

# Chromatic Roots of a Ring of Four Cliques

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## Abstract

For any positive integers  $a, b, c, d$ , let  $R_{a,b,c,d}$  be the graph obtained from the complete graphs  $K_a, K_b, K_c$  and  $K_d$  by adding edges joining every vertex in  $K_a$  and  $K_c$  to every vertex in  $K_b$  and  $K_d$ . This paper shows that for arbitrary positive integers  $a, b, c$  and  $d$ , every root of the chromatic polynomial of  $R_{a,b,c,d}$  is either a real number or a complex number with its real part equal to  $(a + b + c + d - 1)/2$ .

**Keywords:** graph, chromatic polynomial, chromatic root, ring of cliques

## 1 Introduction

A *ring of cliques* is a graph whose vertex set is the disjoint union of cliques, arranged in a cyclic order, such that the vertices of each clique are joined to all the vertices in the two neighbouring cliques. If the cliques have size  $a_1, a_2, \dots, a_n$  then we denote this graph by  $R_{a_1, a_2, \dots, a_n}$ . Figure 1 shows the graph  $R_{2,2,3,3}$ .

Graphs with this structure have occurred several times previously in the study of chromatic polynomials and their roots. In particular, in proving that there are non-chordal graphs with integer chromatic roots, Read [6] considered the graphs in this family with  $a_1 = 1$  (and he also used slightly different notation). Rings of cliques cropped up again recently in a preliminary investigation of the *algebraic properties* of chromatic roots (Cameron [1]) and in the course

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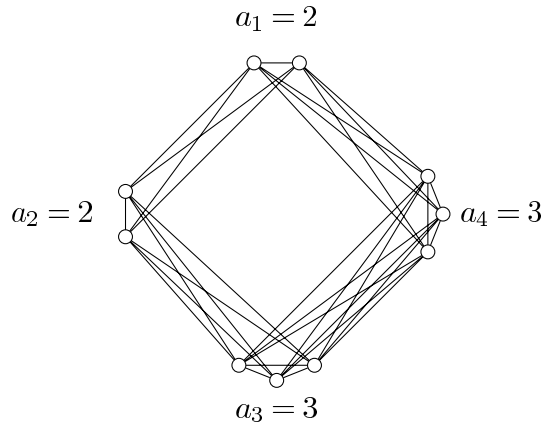


Figure 1: The graph  $R_{2,2,3,3}$

of this investigation, the chromatic roots of many of these graphs were computed. When the chromatic roots of the ring-of-clique graphs with *exactly four cliques* and a fixed number of vertices were plotted, an intriguing pattern was observed — all the complex chromatic roots lie on a single vertical line. Figure 2 shows the union of the chromatic roots of the 12-vertex graphs of the form  $R_{a,b,c,d}$ .

Faced with such a striking empirically-observed pattern, we were led to explain it theoretically. This appears to require a surprisingly intricate argument, but eventually we obtain the following result:

**Theorem 1** *For arbitrary non-negative integers  $a, b, c$  and  $d$  the chromatic roots of  $R_{a,b,c,d}$  are either real, or complex with real part equal to  $(a + b + c + d - 1)/2$ .*

The overall structure of the argument is as follows: the chromatic polynomial  $P(R_{a,b,c,d}, \lambda)$  is first expressed as the product of linear factors and a factor  $Q_{a,b,c,d}(\lambda)$ . It then suffices to show that the complex roots of  $Q_{a,b,c,d}(\lambda)$  all lie on the vertical line  $\Re(\lambda) = (a + b + c + d - 1)/2$  in the complex  $\lambda$ -plane. Next the polynomial  $F_{a,p,q,n}(z)$  is defined to be  $Q_{a,b,c,d}(z + (a + b + c + d - 1)/2)$  thus translating the vertical line supposed to contain the roots to the imaginary axis and also reparameterizing the problem (in a somewhat counterintuitive way). Then  $F_{a,p,q,n}$  is shown to be an *even* polynomial and we define a fourth polynomial  $W_{a,p,q,n}$  by  $W_{a,p,q,n}(z^2) = F_{a,p,q,n}(z)$ . The proof is completed by demonstrating that  $W_{a,p,q,n}$  is real-rooted using polynomial interleaving techniques, and therefore  $F_{a,p,q,n}$  has only real or pure imaginary roots as required.

