Discussion Group 3: Perspectives on multimodality and embodiment in the teaching and learning of mathematics
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The Multimodal Material Mind:

Embodiment in Mathematics Education

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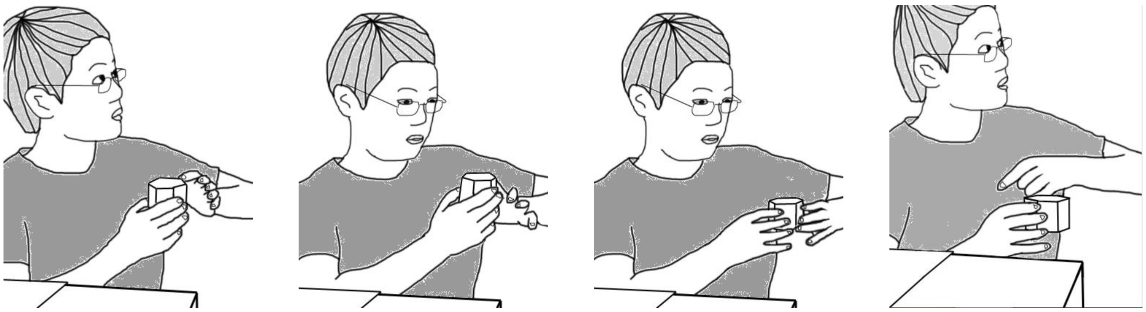
In a Grade 5 class, the students are engaged in investigating the number of faces of a regular prism. The teacher has asked the following question: "If I know the name of a prism, can I deduce the number of its faces?" Addressing the class, Jim, one of the students, summarizes his findings and says:

> Jim: Uh, yes, we can deduce the number of faces if we know the name of the prism because, if we take the example of a hexagonal prism, its bases are hexagons as the name says … (he touches one of the bases of a plastic hexagonal prism that he is holding in his right hand; see Figure 1, Pic 1).
>
> Teacher: Excellent.
>
> Jim: … of the prism. So each edge (touching one of the edges; see Figure 1, Pic 2) has a face so a hexagon has 6 edges (touching with his hands several faces of the hexagon; see Figure 1, Pic 3). So there are 6 lateral faces, and if we count the bases (making a round gesture with the index of his right hand; see Figure 1, Pic 4), it's 8 faces.



Pic 1 Pic 2 Pic 3 Pic 4

Figure 1. Pics 1-4, Jim touching the plastic hexagon and making gestures.

There are several elements in this short passage that have become relevant in contemporary discussions of mathematics education. They deal with a clear understanding of the roles played by the material geometric artifact, the tactile movement of Jim's hands around the physical geometric artifact and the linguistic activity that Jim deploys while touching it, and Jim's perception and imagination in the course of his embodied meaning-making process. Attention to these elements points to the idea that mathematical meanings that arise in teaching and learning are of a multimodal nature. More broadly, attention to these elements comes from new conceptions about human cognition, marked in particular by new understandings of the role of the body, language, and material culture. In contradistinction to traditional approaches, these conceptions highlight the cognitive role of semiotics and embodiment in mathematics thinking, teaching, and learning. Within these new conceptions, gestures, body posture, kinesthetic actions, artifacts, and signs in general are considered as a fruitful array of resources to be taken into account in order to investigate how students learn and how teachers teach (e.g., Arzarello, 2006; Bautista & Roth, 2012; Borba & Villareal, 2006; Edwards, Radford, & Arzarello, 2009; Forest & Mercier, 2012; Radford & D'Amore, 2006; Radford, Schubring, & Seeger, 2008). These sensible and material resources are not considered as a mere epiphenomenon of teaching and learning: they are conceptualized as central elements of the students' and teachers' mathematical thinking.

There are, however, a variety of interpretations of the role that our tactile-kinesthetic bodily experience of the world and our interaction with artifacts and material culture play in the way we think and come to know. These interpretations depend on conceptions of cognition. For instance, some approaches inspired by cognitive linguistics (e.g., Fauconnier and Turner, 2002; Friedrich, 1970) emphasize the metaphoric dimension of language and the integrative constitution of embodied mental spaces (see, e.g., Edwards, 2009; Lakoff & Núñez, 2000; Yoon, Thomas,

& Dreyfus, 2011). Other approaches, inspired by research in phenomenology, emphasize the “fleshy” nature of thought (Bautista & Roth, 2012; Roth, 2010; Thom & Roth, 2011), while others stress the materiality of cognition and its cultural-historical dimensions (de Freitas & Sinclair, 2013; Malafouris, 2012; Radford, 2013). All in all, these approaches claim that meaning and cognition are deeply rooted in physical, embodied existence and try to offer an answer to the question of how meaning arises, and of how thought is related to action, emotion, and perception…

Embodiment in Mathematics Education

The Piagetian Legacy

Piaget’s epistemology has had a significant impact on mathematics education, and has also influenced the conception of embodiment. The influence is particularly patent in the so-called “process-object” theories; that is, theories that conceive of thinking as moving from the learner’s actions to operation knowledge structures. Two examples are APOS theory (Dubinsky, 2002; Dubinsky & McDonald, 2001) and the “three worlds of mathematics” (Tall, 2013). APOS stands for actions, processes, objects, and schemas. The “three worlds of mathematics” refers to:

(1) *conceptual embodiment*, which builds on perception and action to develop mental images that “become perfect mental entities” (2013, p. 16). For instance, “the number line develops in the embodied world from a physical line drawn with pencil and ruler to a ‘perfect’ platonic construction that has length but no thickness” (Tall, 2008, p. 14);

(2) *operational symbolism*, which “grows out” of physical action into more or less flexible mathematical procedures; and

(3) *axiomatic formalism*, which “builds formal knowledge in axiomatic systems specified by set-theoretic definition” (2013, p. 16).

One of the differences between APOS and the “three worlds of mathematics” perspective is the following. While APOS theory focuses on the investigation of schema organization and genesis (Arnon et al., 2014), the “three worlds of mathematics” approach emphasizes the role of symbols and investigates the symbolic compression of processes according to whether the learner’s attention is focused on objects, procedures, or symbols (Gray & Tall, 1994; Tall et al., 2001).

The “three worlds of mathematics” approach includes specific ideas about embodiment and mathematical thinking (de Lima & Tall, 2008; Tall, 2004, 2008, 2013; Tall & Mejia-Ramos, 2010; Watson & Tall, 2002). Thus, following a blend of the empiricist and rationalist philosophical traditions discussed in previous sections, Tall (2013) acknowledges that “Mathematical thinking begins in human sensorimotor perception and action and is developed through language and symbolism” (p. 11). The meaning of the term embodiment in the “three worlds of mathematics” approach is explained as something that is “consistent with the colloquial notion of ‘giving a body’ to an abstract idea” (Tall, 2004, p. 32).

