# AN INTRODUCTION TO CONIC SECTIONS

TOH PEE CHOON

Abstract. A set of introductory notes to conic sections.

We begin with some remarks on notation. As far as possible points are denoted with upper case letters and lines with lower case letters. If $P$ and $Q$ are points, then $PQ$ denotes the line that cuts these two points, while $\overline{PQ}$ denote the line segment joining these two points. We also use $|PQ|$ to denote the distance between points $P$ and $Q$.

## 1. Parabola

Let $F_1$ and $F_2$ be a pair of distinct points. We denote the locus of points equidistant from $F_1$ and $F_2$ by

$$\{P : |PF_1| = |PF_2|\}. \tag{1.1}$$

Geometrically, the locus is the perpendicular bisector of the line segment $\overline{F_1F_2}$.

What about the analogous question of the locus of points equidistant from two lines? Before we answer the question, we require a well-defined measure of distance from a point to a line.

**Definition 1.1.** *If $\ell$ is a line and $P$ is a point, we use $|P_\ell|$ to denote the shortest distance from $P$ to $\ell$. In other words, if we drop a perpendicular from $P$ to $\ell$ and denote by $D$ the foot of this perpendicular,*

$$|P_\ell| = |PD|, \text{ where } D \in \ell \text{ and } \overline{PD} \perp \ell.$$

In the cartesian plane, either two lines intersect or they are parallel and distinct. In the latter case, the locus of points equidistant from a pair of non-intersecting parallel lines $\ell_1$ and $\ell_2$ is a line $\ell_3$ parallel to and "in-between" them. In the former case, if $\lambda_1$ and $\lambda_2$ intersect, the pair of angle bisectors at the point of intersection forms the required locus.

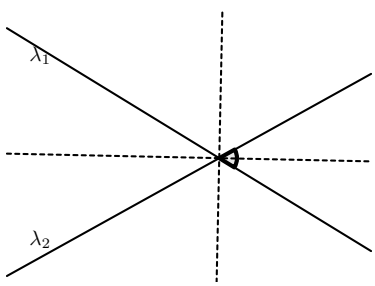


Figure 1.1. Locus of points equidistant from $\lambda_1$, $\lambda_2$.

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**Exercise 1.2.** *Let $\ell$ be a line and $F$ be a point distinct from $\ell$. Try to sketch the locus defined by $\{P : |PF| = |P_\ell|\}$.*

**Definition 1.3.** *Let $d$ be a line and $F$ be a point distinct from $d$. A* **parabola** *is defined geometrically by*

$$\{P : |PF| = |Pd|\}.$$

*The line $d$ is called the* **directrix** *while $F$ is called the* **focus** *of the parabola.*

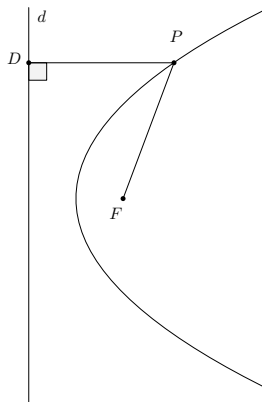


FIGURE 1.2. A parabola defined by directrix $d$ and focus $F$.

Observe that the line perpendicular to $d$ that passes through $F$ forms an **axis of symmetry** for the parabola. The point of intersection of a parabola and its axis of symmetry is called its **vertex**.

In order to recover the usual cartesian equation for the parabola, we let $F = (c, 0)$ and directrix $d$ be the line $x = k$. Let $P = (x, y)$ be a point on the parabola satisfying

$$|PF| = |Pd| \implies \sqrt{(x-c)^2 + y^2} = |k - x|.$$

We can square this equation and simplify the resulting expression to

$$y^2 = 2x(c-k) + k^2 - c^2.$$

We can further simplify by setting $k = -c$ to obtain $y^2 = 4cx$. We record our observation below.