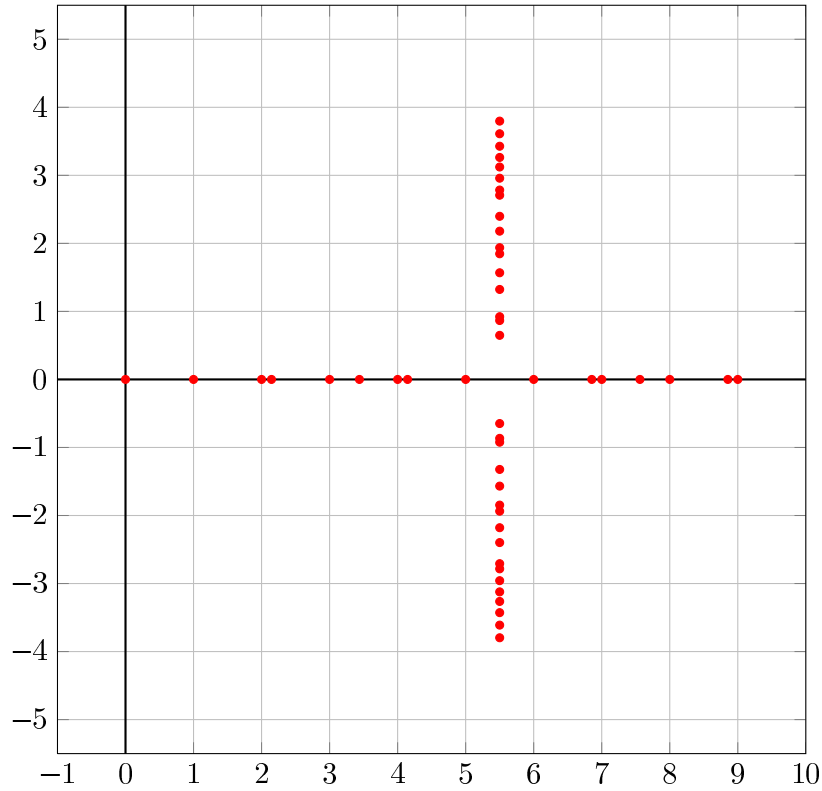


Figure 2: Chromatic roots of the graphs  $R_{a,b,c,d}$  where  $a + b + c + d = 12$ .

## 2 Basics

For any graph  $G$  and any positive integer  $\lambda$ , let  $P(G, \lambda)$  be the number of mappings  $\phi$  from  $V(G)$  to  $\{1, 2, \dots, \lambda\}$  such that  $\phi(u) \neq \phi(v)$  for every two adjacent vertices  $u$  and  $v$  in  $G$ . It is well-known that  $P(G, \lambda)$  is a polynomial in  $\lambda$ , called *the chromatic polynomial* of  $G$ .

The chromatic polynomial of a graph  $G$  has the following properties (see, for instance, [3, 5, 7, 8]), which we will apply later.

**Proposition 1** *Let  $G$  be a simple graph of order at least 2.*

(i) *If  $u$  and  $v$  are two non-adjacent vertices in  $G$ , then*

$$P(G, \lambda) = P(G + uv, \lambda) + P(G/uv, \lambda), \quad (1)$$

*where  $G + uv$  is the graph obtained from  $G$  by adding the edge joining  $u$  and  $v$ , and  $G/uv$  is the graph obtained from  $G$  by identifying  $u$  and  $v$  and removing all parallel edges but one.*

(ii) *If  $u$  is a vertex in  $G$  which is adjacent to all other vertices in  $G$ , then*

$$P(G, \lambda) = \lambda P(G - u, \lambda - 1), \quad (2)$$

where  $G - u$  is the graph obtained from  $G$  by removing  $u$ .

If  $a = 0$ ,  $R_{a,b,c,d}$  is a chordal graph and its chromatic polynomial is

$$P(R_{0,b,c,d}, \lambda) = \frac{(\lambda)_{b+c}(\lambda)_{c+d}}{(\lambda)_c}, \quad (3)$$

and if  $a \geq 1$  and  $c \geq 1$ , then applying Proposition 1 repeatedly yields that

$$P(R_{a,b,c,d}, \lambda) = \lambda P(R_{a-1,b,c,d}, \lambda - 1) + c\lambda P(R_{a-1,b,c-1,d}, \lambda - 1). \quad (4)$$

For a non-negative integer  $a$  and real numbers  $b, c$  and  $d$ , define a polynomial  $Q_{a,b,c,d}(z)$  in  $z$  as follows:  $Q_{0,b,c,d}(z) = 1$  and for  $a \geq 1$ ,

$$Q_{a,b,c,d}(z) = (z - b - c)(z - c - d)Q_{a-1,b,c,d}(z - 1) + c(z - a - c + 1)Q_{a-1,b,c-1,d}(z - 1). \quad (5)$$

It is clear that  $Q_{a,b,c,d}(z)$  is a polynomial of order  $2a$  in  $z$ .