Notice that the notion that ideas exist in an abstract, non-embodied form that can be “expressed” or “receive a body”—in mental imagery (or another type of representation)—is consistent with dualistic theories that separate the realm of ideas from the realm of the material and the sensible. This theoretical commitment, that philosopher of mathematics David Bostock (2009, p. 232) calls “conceptualism,” has a clear implication for the manner in which methodological investigations are conducted. For instance, there is no need for an explicit analysis of the embodied cultural and conceptual sources of such things as symbols, mathematical definitions, or practices like proof. Instead, in the “three worlds of mathematics” approach these elements of thinking about and doing mathematics are analyzed from a taken-as-given world of mathematics and mathematicians. Real numbers, for example, are analyzed by contrasting their meaning and use within the three hierarchical worlds mentioned above: *embodied*, illustrated by a finger tracing “continuous motion” along a number line; *symbolic*, accompanied by “$\sqrt{2}=1.4142\ldots$”; and *formal*, with a definition of a complete ordered field. Tall (2008) explains the hierarchical relationship as follows:

> Physically the number line can be traced with a finger and, as the finger passes from 1 to 2, it feels as if it goes through all the points in between. But when this is represented as decimals, each decimal expansion is a different point (except for the difficult case of recurring nines) and so it does not seem possible to imagine running through *all* the points between 1 and 2 in a finite time . . . Formally, the real numbers $\mathbb{R}$ is an ordered field satisfying the completeness axiom. This involves entering a completely different world where addition is no longer defined by the algorithms of counting or decimal addition, instead it is simply asserted that for each pair of real numbers *a*, *b*, there is a third real number call[ed] the sum of *a* and *b* and denoted by *a*+*b*. (pp. 14-15)

To sum up, APOS theory offers a refined perspective to investigate the genesis of schemas, while the "three world of mathematics" approach offers a powerful framework to study the increasing transformation and compression of symbolism starting from an embodied level and moving towards flexible ways of using symbols and notations. Embodiment, nonetheless, remains a general category; the fate of embodied actions is to be superseded by flexible actions with symbols.

Multimodality

In other approaches, the variety of embodied modalities to which students and teachers resort comes to the forefront. In particular, there is generally a stronger commitment to the essential role of the body even in abstract thought. These approaches may be termed *multimodal*. The term "multimodality" entered the mathematics education field after being borrowed from external research domains, ranging from neuroscience (for example, see Gallese & Lakoff, 2005) to communication studies (Kress, 2001, 2010). As Edwards and Robutti note (2014, p. 7), the "meanings used in these different fields of study are not mutually exclusive but intersect and complement each other." In mathematics education, the term multimodality is often used to underline both the relevance and mutual coexistence of a range of different cognitive, physical, and sensuous (e.g., perceptual, aural, tactile) modalities or resources playing a role in teaching-learning processes, and more broadly in the production of mathematical meanings: "These resources or modalities include both oral and written symbolic communication as well as drawing, gesture, the manipulation of physical and electronic artifacts, and various kinds of bodily motion" (Radford, Edwards, & Arzarello, 2009, pp. 91-92).

An example of a multimodal approach is provided by the work of Abrahamson (2014), in what he terms "embodied design." This process involves the creation of physical tasks and computational environments that allow "proactive multimodal sensorimotor interaction" (Hutto, Kirchoff, & Abrahamson, 2015, p. 375). For example, the Mathematical Imagery Trainer (MIT) allows the learner to engage kinesthetically and perceptually with the idea of proportionality. The student can change the color of a computer screen only when he holds one of his hands twice as far from the table as his other, and maintains this ratio while moving his hands up or down. Thus, the introduction to proportionality is fully embodied in "non-symbolic perceptuomotor schemas" (Abrahamson, 2014, p. 1). Through the use of language, gesture, and, eventually, written inscriptions, the learner is assisted to reconcile his naïve, embodied, enacted experiences with more formal mathematical constructions.

How is embodiment understood here? What is the role of the students' gestures? Embodiment appears as a faculty of the body that has a constructive function in that it helps the creation of mathematical constructs in the course of learning (Alibali & Nathan, 2012; McNeill, 2000, 2005).

Semiotic Bundles

Another example of the multimodal approach has been developed by Arzarello and collaborators. Drawing on Vygotsky's work and neuroscience research, they stress the importance of the multimodal character of the students' semiotic activity in teaching and learning contexts. Here, the emphasis is not on schemas, as is the case of the Piagetian influenced process-object theories mentioned before, but on the *evolution of signs* (Arzarello, 2006). In accordance with Vygotsky's early concept of sign, Arzarello refers to signs as mediating entities of thinking, much as tools are conceived of as mediating entities of labor. Within this context, gestures and other embodied resources to which students and teachers resort become considered as signs, even if they do not present relatively formal rules of production as do language and algebraic and Cartesian graphic symbolism, through explicit grammatical or syntactic rules. In Arzarello's approach multimodality occurs through relationships between sets of signs (e.g., the set of speech language, the set of gestures, the set of algebraic symbols, etc.), produced and transformed according to their (formal or informal) nature and constituting a "semiotic bundle." A semiotic bundle is precisely formed by: "i) A collection of semiotic sets. Ii) A set of relationships between the signs" (Arzarello, 2006, p. 281).

As we can see, the semiotic bundle considers the semiotic resources in a unifying manner allowing for the description of learning through the evolution of signs as they are produced in the classroom by all participants. Arzarello, Paola, Robutti, and Sabena (2009) explain:

> Typically, a semiotic bundle is made of the signs that are produced by a student or by a group of students while solving a problem and/or discussing a mathematical question. Possibly the teacher too participates to this production and so the semiotic bundle may include also the signs produced by the teacher. (p. 100)

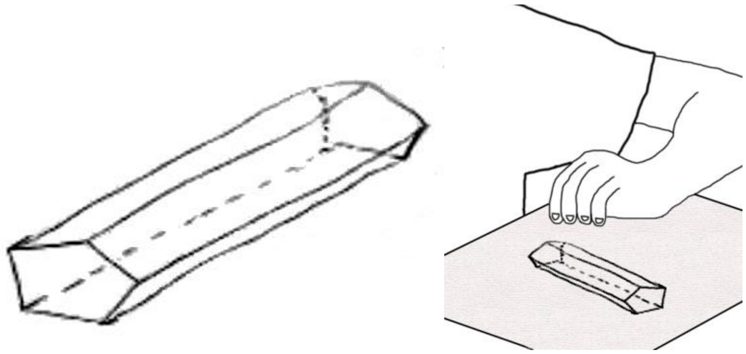
Due to the very general nature of the sign that is considered, the semiotic bundle includes the classical systems of signs (Ernest, 2006) or registers of representation (Duval, 2006) as particular cases, but also gestures and embodied signs. Using the semiotic bundle, two kinds of interrelated analysis can be done: (1) a *synchronic analysis*, which focuses on the relationships between different signs in a certain moment; and (2) a *diachronic analysis*, which focuses on the evolution of signs (and the evolutions of the relationships between signs). While synchronic analysis allows for taking a kind of “picture” of the students’ and teacher’s mathematical activity from a semiotic point of view, diachronic analysis allows for obtaining a sort of multimodal semiotic “movie” of such an activity.

