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ON THE AREA OF CIRCLE

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Abstract

One of the approaches to develop the area formula of circle is by arranging sectors of a circle into an approximate parallelogram. While teaching our preservice teachers, we were asked if it is mathematically correct to state that the area of a circle is equivalent to area of rectangle. In this note, we present three proofs to show that area of rectangle can be used to derive the formula for area of circle.

1. Introduction

According to Van de Walle (2004), one of the approaches to develop the area formula is by arranging sectors of a circle into an approximate parallelogram. "As the number of sectors gets larger, the figure becomes closer and closer to a rectangle" (p. 339). This approach to developing the area of circle is common to some elementary textbooks. While preparing our preservice teachers to teach this concept to elementary school students, one of our students asked us whether it is mathematically correct to say that "area of circle equals to area of rectangle". The question appealed to us. In this paper, we decided to provide the mathematical proof in more than one ways to validate the statement.

2. First Proof

Consider a circle of radius r. Divide the circle into n equal sectors. Cut out $n-1$ of the sectors and divide the last sector into equal halves. Rearrange the pieces as shown in the Figure 1.

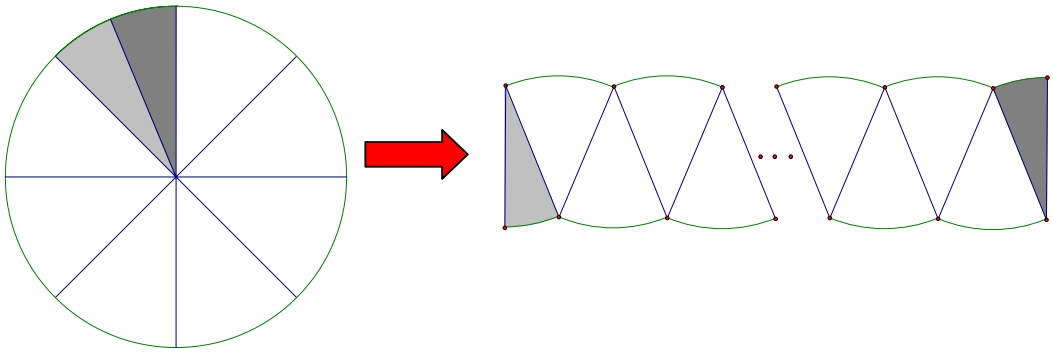

Figure 1. Sectors can be arranged in a "near rectangle."

Consider one sector (Figure 2).

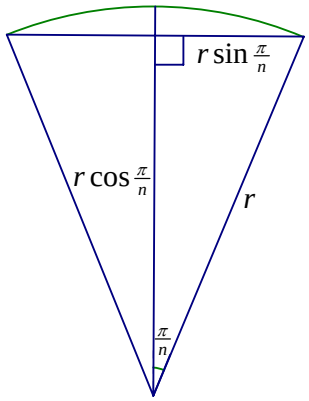

Figure 2. One of the sectors.

The length of the chord is $2r\sin\dfrac{\pi}{n}$. Thus, the sum of the lengths of the chords of half of the sectors = $\dfrac{n}{2}\times 2r\sin\dfrac{\pi}{n} = nr\sin\dfrac{\pi}{n}$.

We now can see that the rearranged circle looks like a rectangle with breadth $r\cos\dfrac{\pi}{n}$ and length $nr\sin\dfrac{\pi}{n}$, with n segments 'capping' the top and bottom of the rectangle (Figure 3).

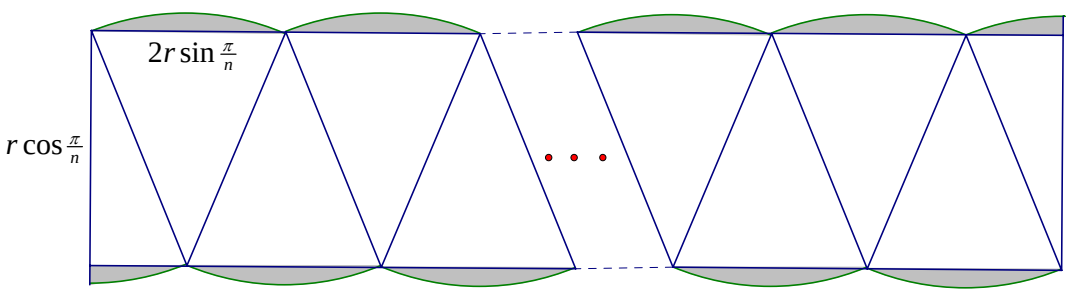

Figure 3. Rearranged circle.