**Proposition 2** *Let  $a, b, c$  and  $d$  be any non-negative integers. Then*

$$P(R_{a,b,c,d}, \lambda) = \frac{(\lambda)_{b+c}(\lambda)_{c+d}}{(\lambda)_{a+c}} Q_{a,b,c,d}(\lambda). \quad (6)$$

*Proof.* If  $a = 0$ , then (6) follows from (3) and the definition of  $Q_{a,b,c,d}(\lambda)$ . Now assume that  $a \geq 1$ . By (4) and induction, we have

$$\begin{aligned} P(R_{a,b,c,d}, \lambda) &= \lambda P(R_{a-1,b,c,d}, \lambda - 1) + c\lambda P(R_{a-1,b,c-1,d}, \lambda - 1) \\ &= \lambda \frac{(\lambda - 1)_{b+c}(\lambda - 1)_{c+d}}{(\lambda - 1)_{a+c-1}} Q_{a-1,b,c,d}(\lambda - 1) \\ &\quad + c\lambda \frac{(\lambda - 1)_{b+c-1}(\lambda - 1)_{c+d-1}}{(\lambda - 1)_{a+c-2}} Q_{a-1,b,c-1,d}(\lambda - 1) \\ &= \frac{(\lambda)_{b+c}(\lambda)_{c+d}}{(\lambda)_{a+c}} [(\lambda - b - c)(\lambda - c - d)Q_{a-1,b,c,d}(\lambda - 1) \\ &\quad + c(\lambda - a - c + 1)Q_{a-1,b,c-1,d}(\lambda - 1)]. \end{aligned} \quad (7)$$

The result then follows.  $\square$

Assume that  $\binom{x}{0} = 1$  and  $\binom{x}{r} = x(x-1)\cdots(x-r+1)/r!$  for any positive integer  $r$  and any complex number  $x$ .

**Proposition 3** *For any non-negative integer  $a$  and real numbers  $b, c$  and  $d$ ,*

$$Q_{a,b,c,d}(\lambda) = a! \sum_{i=0}^a i!(a-i)! \binom{c}{i} \binom{\lambda - b - c}{a-i} \binom{\lambda - c - d}{a-i} \binom{\lambda - a - c + i}{i}. \quad (8)$$

*Proof.* It is trivial if  $a = 0$  as  $Q_{0,b,c,d}(z) = 1$ . Now assume that  $a \geq 1$ . By (5) and induction,

$$\begin{aligned}
& Q_{a,b,c,d}(\lambda) \\
&= (\lambda - b - c)(\lambda - c - d)Q_{a-1,b,c,d}(\lambda - 1) + c(\lambda - a - c + 1)Q_{a-1,b,c-1,d}(\lambda - 1) \\
&= (\lambda - b - c)(\lambda - c - d)(a - 1)! \sum_{i=0}^{a-1} \left\{ i!(a - i - 1)! \binom{c}{i} \binom{\lambda - b - c - 1}{a - i - 1} \right. \\
&\quad \left. \binom{\lambda - c - d - 1}{a - i - 1} \binom{\lambda - a - c + i}{i} \right\} \\
&\quad + c(\lambda - a - c + 1)(a - 1)! \sum_{i=0}^{a-1} \left\{ i!(a - i - 1)! \binom{c - 1}{i} \binom{\lambda - b - c}{a - i - 1} \right. \\
&\quad \left. \binom{\lambda - c - d}{a - i - 1} \binom{\lambda - a - c + i + 1}{i} \right\} \\
&= (a - 1)! \sum_{i=0}^{a-1} i!(a - i)!(a - i) \binom{c}{i} \binom{\lambda - b - c}{a - i} \binom{\lambda - c - d}{a - i} \binom{\lambda - a - c + i}{i} \\
&\quad + (a - 1)! \sum_{i=0}^{a-1} i!(a - i - 1)!(i + 1)^2 \binom{c}{i + 1} \binom{\lambda - b - c}{a - i - 1} \binom{\lambda - c - d}{a - i - 1} \binom{\lambda - a - c + i + 1}{i + 1} \\
&= (a - 1)! \sum_{i=0}^{a-1} i!(a - i)!(a - i) \binom{c}{i} \binom{\lambda - b - c}{a - i} \binom{\lambda - c - d}{a - i} \binom{\lambda - a - c + i}{i} \\
&\quad + (a - 1)! \sum_{i=1}^a (i - 1)!(a - i)! i^2 \binom{c}{i} \binom{\lambda - b - c}{a - i} \binom{\lambda - c - d}{a - i} \binom{\lambda - a - c + i}{i} \\
&= a! \sum_{i=0}^a i!(a - i)! \binom{c}{i} \binom{\lambda - b - c}{a - i} \binom{\lambda - c - d}{a - i} \binom{\lambda - a - c + i}{i}. \tag{9}
\end{aligned}$$