An example of a phenomenon detected with the synchronic view is the gesture-speech relationship in Jim’s activity as described in the initial vignette. In this brief example, gestures and words cannot be considered separate from each other because the meaning of one set completes the meaning of the other one (McNeill, 2000). As pointed out above, we can see for instance in line 3 of the transcript that gestures are co-timed with speech and the sensuous aspects (touching, gazes) are deeply intermingled with speech to jointly express a reasoning: they co-live in the semiotic bundle.

The diachronic analysis is at the heart of the analysis carried out within the semiotic bundle perspective because it allows the researcher to determine whether and how an evolution of meanings occurred during the students’ activity. For instance, going on with the discussion with the teacher and his mates, Jim draws on a sheet of paper a representation of a prism with a pentagonal base and says (Figure 2, Pic 1):

Jim: An edge, uh, um, every edge (he slips his finger on the edge drawn on the sheet) is a side face (Figure 2, pic2) so it has five lateral faces.

So, Jim produces a written diagram, in which we recognize (a variation of) the artifact he had interacted with, performs (a variation of) the gesture he had produced, and produces (a variation of) the reasoning he had done before. The variation consists both in the number of edges (5 for a pentagon and no longer 6 for a hexagon), and in the signs with which he interacts (a drawing and not the plastic artifact): the semiotic bundle has evolved. The changes are a possible hint of the generality with which Jim is reasoning which, in turn, is embodied in the semiotic bundle relationships and evolution.



Pic 1 Pic 2

Figure 2. Example of diachronic analysis: diagram and gesture produced later by Jim.

Looking at the evolution of the students' signs, the teacher can gain clues with respect to the students' understanding: the multimodal aspects of the activity can therefore help her decide whether or not to intervene in order to support the students. A didactic phenomenon reported in the literature is the so-called "semiotic game" (Arzarello et al., 2009), which happens when the teacher attunes to a certain semiotic set employed by the students (typically imitating a certain gesture) and couples it with another set (typically speech words or written mathematical symbols) in order to build a connection between personal and shared mathematical meanings. Therefore, semiotic games constitute an important strategy in the process of the appropriation of the culturally shared meaning of signs…

Phenomenological Approaches

Drawing on experimental and developmental psychology, cognitive science, and neuroscience, Nemirovsky and colleagues propose, like enactivism, a *nondualistic* embodied perspective on mathematical thinking and learning. One of the particularities of their approach is its phenomenological orientation and the important role ascribed to imagination and perceptuomotor integration in the learner's experience (Nemirovsky & Ferrara, 2009; Nemirovsky, Kelton, & Rhodehamel, 2013; Nemirovsky, Rasmussen, Sweeney, & Wawro, 2012). Perceptuomotor integration consists of a deep intertwining of perceptual and motor aspects of tool use or body movements. In particular, Nemirovsky et al. (2013) take a phenomenological perspective based on Husserl's work (1991) to advocate that "(a) mathematical thinking is constituted by bodily activity at varying degrees of overt and covert expression, and (b) mathematical learning consists of transformations in learners' lived bodily engagement in mathematical practices" (p. 376). This approach seeks to overcome not only dualistic views on mind and body, but also some dialectical perspectives on the role of tools in psychology and mathematics education (e.g., Verillon & Rabardel, 1995 and the instrumental approach or Vygotsky's [1978] early theory of mediation). From Nemirovsky's perspective, dialectical approaches are too often inclined to privilege mental structures over bodily experiences. The chosen methodology is based on microethnographic studies, in which the multimodal aspects of the activities are analyzed in great detail over short periods of time (the order of seconds or minutes), taking into account multiple aspects of the situated context. Although perceptuomotor integration includes sociocultural factors, the cultural and material plane remains in the background and is not yet fully integrated in the overall picture provided by this emerging framework (for a similar remark, see Stevens, 2012).

A Materialist Phenomenology

Roth and colleagues adopt also a phenomenological approach (Bautista & Roth, 2012; Hwang & Roth, 2011; Roth, 2012; Thom & Roth, 2011) that they term "materialist phenomenology." They draw on the work of Merleau-Ponty (1962), Maine de Biran (1859), and a new French generation of phenomenologists, such as Marion (2002) and Nancy (2006). Their departure point is a critique of rationalist versions of embodiment and the development of a radical embodied material phenomenology. Indeed, they disagree with some views of embodiment in which gestures and other bodily resources are understood as supplying with theoretical content the words that the students may still be lacking in their conceptual development. The question at stake is that such rationalist understanding of embodiment presupposes, at least to some extent, that the student already has the required conceptualizations and intentions, without being able to express them in a coherent verbal manner (for a similar critique, see Sheets-Johnstone, 2009, pp. 213-216). Instead, "A materialist phenomenological approach theorizes knowing beginning with primitive forms of experiences that precede mind and intention" (Roth, 2010, p. 9). At the root of the primitive forms of experience is not the body; it is the flesh:

> I suggest taking the flesh rather than the body as the ground of all knowing: knowledge as incarnated, enfleshed. Because it is enfleshed, mathematical knowledge also is embodied. It is the flesh where we find tact (touch), contact, and contingency, and therefore, the ground of knowledge so that the sense of the body comes to be the body of sense. (Roth, 2010, p. 9)

While in other phenomenological approaches (e.g., those drawing on Husserl's work) the primacy of senses is attributed to perception, in the radical embodied material phenomenology that Roth et al. propose, touch is the main sense. It is around touch that the sensations from the eyes and the other organs are coordinated, "especially with touch from the hands" (Roth, 2010, p. 11).

Roth discusses the example of Chris, a Grade 2 student, comparing a material cube and a pizza box. The teacher asks what the pizza box would have to have to make it a cube. Prompted by the question, Chris moves his hand along two edges of the pizza box. He utters the word "square" while pointing to the two sides again. Roth (2010) explains:

with the movements and coordination of movements of eyes and hands, the world begins to emerge from touch. Chris's present experience is based on the coordination of hands with eyes, so that seeing the pizza box and moving the hand along one edge, then another edge, is but a realization of the coordination of hands and eyes and the concrete realization of the ability of moving them. (p. 11)

There are certain converging points between the radical embodied materialist phenomenology that Roth advocates and the enactivist approach previously described, but they do not coincide. A central difference is the immanent viewpoint that the materialist epistemology adopts: the existence of an original passivity that living organisms enjoy and which provides them with the possibility to affect something and to be simultaneously affected—in the pizza box example, this immanence expresses itself in the pre-conceptual and pre-intentional movement of Chris' hand along the edge of the pizza box and, later on, a material cube: "It is the flesh, with its capacity of tact (i.e., sense of touch), contact (i.e., touched and being touched), and contingency that is the ground of all senses, sense-making efforts, and, therefore, knowledge" (Roth, 2010, p. 13)

But there is more to the difference between the radical embodied materialist phenomenology and enactivism. Roth (2010) expresses it as follows:

> Varela et al. (1991) propose to look for knowledge at the "interface between body, society, and culture" (p. 179). In the position articulated here there is no interface: mind *is* in society and culture as much as society and culture are in the mind. Similar positions can be found in activity theory from L. S. Vygotsky via A. N. Leont'ev to the present day. Maturana and Varela (1980) take societies to be 'systems of coupled human beings' (p. 118), whereas the position here is the converse: the specifically *human* being is a result of society rather than preceding coupling or, in activity theoretic terms, there is mind because there is society. (p. 16; emphases in the original).