Since for large n, $r\cos\dfrac{\pi}{n}\approx r$ and the length of a chord is almost that of the arc (i.e. the total length of the chords is almost that of the circumference), the figure looks like a rectangle with sides r and πr. We shall prove that indeed this is true for the limit as n tends to infinity.

Alternative 1

We begin with the breadth. Taking limit to infinity, we have $\lim\limits_{n\to\infty} r\cos\dfrac{\pi}{n} = r\cos 0 = r$.

Thus, the breadth tends to r.

Next, the length = $\frac{n}{2}\times 2r\sin\frac{\pi}{n} = nr\sin\frac{\pi}{n}$. Taking limit to infinity, we have

$\lim_{n\to\infty} nr\sin\frac{\pi}{n} = \lim_{n\to\infty}\pi r\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} = \pi r$ since $\lim_{x\to 0}\frac{\sin x}{x} = 1$. Thus, the length tends to πr.

Let the area of a segment subtending an angle of $\frac{2\pi}{n}$ be A_n.

Using the formula for the area of a segment, we have $A_n = \frac{1}{2}r^2\left(\frac{2\pi}{n} - \sin\frac{2\pi}{n}\right)$.

Thus, the area of all the n segments = nA_n = $\frac{nr^2}{2}\left(\frac{2\pi}{n} - \sin\frac{2\pi}{n}\right)$. Taking limit to infinity,

$$\lim_{n\to\infty}\frac{nr^2}{2}\left(\frac{2\pi}{n} - \sin\frac{2\pi}{n}\right) = \lim_{n\to\infty}\pi r^2\left(1 - \frac{n}{2\pi}\sin\frac{2\pi}{n}\right) = \lim_{n\to\infty}\pi r^2\left(1 - \frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right) = \lim_{n\to\infty}\pi r^2\times 0$$

since $\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Hence, the area of the segments tends to 0.
Finally, we work out the area of the circle as follows:

$$\begin{aligned}
\text{Area of circle} &= \text{Area of rectangle} + \text{Area of segments}\\
&= r\cos\frac{\pi}{n}\times nr\sin\frac{\pi}{n} + n\times A_n\\
&= \lim_{n\to\infty}\left(r\cos\frac{\pi}{n}\times nr\sin\frac{\pi}{n} + nA_n\right)\\
&= \lim_{n\to\infty}\left(r\cos\frac{\pi}{n}\right)\lim_{n\to\infty}\left(nr\sin\frac{\pi}{n}\right) + \lim_{n\to\infty} nA_n\\
&= r\times\pi r + 0\\
&= \pi r^2.
\end{aligned}$$

Thus, area of circle = πr^2.

Alternative 2

Let the area of a segment subtending an angle of $\frac{2\pi}{n}$ be A_n.

Using the formula for the area of a segment, we have $A_n = \frac{1}{2}r^2\left(\frac{2\pi}{n} - \sin\frac{2\pi}{n}\right)$.

Hence, we have

$$\begin{aligned}
\text{Area of circle } &= r\cos\frac{\pi}{n}\times nr\sin\frac{\pi}{n}+n\times A_n \\
&= \frac{nr^2}{2}\sin\frac{2\pi}{n}+\frac{1}{2}r^2\left(2\pi-n\sin\frac{2\pi}{n}\right) \\
&= \pi r^2\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}+\frac{1}{2}r^2\left(2\pi-2\pi\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right) \\
&= \lim_{n\to\infty}\pi r^2\frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}}+\lim_{n\to\infty}\pi r^2\left(1-\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}\right) \\
&= \pi r^2+0 \quad \text{since } \lim_{x\to 0}\frac{\sin x}{x}=1
\end{aligned}$$

Thus, area of circle = πr^2.