The result then follows.  $\square$

For any non-negative integer  $a$  and real numbers  $p, q, n$ , define

$$F_{a,p,q,n}(z) = a! \sum_{i=0}^a i!(a - i)! \binom{a + p + q - 1}{i} \binom{z + n + i - 1}{i} \binom{z - p}{a - i} \binom{z - q}{a - i}. \tag{10}$$

Then (8) and (10) implies that  $Q_{a,b,c,d}(z + (a + b + c + d - 1)/2) = F_{a,p,q,n}(z)$ , where

$$\begin{cases} p = (b + c - a - d + 1)/2 \\ q = (c + d - a - b + 1)/2 \\ n = (b + d - a - c + 1)/2. \end{cases} \tag{11}$$

In the next section, we shall show that  $F_{a,p,q,n}(z)$  is an even polynomial in  $z$ , and the polynomial obtained from  $F_{a,p,q,n}(z)$  by replacing  $z^2$  by  $z$  (i.e.,  $W_{a,p,q,n}(z)$  defined on Page 9) has only real roots for an arbitrary positive integer  $a$  and arbitrary real numbers  $p, q, n$  satisfying the condition that  $p + q, p + n$  and  $q + n$  are all non-negative (see Proposition 10). This result implies that every root of  $F_{a,p,q,n}(z)$  is either a real number or a complex number with its real part equal to 0 if  $a$  is a positive integer and  $p + q, p + n$  and  $q + n$  are all non-negative real

numbers. For arbitrary positive integers  $a, b, c, d$ , if  $a \leq \min\{b, c, d\}$  and  $p, q$  and  $n$  are given in (11), then  $p + q = c - a + 1 > 0$ ,  $p + n = b - a + 1 > 0$  and  $q + n = d - a + 1 > 0$ . Since  $Q_{a,b,c,d}(z + (a + b + c + d - 1)/2) = F_{a,p,q,n}(z)$ , where  $p, q$  and  $n$  are given in (11), the following result is obtained.

**Proposition 4** *For arbitrary positive integers  $a, b, c$  and  $d$ , if  $a \leq \min\{b, c, d\}$ , then every root of  $Q_{a,b,c,d}(z)$  is either a real number or a complex number with its real part equal to  $(a + b + c + d - 1)/2$ . Therefore, for arbitrary non-negative integers  $a, b, c$  and  $d$ , every root of  $P(R_{a,b,c,d}, \lambda)$  is either a real number or a complex number with its real part equal to  $(a + b + c + d - 1)/2$ .  $\square$*

### 3 The polynomial $F_{a,p,q,n}(z)$

From the definition of  $F_{a,p,q,n}(z)$ , we have  $F_{0,p,q,n}(z) = 1$  and  $F_{1,p,q,n}(z) = z^2 + pq + pn + qn$ . We shall show that  $F_{a,p,q,n}(z)$  has a recursive expression in terms of  $F_{a-1,p,q,n}(z)$  and  $F_{a-2,p,q,n}(z)$ . We first prove two properties of  $F_{a,p,q,n}(z)$ .

**Proposition 5** *For any integer  $a \geq 1$  and arbitrary real numbers  $p, q, n$ , if  $p + q = 0$ , then*

$$F_{a,p,q,n}(z) = (z - p)(z - q)F_{a-1,p+1,q+1,n}(z). \quad (12)$$

*Proof.* For  $a \geq 1$ ,

$$\begin{aligned} F_{a,p,q,n}(z) &= a! \sum_{i=0}^a i!(a-i)! \binom{a-1}{i} \binom{z+n+i-1}{i} \binom{z-p}{a-i} \binom{z-q}{a-i} \\ &= (z-p)(z-q)(a-1)! \\ &\quad \sum_{i=0}^{a-1} i!(a-1-i)! \binom{a}{i} \binom{z+n+i-1}{i} \binom{z-p-1}{a-1-i} \binom{z-q-1}{a-1-i} \\ &= (z-p)(z-q) \times F_{a-1,p+1,q+1,n}(z). \end{aligned} \quad (13)$$

