
F_{a-1,-q+1,q+1,n}(z) = (z^2 - 5a + 2an + 2a^2 + 3 - 2n - q^2)F_{a-2,-q+1,q+1,n}(z) - (a - 2)(a - 1)(-q - 2 + n + a)(q - 2 + n + a)F_{a-3,-q+1,q+1,n}(z).
\]

By Proposition 5, for any integer \(m \geq 1\),

\[
F_{m,-q,q,n}(z) = (z^2 - q^2)F_{m-1,-q+1,q+1,n}(z).
\]

Hence

\[
F_{a,-q,q,n}(z) = (z^2 - 5a + 2an + 2a^2 + 3 - 2n - q^2)F_{a-1,-q,q,n}(z) - (a - 2)(a - 1)(-q - 2 + n + a)(q - 2 + n + a)F_{a-2,-q,q,n}(z),
\]

implying that (16) holds for \(F_{a,-q,q,n}(z)\).

In the remaining part of this proof, we shall show that if (16) holds for \(F_{a,p,q,n}(z)\), then (16) holds for \(F_{a,p+1,q,n}(z)\). Assume (16) holds for \(F_{a,p,q,n}(z)\), and so

\[
F_{a,p,q,n}(z) = (z^2 + (a - 1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn)F_{a-1,p,q,n}(z) - (a - 1)(p + q + a - 2)(q + n + a - 2)(p + n + a - 2)F_{a-2,p,q,n}(z). \tag{18}
\]

By assumption on \(a\), (16) holds for \(F_{a-1,p+1,q,n}(z)\) and so

\[
F_{a-1,p+1,q,n}(z) = (z^2 + (a - 2)(2p + 2q + 2n + 2a - 3) + (p + 1)(n + q) + (q + 1) + (p + q + a - 2)(q + n + a - 3)(p + n + a - 2)F_{a-3,p+1,q,n}(z).
\]

By Proposition 6, (19) and (19), we have

\[
F_{a,p+1,q,n}(z) = F_{a,p,q,n}(z) + a(a + q + n - 1)F_{a-1,p+1,q,n}(z) = (z^2 + (a - 1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn)F_{a-1,p,q,n}(z) - (a - 1)(p + q + a - 2)(q + n + a - 2)(p + n + a - 2)F_{a-2,p,q,n}(z) + a(a + q + n - 1)F_{a-1,p+1,q,n}(z) = (z^2 + (a - 1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn)(F_{a-1,p+1,q,n}(z) - (a - 1)(a + q + n - 2)F_{a-2,p+1,q,n}(z)) - (a - 1)(p + q + a - 2)(q + n + a - 2)(p + n + a - 2)F_{a-2,p+1,q,n}(z) - (a - 2)(a + q + n - 3)F_{a-3,p+1,q,n}(z) + a(a + q + n - 1)F_{a-1,p+1,q,n}(z)
\]

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Applying Proposition 8, we can get the following result.

Thus (16) holds for $F_{a,p+1,q,n}(z)$. Hence (16) holds for $F_{a,p,q,n}(z)$ for all numbers $p$ in the set \{\(q + r : r = 0, 1, 2, \ldots\)\} and therefore the result is proved. \(\square\)

Since $F_{0,p,q,n}(z) = 1$ and $F_{1,p,q,n}(z) = z^2 + pq + pn + qn$, Proposition 7 implies that $F_{a,p,q,n}(z)$ is an even polynomial in $z$. For any non-negative integer $a$ and real numbers $p, q, n$, let $W_{a,p,q,n}(z)$ be the polynomial in $z$ defined as follows: $W_{0,p,q,n}(z) = 1$, $W_{1,p,q,n}(z) = z + pq + pn + qn$ and for $a \geq 2$,

$$W_{a,p,q,n}(z) = (z + (a - 1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn)W_{a-1,p,q,n}(z) - (a - 1)(p + q + a - 2)(q + n + a - 2)(p + n + a - 2)W_{a-2,p,q,n}(z). \quad (20)$$

Thus it is clear that $F_{a,p,q,n}(z) = W_{a,p,q,n}(z^2)$.

For two non-increasing sequences \((a_1, a_2, \ldots, a_m)\) and \((b_1, b_2, \ldots, b_n)\) of real numbers, we say the first interleaves the second if $m = n + 1$ and \((a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_{n+1})\) is an non-increasing sequence, or $m = n$ and \((a_1, b_1, a_2, b_2, \ldots, a_n, b_n)\) is an non-increasing sequence. If both polynomials $f(x)$ and $g(x)$ in $x$ with real coefficients have only real roots and the non-increasing sequence formed by all roots of $f(x)$ interleaves the non-increasing sequence formed by all roots of $g(x)$, then we say $f(x)$ interleaves $g(x)$. We need to apply the following result given in Section 1.3 of [4]. More details on polynomials with only real roots can be found in [2, 4].