Inclusive Materialism

De Freitas and Sinclair (2014) draw on the work of Barad (2007), Châtelet (2000), and Deleuze (1994) to articulate an approach that they term *inclusive materialism.* Noting that theories of embodiment often remain focused on the individual learner and conceive of mathematical concepts as passive entities, they argue for a reconceptualization of the body that stretches conventional concepts. They suggest conceiving of the body as an assemblage "of human and non-human components" (2014, p. 25)—a heterogeneous assemblage of organic matter, concepts, tools, signs, diagrams, and objects (2014, p. 225).

This materialist ontological stance opens a space to talk of the human body as something that is more than what goes under the skin. It also makes room to talk of the *body of mathematics* and the *body of the tools* one uses in mathematical activity. "The new materialism we propose," they say, "aims to embrace the 'body' of mathematics as that which forms an assemblage with the body of the mathematician, as well as the body of her tools/symbols/diagrams" (2013, p. 454).

The idea of the body that de Freitas and Sinclair propose moves the discussion away from intentionality as one of the chief characteristics of individuals' actions and focuses on the field of agents and agency. As they note, "our aim here is to focus less on human intention and more on distributed agency. We want to problematize some of the ontological tenets underpinning particular conceptions of the human body as the principal administrator of its own participation" (2014, p. 19)

While to be an agent and to be endowed with agency have usually been considered attributes of humans, in inclusive materialism, these attributes are not restricted to humans only. In this perspective, it makes sense to talk about matter as agentic entities. Thus, referring to Roth's (2010) analysis of the cube discussed above, they contend that "The matter of the cube and the matter of the mathematical concepts are also agents" (de Freitas & Sinclair, 2014, p. 24).

Inclusive materialism stretches hence not only the concept of the body but also the concept of agency. The concept of agency must be rethought, because inclusive materialism

> problematizes the premise that any one part of the assemblage is the source of action, intention or will. Such problematizing will mean revising notions such as student agency, as well as advocacy or interventions for improving or supporting student agency. We will need to reconceive agency as operating within the relations of an ever-changing assemblage (de Freitas & Sinclair, 2014, p. 33)

And this is precisely what Roth's (2010) analysis of the cube discussed above does not address adequately. Indeed, de Freitas and Sinclair (2014) argue that Roth's analysis fails to notice that the cube is not an inert object but rather an animated entity in "intra-action" with the student and the mathematical concept in a process of becoming:

> While Roth's cube example sheds light on the role of the body in learning, the analysis fails to do justice to the materiality of either the cube or the mathematics; that is to say, it fails to reckon with the way in which the cube is itself becoming-cube through its encounter with the child, shifting its own boundaries in this process of becoming. Roth treats the nonhuman material in this encounter as passive and inert . . .

> Moreover, the mathematical concept of cube remains untouched and untroubled by the encounter, as though it were indeed an immaterial and inflexible concept that happens to be somehow manifest in this particular instance. (pp. 23-24)

In general, what de Freitas and Sinclair do not see in the current literature on the material aspects of mathematics “is to show how mathematical concepts partake of the material in operative, agential ways” (p. 40).

They locate the “forceful, animate, mobile, alive and material” (de Freitas & Sinclair, 2014, p. 226) nature of mathematical concepts in a conceptual category called the virtual. To understand the meaning of the virtual and virtuality, we need to go back to Deleuze’s concept of the virtual. Deleuze (1994, p. 209) contended that “the virtual must be defined as strictly a part of the real object—as though the object had one part of itself in the virtual into which it is plunged as though into an objective dimension.” This is why “Every object is double without it being the case that the two halves resemble one another, one being a virtual image and the other an actual image” (p. 209). Inclusive materialism expands this idea to mathematical objects as well. As a result, “mathematics cannot be divorced from ‘sensible matter’, and it is the virtual dimension of this matter that animates the mathematical concept. Mathematical entities are thus material objects with *virtual* and *actual* dimensions” (pp. 201-202; emphasis in the original).

An object (mathematical or other) is hence a double object, made up of an actual image and a virtual one, and it is in the virtual one that we find the mobility of the concept. Considered from this post-humanist account, traditional teaching of mathematics does not attend to the virtual; it focuses on the logical. Now, the virtual, as considered in this approach, can be summoned or invoked. The virtual is something that can be “provoked,” “recovered,” “unleashed,” and “conjured,” but also "massacred" (de Freitas & Sinclair, 2014, p. 213). Gestures, diagrams, and mathematical notations are considered as “invoking a dynamic process of excavation that conjures the virtual in sensible matter” (2014, p. 67)….

Cognitive Linguistics

The discipline of cognitive linguistics is based on the theory of embodied cognition, which, like enactivism, holds that the shared experience of existing as biological organisms who are born and grow up in a specific physical (and cultural) world provides the foundation for human language, thought, and meaning (Gibbs, 2006; Johnson, 1987, 2007; Lakoff & Johnson, 1999; Varela et al., 1991). More specifically, cognitive linguistics proposes that the relationship between elements of language and their referents is, in general, not formal and arbitrary, but rather that there is a close linkage between action in the world, language, thought, and meaning (Fauconnier, 1997; Fauconnier & Lakoff, 2009; Fauconnier & Sweetser, 1996; Fauconnier & Turner, 2002; Lakoff, 1987; Lakoff & Johnson, 1980, 1999). For example, if we look back at the opening vignette, our fifth-grade student, Jim, uses a combination of spoken words and gestures to justify his claim that "we can deduce the number of faces if we know the name of the prism." According to a cognitive linguistic framework, neither Jim's language nor his bodily actions are unrelated to the way he thinks about the situation. When he refers to the "faces" of the prism, he is using a term that did not develop arbitrarily within the mathematical community, but rather because of its association with the human face, a somewhat planar feature that we present and respond to in the social world. Thus, our physical form serves as a source for naming a mathematical entity, in a non-arbitrary way.

Similarly, the circular gesture Jim uses to physically encompass all the faces of the prism is related to what cognitive linguistics calls the *image schema* of containment (Johnson, 1987; Lakoff, 1987; Lakoff & Núñez, 2000; Talmy, 2000). Image schemas are "recurrent, stable patterns of sensorimotor experience...[that] preserve the topological structure of the perceptual whole …having internal structures that give rise to constrained inferences" (Johnson, 2007, p. 144). The CONTAINMENT image schema arises from the child's physical experience of filling and emptying containers, experience which builds the notions of "inside," "outside," and "edge" or "boundary." This image schema allows Jim to think about the faces of the prism as members of a collection that can be counted, and his circular gesture indicates the boundary of the collection. The CONTAINMENT image schema provides the foundation for many later understandings, both within and outside of mathematics, including set membership, the domain and range of a function, and bounded regions (Lakoff & Núñez, 2000; Núñez, 2000).