$\square$

**Proposition 6** *For any integer  $a \geq 1$  and arbitrary real numbers  $p, q, n$ ,*

$$F_{a,p+1,q,n}(z) - F_{a,p,q,n}(z) = a(a+n+q-1)F_{a-1,p+1,q,n}(z). \quad (14)$$

*Proof.* For  $a \geq 1$ ,

$$\begin{aligned} &F_{a,p+1,q,n}(z) - F_{a,p,q,n}(z) \\ &= a! \sum_{i=0}^a \left\{ i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \right. \end{aligned}$$

$$\begin{aligned}
& \left( \binom{a+p+q}{i} \binom{z-p-1}{a-i} - \binom{a+p+q-1}{i} \binom{z-p}{a-i} \right) \Big\} \\
= & a! \sum_{i=0}^a \left\{ i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \right. \\
& \left. \left( \binom{a+p+q-1}{i-1} \binom{z-p-1}{a-i} - \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \right) \right\} \\
= & a! \sum_{i=1}^a i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i-1} \binom{z-p-1}{a-i} \\
& - a! \sum_{i=0}^{a-1} i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \\
= & a! \sum_{i=0}^{a-1} (i+1)!(a-i-1)! \binom{z+n+i}{i+1} \binom{z-q}{a-i-1} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \\
& - a! \sum_{i=0}^{a-1} i!(a-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \\
= & (n+q+a-1)a! \\
& \sum_{i=0}^{a-1} i!(a-1-i)! \binom{z+n+i-1}{i} \binom{z-q}{a-1-i} \binom{a+p+q-1}{i} \binom{z-p-1}{a-i-1} \\
= & a(a+n+q-1)F_{a-1,p+1,q,n}(z). \tag{15}
\end{aligned}$$

□

Now we can prove that  $F_{a,p,q,n}(z)$  can be expressed in terms of  $F_{a-1,p,q,n}(z)$  and  $F_{a-2,p,q,n}(z)$ .

**Proposition 7** *Let  $p, q, n$  be arbitrary real numbers. For any integer  $a \geq 2$ ,*

$$\begin{aligned}
F_{a,p,q,n}(z) = & (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn)F_{a-1,p,q,n}(z) \\
& - (a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)F_{a-2,p,q,n}(z). \tag{16}
\end{aligned}$$

*Proof.* By the definition of  $F_{a,p,q,n}(z)$ , we have  $F_{0,p,q,n}(z) = 1$ ,  $F_{1,p,q,n}(z) = z^2 + pq + pn + qn$  and

$$\begin{aligned}
F_{2,p,q,n}(z) = & z^4 + (2q + 2pq + 1 + 2pn + 2p + 2qn + 2n)z^2 + pq^2 + pq \\
& + qn + q^2n + p^2q^2 + p^2n^2 + p^2q + 4pqn + pn^2 + 2p^2qn \\
& + pn + 2pq^2n + 2pqn^2 + qn^2 + q^2n^2 + p^2n. \tag{17}
\end{aligned}$$

Thus it can be verified that (16) holds when  $a = 2$ .

Assume that (16) holds for every integer  $2 \leq a < k$ , where  $k \geq 3$ . Now consider the case that  $a = k$ .

By the definition of  $F_{a,p,q,n}(z)$ ,  $F_{a,p,q,n}(z)$  is also a polynomial of order  $a$  in  $p$ . Let  $q, n, z$  be any fixed real numbers. If (16) holds for all numbers  $p$  in the set  $\{-q+r : r = 0, 1, 2, \dots\}$ , then the result is proven.

By assumption on  $a$ , (16) holds for  $F_{a-1,-q+1,q+1,n}(z)$  and thus

$$\begin{aligned} & F_{a-1,-q+1,q+1,n}(z) \\ = & (z^2 - 5a + 2an + 2a^2 + 3 - 2n - q^2)F_{a-2,-q+1,q+1,n}(z) \\ & - (a-2)(a-1)(-q-2+n+a)(q-2+n+a)F_{a-3,-q+1,q+1,n}(z). \end{aligned}$$

By Proposition 5, for any integer  $m \geq 1$ ,

$$F_{m,-q,q,n}(z) = (z^2 - q^2)F_{m-1,-q+1,q+1,n}(z).$$

Hence

$$\begin{aligned} F_{a,-q,q,n}(z) &= (z^2 - 5a + 2an + 2a^2 + 3 - 2n - q^2)F_{a-1,-q,q,n}(z) \\ &\quad - (a-2)(a-1)(-q-2+n+a)(q-2+n+a)F_{a-2,-q,q,n}(z), \end{aligned}$$

implying that (16) holds for  $F_{a,-q,q,n}(z)$ .

In the remaining part of this proof, we shall show that if (16) holds for  $F_{a,p,q,n}(z)$ , then (16) holds for  $F_{a,p+1,q,n}(z)$ . Assume (16) holds for  $F_{a,p,q,n}(z)$ , and so

$$\begin{aligned} F_{a,p,q,n}(z) &= (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn)F_{a-1,p,q,n}(z) \\ &\quad - (a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)F_{a-2,p,q,n}(z). \end{aligned} \quad (18)$$

By assumption on  $a$ , (16) holds for  $F_{a-1,p+1,q,n}(z)$  and so

$$\begin{aligned} & F_{a-1,p+1,q,n}(z) \\ = & (z^2 + (a-2)(2p+2q+2n+2a-3) + (p+1)(n+q) + qn)F_{a-2,p+1,q,n}(z) \\ & - (a-2)(p+q+a-2)(q+n+a-3)(p+n+a-2)F_{a-3,p+1,q,n}(z). \end{aligned} \quad (19)$$