**Theorem 1.4.** *The parabola with focus $F = (c, 0)$ and directrix $x = -c$ has the following cartesian form*

$$y^2 = 4cx.$$

This is the cartesian equation of a parabola in **standard position**. This means that the axis of symmetry of the parabola ($y = 0$ in this case) lies on either the $x$ or $y$ axes. Analogously, we also have $x^2 = 4cy$, if the focus $F = (0, c)$ and $y = -c$ is the directrix. We observe that in either situation, the vertex of the parabola lies at the origin. We can of course perform translation to obtain the following general cartesian equations for parabolas, which are occasionally known as parabolas in standard form:

$$(x - x_0)^2 = 4c(y - y_0);$$
$$(y - y_0)^2 = 4c(x - x_0).$$

We can also work backwards to obtain the geometric description of a parabola from its cartesian equation. For example, consider the equation $y = cx^2$. Since

$$x^2 = \frac{y}{c} \implies x^2 + \left(y - \frac{1}{4c}\right)^2 = \frac{y}{c} + \left(y - \frac{1}{4c}\right)^2.$$

We can simplify the right hand side and then take square roots to obtain

$$\sqrt{x^2 + \left(y - \frac{1}{4c}\right)^2} = \left|y + \frac{1}{4c}\right|.$$

The above can be interpreted as $|PF| = |P_d|$ where $F = (0, \frac{1}{4c})$ and the directrix, $d$, is $y = -\frac{1}{4c}$.

We can use **GeoGebra**, a dynamic geometry software to help us better understand the parabola.

**GeoGebra Example 1.5.** *We can verify the focus-directrix property for the parabola defined by the equation $y^2 = 4xc$.*

(1) Create a slider named $c$ with range $-5 \leq c \leq 5$.
(2) In the Input window, enter
```
y^2=4*x*c
```
(3) From Theorem 1.4 the focus is $F = (c, 0)$ and the directrix is $x = -c$. We can create these objects by typing the following in the Input window.
```
F=(c,0)
x=-c
```
(4) Vary the value of $c$ by dragging the slider and observe the change in the parabola, the focus and the directrix.
(5) Create a point $P$ on the parabola. (We can right-click on the point to rename it.)
(6) Use the distance tool to measure the length $|PF|$.
(7) To measure distance $|P_d|$, we need to construct the foot of the perpendicular from $P$ to the directrix $d$. An alternative is to create the a point D with $-c$ as its $x$-coordinate and the $y$-coordinate of $P$ as its $y$-coordinate. We do this with the following input.
```
D=(-c,y(P))
```
(8) Use the distance tool to measure the length $|PD|$ and verify that it always equals $|PF|$ by varying the position of $P$.

It is slightly more tricky to create a parabola directly from its focus and directrix. GeoGebra has a conics tool which accomplishes this but it is probably more instructive to do so from first principles. Let us first fix the focus as $F = (1, 0)$ and the directrix, $d : x = -1$. We wish to construct a point $P$ such that $|PF| = |P_d|$. Note that $|P_d|$ depends on a point $D$ which is the foot of the perpendicular from $P$ to the directrix. (In this case, it would be more precise to use $D_p$ to indicate the dependency on $P$ but we elect to sacrifice precision for clarity.) There is actually a one-to-one correspondence, i.e. every point $D$ on the directrix corresponds to exactly one point $P$ on the parabola. Since $|PF| = |PD|$, $DPF$ forms an isosceles triangle. This means that the perpendicular line from the vertex $P$ to the base $\overline{DF}$ of this isosceles triangle is actually the perpendicular bisector of the base. So in order to locate the point $P$, we can begin with an arbitrary point $D$ on the directrix and construct the perpendicular bisector of $\overline{DF}$. $P$ must lie on the intersection of this bisector and the line through $D$ that is perpendicular to the directrix.

**GeoGebra Example 1.6.** *We use the focus-directrix property to construct a parabola.*

(1) Create a sliders named $c$ and $k$ with ranges $-5 \leq c, k \leq 5$.
(2) Create the focus $F = (c, 0)$ and the directrix $x = k$ and label it as $d$ with the following input.

```
F=(c,0)
d:x=k
```

(3) Set $c = 1$ and $k = -1$. We can drag the slider or type directly into the input box.
(4) Create a point $D$ on the directrix. (We can right-click on the point to rename it.)
(5) Use the perpendicular bisector tool to construct the perpendicular bisector of $\overline{DF}$.
(6) Use the perpendicular line tool to construct the perpendicular line to $d$ from the point $D$.
(7) Create $P$ as the intersection of these two lines.
(8) Drag the point $D$ along the directrix and observe how $P$ traces out a parabola. We can right-click on $P$ and toggle trace on or off for better visualization. (We can use the Move Graphics View tool to move the window slightly to remove the traces.)
(9) Alternatively, we can use the locus tool. Select the locus point $P$ first, followed by $D$.
(10) Vary $c$ and $k$ to observe how the parabola changes with the focus and directrix.