3. Second proof

Consider a circle of radius r. Divide the circle into $2n$ equal sectors and rearrange the pieces as shown in Figure 4. The rearranged pieces are bounded by 2 rectangles ABCD and PQRS. We shall prove that for sufficiently large n, area ABCD and area PQRS are approximately the same. We give a rigorous proof of this claim below. The results of the proof implies that the area of circle can be very well approximated by either of the 2 rectangles that bound the rearranged sectors as shown in Figure 4.

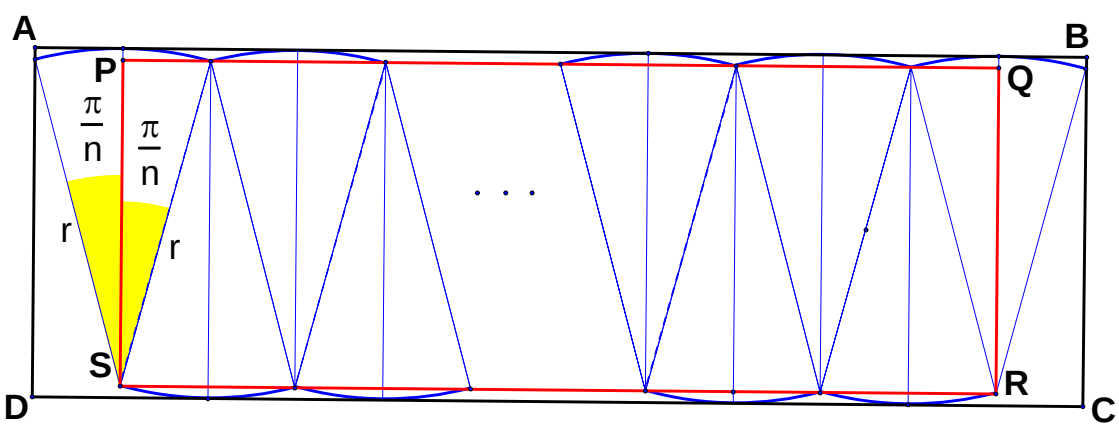

Figure 4. Rearranged circle.

According to Figure 4,

Area of rectangle $PQRS \le$ Area of circle $\le$ Area of rectangle $ABCD$;

that is,

$$(2n-2)\times\frac{r^2}{2}\sin\frac{\pi}{n}\cos\frac{\pi}{n}\le \text{Area of circle}\le (2n)\times\frac{r^2}{2}\sin\frac{\pi}{n}$$

or

$$(n-1)\times\frac{r^2}{2}\sin\frac{2\pi}{n}\le \text{Area of circle}\le r^2 n\sin\frac{\pi}{n}. \qquad (1)$$

Since a direct calculation shows that

$$\lim_{n\to\infty}(n-1)\times\frac{r^2}{2}\sin\frac{2\pi}{n}=\lim_{n\to\infty}r^2 n\sin\frac{\pi}{n}=\pi r^2,$$

the desired result follows from (1):

$$\text{Area of circle}=\pi r^2.$$

4. Concluding remarks

Is area of circle legitimately area of rectangle? We were attracted to this question as we felt a strong need to justify the result to provide grounds for its plausibility. In this paper, we look to proofs to answer our question because "proof is the way by which mathematics establishes truth" (Hanna & Jahnke, 1996, p. 892). There are many ways to prove that area of circle is equivalent to area of rectangle. The 3 proofs provided in this article point out the place of proof in mathematics teacher education and has deep implications for mathematics teaching.

From a pedagogical point of view, the proofs of these kind help preservice teachers reinforce multiple concepts learned in college-level classes such as trigonometry and calculus. The process of proving provides opportunity for preservice teachers to understand the meaning of "the statement being proved: to see not only that it is true (verification), but also why it is true (explanation)" (p. 902). In this sense, the process of proving adds to the mathematical understanding in the preservice mathematics teacher education. By providing convincing mathematical arguments about one approach to the formula for area of circle, we hope to encourage preservice teachers to understand and think more clearly and effectively about mathematics.

The proofs provided in this paper provide a legitimate and convincing way to view area of circle as area of rectangle. This legitimation allows the accurate transmission of mathematical knowledge in the mathematics classroom, and to the "formalisation of a body of mathematical knowledge" (p. 902).

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