By Proposition 6, (18) and (19), we have

$$\begin{aligned} & F_{a,p+1,q,n}(z) \\ = & F_{a,p,q,n}(z) + a(a+q+n-1)F_{a-1,p+1,q,n}(z) \\ = & (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn)F_{a-1,p,q,n}(z) \\ & - (a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)F_{a-2,p,q,n}(z) \\ & + a(a+q+n-1)F_{a-1,p+1,q,n}(z) \\ = & (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn) \\ & (F_{a-1,p+1,q,n}(z) - (a-1)(a+q+n-2)F_{a-2,p+1,q,n}(z)) \\ & - (a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2) \\ & (F_{a-2,p+1,q,n}(z) - (a-2)(a+q+n-3)F_{a-3,p+1,q,n}(z)) \\ & + a(a+q+n-1)F_{a-1,p+1,q,n}(z) \end{aligned}$$



$$\begin{aligned}
&= (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn) \\
&\quad (F_{a-1,p+1,q,n}(z) - (a-1)(a+q+n-2)F_{a-2,p+1,q,n}(z)) \\
&\quad - (a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)F_{a-2,p+1,q,n}(z) \\
&\quad + (a-1)(a+q+n-2)(-F_{a-1,p+1,q,n}(z) + \\
&\quad (z^2 + (a-2)(2p+2q+2n+2a-3) + (p+1)(n+q) + qn)F_{a-2,p+1,q,n}(z)) \\
&\quad + a(a+q+n-1)F_{a-1,p+1,q,n}(z) \\
&= (z^2 + (a-1)(2p+2q+2n+2a-1) + (p+1)(n+q) + qn)F_{a-1,p+1,q,n}(z) \\
&\quad - (a-1)(p+q+a-1)(q+n+a-2)(p+n+a-1)F_{a-2,p+1,q,n}(z).
\end{aligned}$$

Thus (16) holds for  $F_{a,p+1,q,n}(z)$ . Hence (16) holds for  $F_{a,p,q,n}(z)$  for all numbers  $p$  in the set  $\{q+r : r=0,1,2,\dots\}$  and therefore the result is proved.  $\square$

Since  $F_{0,p,q,n}(z) = 1$  and  $F_{1,p,q,n}(z) = z^2 + pq + pn + qn$ , Proposition 7 implies that  $F_{a,p,q,n}(z)$  is an even polynomial in  $z$ . For any non-negative integer  $a$  and real numbers  $p, q, n$ , let  $W_{a,p,q,n}(z)$  be the polynomial in  $z$  defined as follows:  $W_{0,p,q,n}(z) = 1$ ,  $W_{1,p,q,n}(z) = z + pq + pn + qn$  and for  $a \geq 2$ ,

$$\begin{aligned}
W_{a,p,q,n}(z) &= (z + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn)W_{a-1,p,q,n}(z) \\
&\quad - (a-1)(p+q+a-2)(q+n+a-2)(p+n+a-2)W_{a-2,p,q,n}(z). \quad (20)
\end{aligned}$$

Thus it is clear that  $F_{a,p,q,n}(z) = W_{a,p,q,n}(z^2)$ .

For two non-increasing sequences  $(a_1, a_2, \dots, a_m)$  and  $(b_1, b_2, \dots, b_n)$  of real numbers, we say the first *interleaves* the second if  $m = n+1$  and  $(a_1, b_1, a_2, b_2, \dots, a_n, b_n, a_{n+1})$  is a non-increasing sequence, or  $m = n$  and  $(a_1, b_1, a_2, b_2, \dots, a_n, b_n)$  is a non-increasing sequence. If both polynomials  $f(x)$  and  $g(x)$  in  $x$  with real coefficients have only real roots and the non-increasing sequence formed by all roots of  $f(x)$  interleaves the non-increasing sequence formed by all roots of  $g(x)$ , then we say  $f(x)$  *interleaves*  $g(x)$ . We need to apply the following result given in Section 1.3 of [4]. More details on polynomials with only real roots can be found in [2, 4].

**Proposition 8 ([4])** *Let  $f(x)$  and  $g(x)$  be polynomials with real coefficients and with positive leading coefficients and  $u$  and  $v$  be any real numbers. If  $f(x)$  interleaves  $g(x)$  and  $v \leq 0$ , then  $(x-u)f(x) + vg(x)$  interleaves  $f(x)$ .  $\square$*

Applying Proposition 8, we can get the following result.