Image schemas can help account for the fact that many mathematical expressions and some symbols evoke space and spatial relationships, even when the subject is not geometry (e.g., "limit," "field," "onto," $\Leftrightarrow$). When we talk about "balancing" an equation, or "supporting" an argument, from the perspective of cognitive linguistics, this is comprehensible because of our shared experience of balancing and supporting our bodies (as well as building blocks, bicycles, and so on).

An image schema can serve as the source domain for a powerful mechanism in cognitive linguistics, *conceptual metaphor*. Conceptual metaphors are unconscious mappings between two conceptual domains, in which the inferential structure of the first domain is mapped onto the second (Johnson, 1987; Lakoff, 1987, 1992; Lakoff & Johnson, 1980, 1999). As an example, a common image structure, SOURCE-PATH-GOAL, is based on our physical experience of traveling from one location (the SOURCE) to another (the GOAL), along a given PATH (Johnson, 1987). This image schema can be found in multiple areas of mathematics, from addition using the number line (Lakoff & Núñez, 2000) to functions and graphing (Bazzini, 2001; Ferrara, 2003, 2014; Font, Bolite, & Acevedo, 2010) to continuity (Núñez, Edwards, & Matos, 1999). It can even provide the source domain for understanding proof, a concept with no obvious relationship to movement through space. Figure 6 illustrates the metaphorical mapping from the internal structure of the SOURCE-PATH-GOAL schema to the explicit form of a mathematical proof (Edwards, 2010).

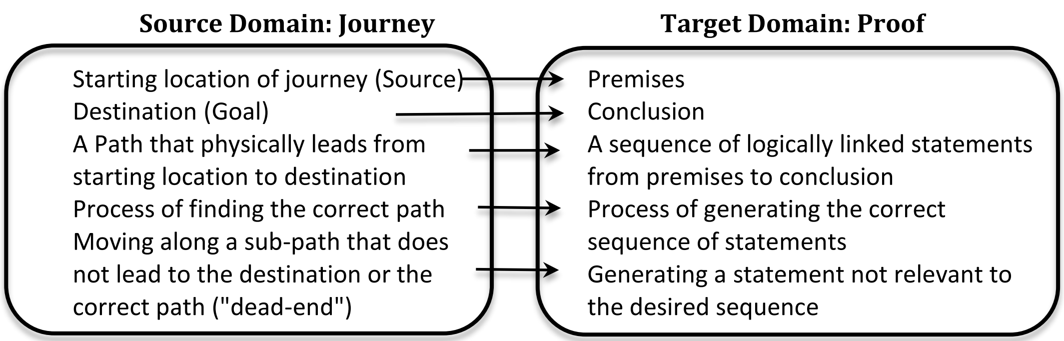


Figure 6. Conceptual metaphor A PROOF IS A JOURNEY.

Empirical support for this metaphor can be found in both the speech and gestures of a doctoral student talking about mathematical proof, shown in Figure 7 (Edwards, 2010):

> Student: ’cause you start figuring out, I’m starting at **point a and ending up at point b**. There’s gonna be **some road//where does it go through**? And can I show that **I can get through there**? (bold indicates speech coordinated with gestures)

Figure 7. Gesture indicating the SOURCE-PATH-GOAL image schema.

The source domain for a conceptual metaphor can be drawn from experience in the physical world (in which case it is called a grounding metaphor), or it can be drawn from an existing conceptualization (creating a linking metaphor, which can yield more abstract concepts—connecting, for example, subdomains within mathematics; Lakoff & Núñez, 2000; Núñez, 2008). An example of conceptual metaphor at a more sophisticated level of mathematics can be found in an analysis of undergraduate students' metaphors for limits. Oehrtman (2009) used interviews and written assignments to ask students in an introductory calculus class about limits. He identified five clusters of metaphors for limits used consistently by his students: "(a) a collapse in dimension, (b) approximation and error analyses, (c) proximity in a space of point-locations, (d) a small physical scale beyond which nothing exists, and (e) the treatment of infinity as a number" (p. 396). For example, students who were asked about the Taylor series for *sin x* used the "proximity" or "physical closeness" metaphor, in statements such as: "the closer the polynomials will *wrap themselves* around the original function" and "the polynomial becomes more and more *loosely fitted* around the curve" (italics added; p. 417).

The examples above illustrate the use of the theory of conceptual metaphor to analyze specific mathematical topics and concepts; however, there is a more fundamental metaphor underlying mathematical discourse, one which may have contributed to perennial arguments over the ontological status of mathematics. This is the metaphor we use when we talk about mathematical entities as if they had a physical existence, i.e., as objects (Font, Godino, Planas, & Acevedo, 2009; Lakoff & Núñez, 2000). This metaphor is evident when we talk about "manipulating" an equation, or ask "how many fives in twenty?" Lakoff and Núñez (2000) spelled out this use of language in what they call the ontological metaphor, which takes physical objects as the source domain for conceptualizing mathematical entities; however, this phenomenon was noted earlier by Pimm (1987) and Sfard (1994). Sfard (1994) noted, "The fact that we use the word 'existence' with reference to abstract objects (as in existence theorems) reflects in the most persuasive way the metaphorical nature of the world of abstract ideas" (p.

48). The objectual metaphor, as it is termed by Font and colleagues (Font et al., 2009), offers great advantages—it allows someone carrying out mathematical work to treat symbols as well as abstract ideas as objects, "moving" and "transforming" them, thus radically reducing the cognitive load that would be required if every mathematical sign had to be grounded in its logical mathematical definition.

The construct of conceptual integration emerged within cognitive linguistics at about the same time as conceptual metaphor; however, where metaphor comprises a unidirectional mapping from exactly one source domain to exactly one target domain, conceptual integration can involve multiple input spaces (Fauconnier & Lakoff, 2009). Conceptual integration, also known as conceptual blending, "connects input spaces, projects selectively to a blended space, and develops emergent structure" (Fauconnier & Turner, 2002, p. 89).

Conceptual integration often builds on existing blends, in a manner similar to linking metaphors (Lakoff & Núñez, 2000). For example, the conceptual mapping for the complex numbers relies on the existence of the blend for the number line as well as the blend for the Cartesian coordinate plane, each of which are blends themselves (Fauconnier & Turner, 2002; Lakoff & Núñez, 2000). The first input space for the "complex numbers" blend consists of the oriented coordinate plane with vector arithmetic. For the second input space, the blend draws on positive and negative real numbers and their associated operations and properties. The blended space yields the complex numbers in the complex plane, in which each element is simultaneously a number and a vector, an emergent quality that was not present in either of the input spaces. The blend creates other new, emergent structures; for example, the blend of the coordinate axes with the positive and negative numbers yields a complex number made up of a real part, displayed on the x axis, and a complex part on the y axis. In addition, "running the blend" (working out its entailments through a mechanism called "elaboration") allows addition and multiplication to be redefined to work consistently and coherently in the new space (Fauconnier & Turner, 2002). Fauconnier and Turner (2002) point out that the generic space for this blend (that is, the elements that the two spaces share in common) is made up of two operations with a specific set of properties, namely, associativity, commutativity, identities, inverses, and distributivity of one operation over the other. This combination of operations and properties has come to be seen, and labeled, as a mathematical entity in itself, the commutative ring.