In addition to the geometric and cartesian form, a parabola can also be represented by its **polar form**. Suppose we have a parabola with focus $F$ at the origin and directrix $x = k$ for some $k > 0$. An arbitrary point $P$ on the parabola can be represented by the ordered pair $(r, \theta)$ where $r = |PF|$ and $\theta$ is the angle made by $\overline{PF}$ and the $x$-axis. In this case, we can see that

$$|P_d| = k - r\cos(\theta).$$

Equating $|P_d| = |PF|$ and solving for $r$ gives the following result.

**Theorem 1.7.** *The parabola with focus $F = (0, 0)$ and directrix $x = k$, $k > 0$ has the following polar form*

$$r = \frac{k}{1 + \cos(\theta)}.$$

**Exercise 1.8.** *Show that the cartesian equation of the parabola with polar form: $r = \dfrac{k}{1 + \cos(\theta)}$ is given by*

$$y^2 = -k(2x - k).$$

**Exercise 1.9.** *Let $F = (0, 0)$ and $k > 0$. For each of the three possible directrices: $x = -k$, $y = k$ and $y = -k$, find the corresponding polar form.*

**Exercise 1.10.** *Given a parabola defined by $y^2 = 4cx$, show that the polar form is given by*

$$r = 4c\cot(\theta)\mathrm{cosec}(\theta).$$

The **tangent line** is an important concept in the study of conic sections. We shall define the tangent line to a parabola without reference to derivatives and subsequently show that this definition is aligned with the usual definition from calculus.

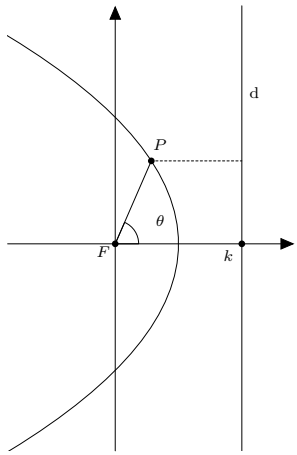


FIGURE 1.3. A parabola with the polar form $r = \dfrac{k}{1+\cos(\theta)}$.

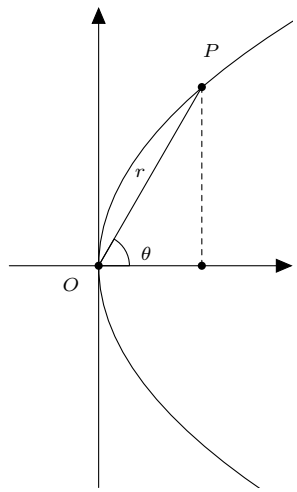


FIGURE 1.4. Polar form of the parabola $y^2 = 4cx$.

**Definition 1.11.** *Given a parabola and a point $P$ on the parabola. The tangent line to the parabola at $P$ is the unique line that intersects the parabola at exactly $P$.*

Consider the parabola $y^2 = 4cx$ and the straight line defined by $y = mx + k$. We wish to study the possible intersections of these two curves with respect to the parameters $m$ and $k$. Eliminating $y$ gives

$$(mx+k)^2 = 4cx \implies m^2x^2 + 2(mk-2c)x + k^2 = 0.$$

The discriminant of the quadratic is given by

$$4(mk-2c)^2 - 4m^2k^2 = 16c(c-mk).$$

So if $c < mk$ then there are no intersections but if $c > mk$, there are two points of intersection. The most interesting case is when there is a unique point of intersection. Here $c = mk$ and we see that the line

$$y = mx + \frac{c}{m}$$

touches the parabola at exactly one point for all values of $m$. Let $P = (x_0, y_0)$ be a point on the parabola. If $P$ is the vertex, i.e. $x_0 = 0$, then the tangent line is $x = 0$. For any other point $P$ on the parabola, the gradient $m$ can be found since

$$m^2 x_0 - y_0 m = -c \implies \left(m - \frac{y_0}{2x_0}\right)^2 = -\frac{c}{x_0} + \frac{y_0^2}{4x_0^2}.$$

Since $P$ lies on the parabola, we have $y_0^2 = 4cx_0$, this means that the right hand side of the above equation is zero. So

$$m = \frac{y_0}{2x_0} = \frac{2c}{y_0}$$

is the unique solution and the tangent line is given by

$$y = \frac{2c}{y_0}x + \frac{cy_0}{2c} \implies yy_0 = 2cx + 2cx_0.$$

We record this observation below.