**Proposition 8** ([4]) \ Let $f(x)$ and $g(x)$ be polynomials with real coefficients and with positive leading coefficients and $u$ and $v$ be any real numbers. If $f(x)$ interleaves $g(x)$ and $v \leq 0$, then $(x - u)f(x) + vg(x)$ interleaves $f(x)$. \(\square\)

Applying Proposition 8, we can get the following result.

**Proposition 9** \ Let $a$ be any positive integer and $p, q, n$ be any real numbers.

(i) If $(p + q)(n + q)(n + p) \geq 0$, then $W_{2,p,q,n}(z)$ interleaves $W_{1,p,q,n}(z)$. 

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(ii) If \( a \geq 3 \), \((p + q + a - 2)(q + n + a - 2)(p + n + a - 2) \geq 0\) and \( W_{a-1,p,q,n}(z) \) interleaves \( W_{a-2,p,q,n}(z) \), then \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \).

Proof. By the definition of \( W_{a,p,q,n}(z) \), \( W_{1,p,q,n}(z) = z + pq + pn + qn \) and

\[
W_{2,p,q,n}(z) = (z + 2p + 2q + 2n + 1 + pq + pn + qn)(z + pq + pn + qn) - (p + q)(q + n)(p + n).
\]  

(21)

As the only root of \( W_{1,p,q,n}(z) \) is \(-pq - pn - qn\) and \( W_{2,p,q,n}(-pq - pn - qn) = -(p + q)(n + q)(n + p) \leq 0\), \( W_{2,p,q,n}(z) \) interleaves \( W_{1,p,q,n}(z) \). So (i) holds.

By Proposition 7,

\[
F_{a,p,q,n}(z) = (z^2 + (a - 1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn)F_{a-1,p,q,n}(z) - (a - 1)(p + q + a - 2)(q + n + a - 2)(p + n + a - 2)F_{a-2,p,q,n}(z).
\]  

(22)

Since \(-(a - 1)(p + q + a - 2)(q + n + a - 2)(p + n + a - 2) \leq 0\) and \( W_{a-1,p,q,n}(z) \) interleaves \( W_{a-2,p,q,n}(z) \), Proposition 8 implies that \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \). Hence (ii) holds.

\( \square \)

Notice that \( W_{a,p,q,n}(z) = W_{a,q,p,n}(z) = W_{a,n,q,p}(z) \) holds for arbitrary real numbers \( p, q, n \) and non-negative integer \( a \), we assume that \( p \leq q \leq n \) in the following.

Proposition 10 Let \( p, q, n \) be arbitrary real numbers with \( p \leq q \leq n \) and \( p + q \geq 0 \). Then, for every integer \( a \geq 2 \), \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \). Therefore, for every positive integer \( a \), \( W_{a,p,q,n}(z) \) has only real roots and every root of \( F_{a,p,q,n}(z) \) is either a real number or a complex number with its real part equal to 0.

Proof. Since \( p + q \geq 0 \) and \( p \leq q \leq n \), we have \( q + n \geq p + n \geq 0 \) and so Proposition 9 (i) implies that \( W_{2,p,q,n}(z) \) interleaves \( W_{1,p,q,n}(z) \). Then, by Proposition 9 (ii), \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \) for every integer \( a \geq 3 \).

\( \square \)

By the discussion immediately preceding Proposition 4, it follows that for all positive integers \( a, b, c, d \) with \( a \leq \min\{b, c, d\} \), the hypotheses of Proposition 10 are satisfied and hence we have proved Theorem 1.

4 Further properties of \( F_{a,p,q,n} \) and \( W_{a,p,q,n} \)

Even if \( p + q < 0 \), there are still some situations in which \( W_{a,p,q,n}(z) \) has only real roots. In this section we consider these, although they do not correspond to values of the parameters \( a, p, q \) and \( n \) that arise from rings of cliques. We need to apply the following result on the factorization of \( F_{a,p,q,n}(z) \) when \( a + p + n = 1 \) or \( a + p + n = 2 \).
Proposition 11 Let $a$ be an integer with $a \geq 1$ and $p, q, n$ be arbitrary real numbers.