Another example of conceptual integration is drawn from the work of Zandieh, Roh, and Knapp (2014), who analyzed the logical frameworks that a group of students used in working together to create a proof. The task set for the students was to create a proof showing that one conditional statement implied a second conditional statement, specifically, that “either Euclid’s Fifth Postulate (EFP) implies Playfair’s Parallel Postulate (PPP) or PPP implies EFP” (p. 213). The researchers noted two different conceptual blends for the logical framing of their proof: “a simple proving frame” (which was inadequate for this particular task) and a “conditional implies conditional proving frame” (p. 214). The researchers also analyzed the ways in which the students utilized the visual information associated with each postulate, and proposed that the mechanism of conceptual integration allowed the students to merge this information to find the key idea needed for their proof. They also note that conceptual blending does not always lead to correct thinking, offering an example in which three students “condense the premise and conclusion of EFP in a way that loses the implication structure” (p. 228).

The research of Yoon et al. (2011) provides another example of how the analysis of gesture has been integrated into cognitive linguistics. They have investigated what they refer to as “virtual mathematical constructs – constructs that are created via sensuous cognition in a mathematical gesture space through the multimodal use of gestures, speech and other linked semiotic systems” (p. 893). That is, they point out that a student or teacher can utilize the affordances offered by the body, specifically the hand and arm, to establish mathematical meanings through linked gesture and speech (McNeill, 2000, 2005), giving the example of a student using a straight hand held at a (varying) angle to represent the changing gradient of an antiderivative graph. This is possible because elements of the physical world (in this case the hand and arm) can be recruited to serve as one input space for a conceptual blend, a particular type of input called *Real Space* (Liddell, 1998). The (student’s understanding of the) mathematical domain is the second input space. In the blend, the student’s flat or angled palm and fingers are mapped to the gradient of the tangent line, and the hand’s motion and location to movement along the antiderivative graph (Yoon et al., 2011). Via conceptual integration, physical action becomes an important resource for the students in constructing an understanding of the mathematical content.

The field of cognitive linguistics offers a powerful theoretical framework and a set of productive tools for understanding mathematics that can be applied equally well to a child’s earliest construction of number sense or a mathematician’s elaboration of abstract structures. The theory of cognitive linguistics is conceptually coherent, supported empirically via multiple methodologies, and connects with other advances in cognitive science, including neurological research (e.g., Fields, 2013; Guhe et al., 2011; Winter, Marghetis, & Matlock, 2015). One of the principles of embodied cognition is that of cognitive continuity (Johnson, 2007), under which mathematics is not ontologically different from other realms of cognition and action. Instead, the cognitive mechanisms that allowed humans to survive and thrive over millennia have also supported the development of mathematical thought and other conceptual domains.

References

Abrahamson, D. (2014). Building educational activities for understanding: An elaboration on the embodied-design framework and its epistemic grounds. *International Journal of Child-Computer Interaction*, *2*(1), 1-16.

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences*, *21*(2), 247-286.

Arnauld, A. (1861). *The Port-Royal logic.* Edinburgh, Scotland: James Gordon.

Arnon, I., Cottrill, J., Dubinsky, E., Oktaç, A., Fuentes, S., Trigueros, M., & Weller, K. (2014). *APOS theory.* New York, NY: Springer.

Arzarello, F. (2006). Semiosis as a multimodal process. In L. Radford & B. D'Amore (Guest Eds.), *Revista Latinoamericana de Investigación en Matemática Educativa, Special Issue on Semiotics, Culture, and Mathematical Thinking* (pp. 267-299).

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97-109.

Bautista, A., & Roth, W.-M. (2012). Conceptualizing sound as a form of incarnate mathematical consciousness. *Educational Studies in Mathematics*, *79*(1), 41-59.

Bazzini, L. (2001). From grounding metaphors to technological devices: A call for legitimacy in school mathematics. *Educational Studies in Mathematics*, *47*(3), 259-271.

Beth, E. W., & Piaget, J. (1966). *Mathematical epistemology and psychology.* Dordrecht, The Netherlands: D. Reidel.

Borba, M., & Villareal, M. (2006). *Humans-with-media and the reorganization of mathematical thinking.* New York, NY: Springer.

Bostock, D. (2009). *Philosophy of mathematics.* Malden, MA: Wiley-Blackwell.

de Freitas, E. (2016). Material encounters and media events: What kind of mathematics can a body do? *Educational Studies in Mathematics*, *91*, 185-202.

de Freitas, E., & Sinclair, N. (2013). New materialist ontologies in mathematics education: The body in/of mathematics. *Educational Studies in Mathematics*, *83*, 453-470.

de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body.* Cambridge, England: Cambridge University Press.

de Lima, R. N., & Tall, D. (2008). Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics*, *67*(1), 3-18.

Deleuze, G. (1994). *Difference and repetition.* (P. Patton, Trans.). New York: Columbia University Press. (Original work published 1968)

Descartes, R. (1982). *Méditations [Meditations]*. Paris, France: Librairie philosophique Vrin. (Original work published 1641)

Dubinsky, E. (2002). Reflective abstraction in advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). New York, NY: Kluwer.

Dubinsky, E., & McDonald, M. (2001). APOS: A constructivist theory of learning in undergraduate mathematics education research. In *The teaching and learning of mathematics at university level: An ICMI study* (pp. 275-282). Dordrecht, The Netherlands: Kluwer.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, *61*, 103-131.

Edwards, L. D. (2009). Gestures and conceptual integration in mathematical talk. *Educational Studies in Mathematics*, *70*(2), 127-141. doi:10.1007/s10649-008-9124-6.

Edwards, L. D. (2010). Doctoral students, embodied discourse and proof. In M. M. F. Pinto & T. F. Kawasaki (Eds), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 329 -336). Belo Horizonte, Brazil: PME.

Edwards, L. D. (2011). Embodied cognitive science and mathematics. In B. Ubuz (Ed), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 297–304). Ankara, Turkey: PME.

Edwards, L., Radford, L., & Arzarello, F. (2009). Gestures and multimodality in the teaching and learning of mathematics. *Educational Studies in Mathematics, 70*(2), 91-215.

Edwards, L. D., & Robutti, O. (2014). Embodiment, modalities and mathematical affordances. In L. D. Edwards, F. Ferrara, & D. Moore-Russo (Eds.), *Emerging perspectives on gesture and embodiment in mathematics* (pp. 1-23). Charlotte, NC: Information Age Publishers.

Ernest, P. (2006). A semiotic perspective of mathematical activity. *Educational Studies in Mathematics*, *61*, 67-101.

Fauconnier, G. (1997). *Mappings in thought and language*. Cambridge, England: Cambridge University Press.

