Expressions for Matching Polynomials*

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Abstract

This paper studies representations of the matching polynomial $m(G, x)$ of a simple graph $G$. It is well known that $m(G, x) = \det(xI_n - A)$ if $G$ does not contain cycles, where $n$ is the order of $G$ and $A$ is the adjacency matrix of $G$. Aihara in 1979 posed an open problem: does every graph $G$ of order $n$ have a Hermitian matrix $H$ such that $m(G, x) = \det(xI_n - H)$, where a Hermitian matrix $(a_{j,k})$ is a square matrix with the property that each entry is a complex number and $a_{j,k}$ and $a_{k,j}$ are conjugates for all $j, k$? He solved this problem for unicyclic graphs. Yan, Yeh and Zhang recently found solutions for graphs which contain a small number of odd cycles but no even cycles. In this paper we solve this problem for any graph in which all cycles are edge-disjoint. For a general graph $G$, Godsil and Gutman in 1978 and Yan, Yeh and Zhang in 2005 established different expressions for $m(G, x)$ in terms of $\det(xI_n - H)$ for some families of matrices $H$. This paper generalizes their results and greatly simplifies the computation of $m(G, x)$.

Keywords: graph, adjacency matrix, matching polynomial, characteristic polynomial

1 Introduction

In this paper we consider simple graphs only. For any graph $G$, let $V(G)$, $E(G)$ and $v(G)$ be its vertex set, edge set and order (i.e., $v(G) = |V(G)|$). If it is not mentioned elsewhere, we always assume that $G$ is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E$. A matching of $G$ is a subset $M$ of $E$ such that each vertex of $G$ is incident with at most one edge in $M$. For any integer $k \geq 0$, let $\phi_k(G)$ denote the number of matchings $M$ of $G$ with $|M| = k$. Thus $\phi_0(G) = 1$, $\phi_1(G) = |E|$ and $\phi_k(G) = 0$ if $k \notin \{0, 1, 2, \ldots, [v(G)/2]\}$. In particular, let $\phi_G$ be the number of perfect matchings of $G$, i.e., $\phi_G = \phi_{v(G)/2}(G)$.

The matching polynomial of $G$ (see [4]), denoted by $m(G, x)$, is defined as follows:

\[
m(G, x) = \sum_{k=0}^{[v(G)/2]} (-1)^k \phi_k(G) x^{v(G) - 2k}.
\]

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