**Proposition 9** *Let  $a$  be any positive integer and  $p, q, n$  be any real numbers.*

- (i) *If  $(p+q)(n+q)(n+p) \geq 0$ , then  $W_{2,p,q,n}(z)$  interleaves  $W_{1,p,q,n}(z)$ .*

- (ii) If  $a \geq 3$ ,  $(p + q + a - 2)(q + n + a - 2)(p + n + a - 2) \geq 0$  and  $W_{a-1,p,q,n}(z)$  interleaves  $W_{a-2,p,q,n}(z)$ , then  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ .

*Proof.* By the definition of  $W_{a,p,q,n}(z)$ ,  $W_{1,p,q,n}(z) = z + pq + pn + qn$  and

$$\begin{aligned} W_{2,p,q,n}(z) &= (z + 2p + 2q + 2n + 1 + pq + pn + qn)(z + pq + pn + qn) \\ &\quad - (p + q)(q + n)(p + n). \end{aligned} \quad (21)$$

As the only root of  $W_{1,p,q,n}(z)$  is  $-pq - pn - qn$  and  $W_{2,p,q,n}(-pq - pn - qn) = -(p + q)(n + q)(n + p) \leq 0$ ,  $W_{2,p,q,n}(z)$  interleaves  $W_{1,p,q,n}(z)$ . So (i) holds.

By Proposition 7,

$$\begin{aligned} F_{a,p,q,n}(z) &= (z^2 + (a - 1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn)F_{a-1,p,q,n}(z) \\ &\quad - (a - 1)(p + q + a - 2)(q + n + a - 2)(p + n + a - 2)F_{a-2,p,q,n}(z). \end{aligned} \quad (22)$$

Since  $-(a - 1)(p + q + a - 2)(q + n + a - 2)(p + n + a - 2) \leq 0$  and  $W_{a-1,p,q,n}(z)$  interleaves  $W_{a-2,p,q,n}(z)$ , Proposition 8 implies that  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ . Hence (ii) holds.  $\square$

Notice that  $W_{a,p,q,n}(z) = W_{a,q,p,n}(z) = W_{a,n,q,p}(z)$  holds for arbitrary real numbers  $p, q, n$  and non-negative integer  $a$ , we assume that  $p \leq q \leq n$  in the following.

**Proposition 10** *Let  $p, q, n$  be arbitrary real numbers with  $p \leq q \leq n$  and  $p + q \geq 0$ . Then, for every integer  $a \geq 2$ ,  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ . Therefore, for every positive integer  $a$ ,  $W_{a,p,q,n}(z)$  has only real roots and every root of  $F_{a,p,q,n}(z)$  is either a real number or a complex number with its real part equal to 0.*

*Proof.* Since  $p + q \geq 0$  and  $p \leq q \leq n$ , we have  $q + n \geq p + n \geq 0$  and so Proposition 9 (i) implies that  $W_{2,p,q,n}(z)$  interleaves  $W_{1,p,q,n}(z)$ . Then, by Proposition 9 (ii),  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$  for every integer  $a \geq 3$ .  $\square$

By the discussion immediately preceding Proposition 4, it follows that for all positive integers  $a, b, c, d$  with  $a \leq \min\{b, c, d\}$ , the hypotheses of Proposition 10 are satisfied and hence we have proved Theorem 1.

## 4 Further properties of $F_{a,p,q,n}$ and $W_{a,p,q,n}$

Even if  $p + q < 0$ , there are still some situations in which  $W_{a,p,q,n}(z)$  has only real roots. In this section we consider these, although they do not correspond to values of the parameters  $a, p, q$  and  $n$  that arise from rings of cliques. We need to apply the following result on the factorization of  $F_{a,p,q,n}(z)$  when  $a + p + n = 1$  or  $a + p + n = 2$ .

**Proposition 11** Let  $a$  be an integer with  $a \geq 1$  and  $p, q, n$  be arbitrary real numbers.

(i) If  $a + p + n = 1$ , then

$$F_{a,p,q,n}(z) = \prod_{j=0}^{a-1} (z^2 - (n+j)^2). \quad (23)$$

(ii) If  $a + p + n = 2$ , then

$$F_{a,p,q,n}(z) = (z^2 + (p-1)(n-1) + aq) \prod_{j=0}^{a-2} (z^2 - (n+j)^2). \quad (24)$$

*Proof.* (i) If  $a + p + n = 1$ , then

$$i!(a-i)! \binom{z-p}{a-i} \binom{z+n+i-1}{i} = \prod_{j=0}^{a-1} (z+n+j).$$

Thus

$$\begin{aligned} F_{a,p,q,n}(z) &= a! \sum_{i=0}^a i!(a-i)! \binom{a+p+q-1}{i} \binom{z-p}{a-i} \binom{z-q}{a-i} \binom{z+n+i-1}{i} \\ &= a! \prod_{j=0}^{a-1} (z+n+j) \sum_{i=0}^a \binom{a+p+q-1}{i} \binom{z-q}{a-i} \\ &= a! \prod_{j=0}^{a-1} (z+n+j) \binom{a+p+q-1+z-q}{a} \\ &= a! \prod_{j=0}^{a-1} (z+n+j) \binom{z-n}{a} \\ &= \prod_{j=0}^{a-1} (z^2 - (n+j)^2). \end{aligned} \quad (25)$$

Thus (i) holds.