Fauconnier, G., & Lakoff, G. (2009). On metaphor and blending. *Cognitive Semiotics*, *5*(1-2), 393-399.

Fauconnier, G., & Sweetser, E. (1996). *Spaces, worlds, and grammar*. Chicago, IL: University of Chicago Press.

Fauconnier, G., & Turner, M. (2002). *The way we think: Conceptual blending and the mind's hidden complexities.* New York, NY: Basic Books.

Ferrara, F. (2003). Bridging perception and theory: What role can metaphors and imagery play? In I. Schwank (Ed.), *Proceedings of the Third Conference of the European Society in Mathematics Education*. Bellaria, Italy: University of Pisa. Retrieved from http://www.dm.unipi.it/~didattica/CERME3/proceedings/Groups/TG1/TG1_ferrara_cerme3.pdf

Ferrara, F. (2014). How multimodality works in mathematical activity: Young children graphing motion. *International Journal of Science and Mathematics Education, 12*(4), 917-939.

Fields, C. (2013). Metaphorical motion in mathematical reasoning: Further evidence for pre-motor implementation of structure mapping in abstract domains. *Cognitive Processing*, *14*(3), 217-229.

Fischbach, F. (2014). *La production des hommes* [*The production of men*]. Paris, France: Vrin.

Font, V., Bolite, J., & Acevedo, J. (2010). Metaphors in mathematics classrooms: Analyzing the dynamic process of teaching and learning of graph functions. *Educational Studies in Mathematics, 75*, 131-152.

Font, V., Godino, J., Planas, N., & Acevedo, J. (2009). The existence of mathematical objects in the classroom discourse. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education* (pp. 984-995). Lyon, France: INRP. Retrieved from http://www.inrp.fr/editions/editions-electroniques/cerme6/.

Forest, D., & Mercier, A. (2012). Classroom video data and resources for teaching: Some thoughts on teacher education. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From text to 'lived' resources* (pp. 215-230). Dordrecht, The Netherlands: Springer.

Friedrich, P. (1970). Shape in grammar. *Language*, *46*(2), 379-407.

Gallese, V., & Lakoff, G. (2005). The brain's concepts: The role of the sensory-motor system in conceptual knowledge. *Cognitive Neuropsychology*, *22*(3/4), 455-479.

Gibbs, R. W. (2006). *Embodiment and cognitive science.* Cambridge, England: Cambridge University Press.

Gray, E., & Tall, D. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic. *Journal for Research in Mathematics Education*, *26*(2), 115-141.

Guhe, M., Pease, A., Smaill, A., Martinez, M., Schmidt, M., Gust, H., . . . Krumnack, U. (2011). A computational account of conceptual blending in basic mathematics. *Cognitive Systems Research*, *12*(3/4), 249-265. doi:10.1016/j.cogsys.2011.01.004

Hume, D. (1921). *An enquiry concerning human understanding and selections from A treatise of human nature.* Chicago, IL: The Open Court Publishing Co. (Original works published 1748 and 1739, respectively)

Hume, D. (1965). *A treatise of human nature.* (L. A. Selby-Bigge, Ed). Oxford: Clarendon Press. (Original work published 1739)

Husserl, E. (1991). *Collected works: Vol. IV. On the phenomenology of the consciousness of internal time.* (R. Bernet, Ed. & J. B. Brough, Trans.). Dordrecht, The Netherlands: Kluwer.

Hutto, D. D., Kirchhoff, M. D., & Abrahamson, D. (2015). The enactive roots of STEM: Rethinking educational design in mathematics. *Educational Psychology Review*, *27*(3), 371-389.

Hwang, S., & Roth, W-M. (2011). *Scientific & mathematical bodies. The interface of culture and mind.* Rotterdam, The Netherlands: Sense Publishers.

Ilyenkov, E. V. (1982). *The dialectic of the abstract and the concrete in Marx's capital.* Moscow, Soviet Union: Progress Publishers.

Johnson, M. (1987). *The body in the mind*. Chicago, IL: University of Chicago Press.

Johnson, M. (2007). *The meaning of the body: Aesthetics of human understanding*. Chicago, IL: University of Chicago Press.

Kant, I. (1894). *Inaugural dissertation.* New York, NY: Columbia College. (Original work published 1770)

Kant, I. (1992). *Theoretical philosophy, 1755-1770.* (D. Walford, Trans. & Ed.). Cambridge, England: Cambridge University Press.

Kant, I. (2003). *Critique of pure reason.* (N. K. Smith, Trans.) New York, NY: St. Marin's Press. (Original work published 1781)

Kress, G. (2001). *Multimodal discourse: The modes and media of contemporary communication*. London, England: Arnold Publishers.

Kress, G. (2010). *Multimodality. A social semiotic approach to contemporary communication.* London, England: Routledge.

Lakoff, G. (1987). *Women, fire, and dangerous things*. Chicago, IL: University of Chicago Press.

Lakoff, G. (1992). The contemporary theory of metaphor. In A. Ortony (Ed.), *Metaphor and thought* (2nd ed., pp. 203-204). Cambridge, England: Cambridge University Press.

Lakoff, G., & Johnson, M. (1980). *Metaphors we live by*. Chicago, IL: University of Chicago Press.

Lakoff, G., & Johnson, M. (1999). *Philosophy in the flesh.* New York, NY: Basic Books.

Lakoff, G., & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York, NY: Basic Books.

Lefebvre, H. (2009). *Dialectical materialism.* Minneapolis, MN: University of Minnesota Press.

Leibniz [or Leibnitz], G. W. (1949). *New essays concerning human understanding.* La Salle, IL: The Open Court. (Original work published 1705)

Leont'ev, A. N. (1978). *Activity, consciousness, and personality.* Englewood Cliffs, NJ: Prentice-Hall.

Liddell, S. K. (1998). Grounded blends, gestures and conceptual shifts. *Cognitive Linguistics, 9*(3), 283-314.

Malafouris, L. (2012). Prosthetic gestures: How the tool shapes the mind. *The Behavioral and Brain Sciences*, *35*(4), 230-1.

Maturana, H. R., & Varela, F. J. (1992). *The tree of knowledge: the biological roots of human understanding*. Boston, MA: Shambhala Publications.

McNeill, D. (2000). *Language and gesture.* Cambridge, England: Cambridge University Press.

McNeill, D. (2005). *Gesture and thought.* Chicago, IL: University of Chicago Press.

Merleau-Ponty, M. (1962). *Phenomenology of perception*. London: Routledge & Kegan Paul. (Original work published 1945)

Mikhailov, F. T. (1980). *The riddle of the self.* Moscow, Russia: Progress Publishers.

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics*, *70*, 159-174.

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education*, *44*(2), 372-415.

Nemirovsky, R., Rasmussen, C., Sweeney, G., & Wawro, M. (2012). When the classroom floor becomes the complex plane: Addition and multiplication as ways of bodily navigation. *Journal of the Learning Sciences*, *21*, 287-323.