(i) If $a + p + n = 1$, then

$$F_{a,p,q,n}(z) = \prod_{j=0}^{a-1} (z^2 - (n + j)^2).$$

(ii) If $a + p + n = 2$, then

$$F_{a,p,q,n}(z) = (z^2 + (p-1)(n-1) + aq) \prod_{j=0}^{a-2} (z^2 - (n + j)^2).$$

Proof. (i) If $a + p + n = 1$, then

$$i!(a - i)! \binom{z - p}{a - i} \binom{z + n + i - 1}{i} \prod_{j=0}^{a-1} (z + n + j).$$

Thus

$$F_{a,p,q,n}(z) = \frac{a!}{a} \sum_{i=0}^{a} i!(a - i)! \binom{a + p + q - 1}{i} \binom{z - p}{a - i} \binom{z - q}{a - i} \binom{z + n + i - 1}{i} \prod_{j=0}^{a-1} (z + n + j).$$

(ii) Now let $a + p + n = 2$. Since $F_{1,p,q,n}(z) = z^2 + pq + pn + qn$, it is easy to verify that (ii) holds when $a = 1$. Assume that (ii) holds for any integer $1 \leq a < k$, where $k \geq 2$. Now let $a = k$.

Since $a + p + n = 2$, by Proposition 7,

$$F_{a,p,q,n}(z) = (z^2 + (a - 1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn)F_{a-1,p,q,n}(z).$$

As $a - 1 + p + n = 1$, by (i) of this result, we have

$$F_{a-1,p,q,n}(z) = \prod_{j=0}^{a-2} (z^2 - (n + j)^2).$$
Since \( p + n + a = 2 \), it can be verified that

\[
(a - 1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn = (p - 1)(n - 1) + aq.
\]

Hence (ii) also holds.

\[ \square \]

**Proposition 12**  
Let \( p, q, n \) be arbitrary real numbers with \( p \leq q \leq n \).

(i) If \( p + q \) is a negative integer, then for every integer \( a \) with \( a \geq 2 - p - q \), \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \).

(ii) If \( q + n \) is an integer, then for every integer \( a \) with \( \max\{2, 2 - q - n\} \leq a \leq 2 - p - n \), \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \).

**Proof.**  
(i) First consider the case that \( a = 2 - p - q \). Since \( p + q \leq -1 \), we have \( a \geq 3 \). Proposition 11 implies that \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \).

Now assume that \( a > 2 - p - q \) and \( W_{a-1,p,q,n}(z) \) interleaves \( W_{a-2,p,q,n}(z) \). Since \( a > 2 - p - q \), we have \( a + p + q - 2 \geq 1 \) and so \( a + q + n - 2 \geq a + p + n - 2 \geq a + p + q - 2 \geq 1 \). Thus Proposition 9 (ii) implies that \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \). Therefore (i) holds.

(ii) The result is trivial if \( \max\{2, 2 - q - n\} > 2 - p - n \). Now assume that \( \max\{2, 2 - q - n\} \leq 2 - p - n \).

Let \( a = \max\{2, 2 - q - n\} \). Then \( a \geq 2 - q - n \), implying that \( a + q + n - 2 \geq 0 \). We also have \( a \leq 2 - p - n \), implying that \( a + p + n - 2 \leq 0 \) and so \( a + p + q - 2 \leq 0 \). If \( a = \max\{2, 2 - q - n\} = 2 \), then Proposition 9 (i) implies that \( W_{2,p,q,n}(z) \) interleaves \( W_{1,p,q,n}(z) \), i.e., \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \). If \( a = \max\{2, 2 - q - n\} = 2 - q - n \), then Proposition 11 implies that \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \).

Now assume that \( \max\{2, 2 - q - n\} < a \leq 2 - p - n \) and \( W_{a-1,p,q,n}(z) \) interleaves \( W_{a-2,p,q,n}(z) \). Note that \( \max\{2, 2 - q - n\} < a \leq 2 - p - n \) implies that \( a + q + n - 2 > 0 \) and \( a + p + q - 2 \leq a + p + n - 2 \leq 0 \). Thus Proposition 9 (ii) implies that \( W_{a,p,q,n}(z) \) interleaves \( W_{a-1,p,q,n}(z) \). Therefore (ii) holds.

By Proposition 12, the following result is obtained.

**Proposition 13**  
Let \( a \) be a positive integer and \( p, q, n \) be arbitrary real numbers with \( p \leq q \leq n \). If one of the following conditions holds, then \( W_{a,p,q,n}(z) \) has only real roots and therefore every root of \( F_{a,p,q,n}(z) \) is either a real number or a complex number with its real part equal to 0:

(i) \( p + q \) is a negative integer and \( a \geq 1 - p - q \); and
(ii) $q + n$ is an integer and $\max\{1, 1 - q - n\} \leq a \leq 2 - p - n.$

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References