(ii) Now let  $a + p + n = 2$ . Since  $F_{1,p,q,n}(z) = z^2 + pq + pn + qn$ , it is easy to verify that (ii) holds when  $a = 1$ . Assume that (ii) holds for any integer  $1 \leq a < k$ , where  $k \geq 2$ . Now let  $a = k$ .

Since  $a + p + n = 2$ , by Proposition 7,

$$F_{a,p,q,n}(z) = (z^2 + (a-1)(2p+2q+2n+2a-3) + pq + pn + qn) F_{a-1,p,q,n}(z).$$

As  $a-1 + p + n = 1$ , by (i) of this result, we have

$$F_{a-1,p,q,n}(z) = \prod_{j=0}^{a-2} (z^2 - (n+j)^2).$$

Since  $p + n + a = 2$ , it can be verified that

$$(a - 1)(2p + 2q + 2n + 2a - 3) + pq + pn + qn = (p - 1)(n - 1) + aq.$$

Hence (ii) also holds.  $\square$

**Proposition 12** *Let  $p, q, n$  be arbitrary real numbers with  $p \leq q \leq n$ .*

- (i) *If  $p + q$  is a negative integer, then for every integer  $a$  with  $a \geq 2 - p - q$ ,  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ .*
- (ii) *If  $q + n$  is an integer, then for every integer  $a$  with  $\max\{2, 2 - q - n\} \leq a \leq 2 - p - n$ ,  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ .*

*Proof.* (i) First consider the case that  $a = 2 - p - q$ . Since  $p + q \leq -1$ , we have  $a \geq 3$ . Proposition 11 implies that  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ .

Now assume that  $a > 2 - p - q$  and  $W_{a-1,p,q,n}(z)$  interleaves  $W_{a-2,p,q,n}(z)$ . Since  $a > 2 - p - q$ , we have  $a + p + q - 2 \geq 1$  and so  $a + q + n - 2 \geq a + p + n - 2 \geq a + p + q - 2 \geq 1$ . Thus Proposition 9 (ii) implies that  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ . Therefore (i) holds.

(ii) The result is trivial if  $\max\{2, 2 - q - n\} > 2 - p - n$ . Now assume that  $\max\{2, 2 - q - n\} \leq 2 - p - n$ .

Let  $a = \max\{2, 2 - q - n\}$ . Then  $a \geq 2 - q - n$ , implying that  $a + q + n - 2 \geq 0$ . We also have  $a \leq 2 - p - n$ , implying that  $a + p + n - 2 \leq 0$  and so  $a + p + q - 2 \leq 0$ . If  $a = \max\{2, 2 - q - n\} = 2$ , then Proposition 9 (i) implies that  $W_{2,p,q,n}(z)$  interleaves  $W_{1,p,q,n}(z)$ , i.e.,  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ . If  $a = \max\{2, 2 - q - n\} = 2 - q - n$ , then Proposition 11 implies that  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ .

Now assume that  $\max\{2, 2 - q - n\} < a \leq 2 - p - n$  and  $W_{a-1,p,q,n}(z)$  interleaves  $W_{a-2,p,q,n}(z)$ . Note that  $\max\{2, 2 - q - n\} < a \leq 2 - p - n$  implies that  $a + q + n - 2 > 0$  and  $a + p + q - 2 \leq a + p + n - 2 \leq 0$ . Thus Proposition 9 (ii) implies that  $W_{a,p,q,n}(z)$  interleaves  $W_{a-1,p,q,n}(z)$ . Therefore (ii) holds.  $\square$

By Proposition 12, the following result is obtained.

**Proposition 13** *Let  $a$  be a positive integer and  $p, q, n$  be arbitrary real numbers with  $p \leq q \leq n$ . If one of the following conditions holds, then  $W_{a,p,q,n}(z)$  has only real roots and therefore every root of  $F_{a,p,q,n}(z)$  is either a real number or a complex number with its real part equal to 0:*

- (i)  *$p + q$  is a negative integer and  $a \geq 1 - p - q$ ; and*

(ii)  $q + n$  is an integer and  $\max\{1, 1 - q - n\} \leq a \leq 2 - p - n$ . □

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