Núñez, R. E. (2000). Mathematical idea analysis: What embodied cognitive science can say about the human nature of mathematics. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 24th International Conference for the Psychology of Mathematics Educatio*n (Vol. 1, pp. 3-22). Hiroshima, Japan: PME.

Núñez, R. E. (2008). Mathematics, the ultimate challenge to embodiment: Truth and the grounding of axiomatic systems. In P. Calvo & A. Gomila (Eds.), *Handbook of cognitive science: An embodied approach* (pp. 333-353). Philadelphia, PA: Elsevier.

Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, *39*(1-3), 45-65.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit. *Journal for Research in Mathematics Education*, *40*(4), 396-426.

Piaget, J. (1924). L'expérience humaine et la causalité physique [Human experience and physical causality]. *Journal de Psychologie Normale et Pathologique*, *21*, 586-607.

Piaget, J. (1970). *Genetic epistemology.* New York, NY: W. W. Norton.

Piaget, J. (1973). *Introduction à l'épistémologie génétique* (Vol. 1) [Introduction to genetic epistemology (Vol. 1)]. Paris, France: Presses Universitaires de France.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London, UK, Routledge and Kegan Paul.

Radford, L. (2005). The semiotics of the schema. Kant, Piaget, and the calculator. In M. H. G. Hoffmann, J. Lenhard, & F. Seeger (Eds.), *Activity and sign. Grounding mathematics education* (pp. 137-152). New York, NY: Springer.

Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM - The International Journal on Mathematics Education*, *40*(1), 83-96.

Radford, L. (2009a). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, *70*(2), 111-126.

Radford, L. (2009b). ‘‘No! He starts walking backwards!’’: Interpreting motion graphs and the question of space, place and distance. *ZDM - The International Journal on Mathematics Education*, *41*, 467-480.

Radford, L. (2010). The eye as a theoretician: Seeing structures in generalizing activities. *For the Learning of Mathematics*, *30*(2), 2-7.

Radford, L. (2012). On the development of early algebraic thinking. *PNA*, *6*(4), 117-133.

Radford, L. (2013). Sensuous cognition. In D. Martinovic, V. Freiman, & Z. Karadag (Eds.), *Visual mathematics and cyberlearning* (pp. 141-162). New York, NY: Springer.

Radford, L. (2014a). Towards an embodied, cultural, and material conception of mathematics cognition. *ZDM - The International Journal on Mathematics Education*, *46*, 349-361.

Radford, L. (2014b). The progressive development of early embodied algebraic thinking. *Mathematics Education Research Journal*, *26*(2), 257-277.

Radford, L., & D'Amore, B. (2006). Semiotics, culture, and mathematical thinking. *Special issue of Revista latinoamericana de investigación en matemática educativa* Retrieved from http://luisradford.ca

Radford, L., Demers, S., Guzmán, J., & Cerulli, M. (2003). Calculators, graphs, gestures, and the production of meaning. In N. A. Pateman, B. Dougherty, & J. Zilliox (Eds.), *Proceedings of the 27th Conference of the International Group for the Psychology of Mathematics Education (PME27 -PMENA25)* (Vol. 4, pp. 55-62). Hilo, HI: University of Hawaii.

Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond words. *Educational Studies in Mathematics*, *70*(2), 91-95.

Radford, L., Schubring, G., & Seeger, F. (2008). *Semiotics in mathematics education: Epistemology, history, classroom, and culture.* Rotterdam, The Netherlands: Sense Publishers.

Reid, D., & Mgombelo, J. (2015). Survey of key concepts in enactivist theory and methodology. *ZDM - The International Journal on Mathematics Education*, *47*, 171-183.

Roth, W.-M. (2010). Incarnation: Radicalizing the embodiment of mathematics. *For the Learning of Mathematics*, *30*(2), 8-17.

Roth, W. M. (2012). Tracking the origins of signs in mathematical activity: A material phenomenological approach. In M. Bockarova, M. Danesi, & R. Núñez (Eds.), *Cognitive science and interdisciplinary approaches to mathematical cognition* (pp. 209-247). Munich, Germany: Lincom Europa.

Seitz, J. A. (2000). The bodily basis of thought. *New Ideas in Psychology*, *18*, 23-40.

Sfard, A. (1994). Reification as the birth of metaphor. *For the Learning of Mathematics*, *14*(1), 44-55.

Sheets-Johnstone, M. (2009). *The corporeal turn*. Exeter, Devon, UK: imprint-academic.com.

Stevens, R. (2012). The missing bodies of mathematical thinking and learning have been found. *The Journal of the Learning Sciences*, *21*, 337-346.

Tall, D. (2004). Building theories: The three worlds of mathematics. *For the Learning of Mathematics*, *24*(1), 29-32.

Tall, D. (2008). The transition to formal thinking in mathematics. *Mathematics Education Research Journal*, *20*(2), 5-24.

Tall, D. (2013). *How humans learn to think mathematically.* Cambridge, England: Cambridge University Press.

Tall, D., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., . . . Yusof, Y. (2001). Symbols and the bifurcation between procedural and conceptual thinking. *Canadian Journal of Science, Mathematics, and Technology Education*, *1*(1), 81-104.

Tall, D., & Mejia-Ramos, J. P. (2010). The long-term cognitive development of reasoning and proof. In G. Hanna, H. N. Jahnke, & H. Pulte (Eds.), *Explanation and proof in mathematics* (pp. 137-149). New York, NY: Springer.

Talmy, L. (2000). *Toward a cognitive semantics. Volume I: Concept structuring systems.* Cambridge, MA: MIT Press.

Thom, J., & Roth, W. (2011). Radical embodiment and semiotics: Towards a theory of mathematics the flesh. *Educational Studies in Mathematics*, *77*(2-3), 267-284.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience*. Cambridge, MA: MIT Press.

Verillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education*, *10*, 77-101.

Vygotsky, L. S. (1978). *Mind in society.* Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1987). *Collected works* (Vol. 1). (R. W. Rieber & A. S. Carton, Eds.). New York, NY: Plenum.

Wartofsky, M. (1968). *Conceptual foundations of scientific thought.* New York, NY: The Macmillan Company.

Wartofsky, M. (1979). *Models, representation and the scientific understanding.* Dordrecht, The Netherlands: D. Reidel.

Watson, A., & Tall, D. (2002). Embodied action, effect, and symbol in mathematical growth. In A. Cockburn & E. Nardi (Eds), *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education, Norwich, UK* (Vol. 4, pp. 369-376). Norwich, England: PME.

Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review*, *9*(4), 625-636.

Winter, B., Marghetis, T., & Matlock, T. (2015). Of magnitudes and metaphors: Explaining cognitive interactions between space, time, and number. *Cortex*, *64*, 209-224.

Yoon, C., Thomas, M., & Dreyfus, T. (2011). Grounded blends and mathematical gesture spaces: Developing mathematical understandings via gestures. *Educational Studies in Mathematics*, *78*, 371-393.

Zandieh, M., Roh, K. H., & Knapp, J. (2014). Conceptual blending: Student reasoning when proving “conditional implies conditional” statements. *Journal of Mathematical Behavior*, *33*, 209-229.