**Theorem 1.12.** *Let $P = (x_0, y_0)$, where $x_0 \neq 0$, be a point on the parabola $y^2 = 4cx$. Then the tangent line to the parabola at $P$ has gradient $\frac{2c}{y_0}$ and is given by the equation*

$$yy_0 = 2cx + 2cx_0.$$

We remark that the gradient can be found easily using implicit differentiation.

$$2y \cdot \frac{dy}{dx} = 4c \implies \left.\frac{dy}{dx}\right|_{P=(x_0,y_0)} = \frac{2c}{y_0}.$$

We further note that in our previous calculations, while both $x_0$ and $y_0$ appear in the equation, they are not independent and are related by $y_0^2 = 4cx_0$. This relation was used on several occasions to simplify the computations. This can be potentially confusing and one possible alternative is to introduce the parametrization

$$x = ct^2 \text{ and } y = 2ct,$$

which holds for the parabola $y^2 = 4cx$. By the chain rule, we also have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2c}{2ct} = \frac{1}{t} \implies \left.\frac{dy}{dx}\right|_{P=(x_0,y_0)} = \frac{1}{t_0} = \frac{2c}{y_0}.$$

**Exercise 1.13.** *Let $\ell$ be the tangent line to the parabola $y^2 = 4cx$ at $P$. Let $N$ be the intersection of $\ell$ and the $x$-axis. Show that $y$-axis bisects $\overline{PN}$.*

**Exercise 1.14.** *Let $P$ be a point on a parabola with focus $F$. If the tangent line at $P$ intersects the directrix at $R$, show that $\angle RFP = 90°$.*

We next discuss an important property of a parabola known as the **reflection property**. It can be phrased as:"*light coming parallel to the axis of a parabolic mirror is reflected at the parabola to pass through the focus.*" Likewise, the reflection of a light ray coming from the focus will be parallel to the axis of symmetry of the parabola. The reflection property lies at the heart of many applications, for example, the design of a torchlight, reflecting telescopes and satellite dishes.

**GeoGebra Example 1.15.** *We verify the reflection property of a parabola.*

(1) Create a slider $-5 \leq c \leq 5$ and a point $T = (c, 0)$.
(2) Create a parabola $y^2 = 4x$. Note that the focus is at $(1, 0)$.
(3) Create a point $P$ on the parabola and the line $TP$
(4) Use the tangent tool to construct a tangent at $P$ to the parabola.

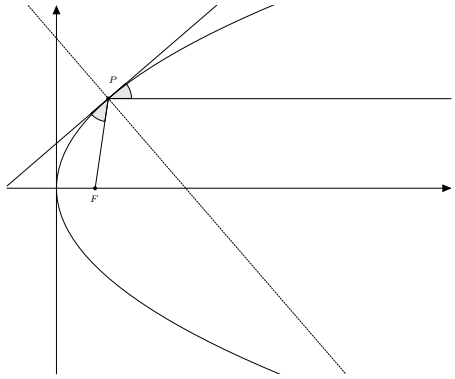


FIGURE 1.5. Light parallel to the axis of symmetry is reflected to pass through the focus.

(5) The reflection of the line $TP$ at $P$ is actually the reflection of the line about the normal to the parabola at $P$.
(6) Construct the normal at $P$ with the perpendicular line tool.
(7) Use the reflect about line tool to reflect $TP$ about the normal at $P$.
(8) Vary $c$ and observe the angle of the reflected line.

The reflection property is equivalent to the following result.

**Theorem 1.16.** *Let $P$ be a point on the parabola defined by $y^2 = 4cx$ with $c > 0$, focus $F$ and directrix $d$. Let $D$ be a point on $d$ such that $\overline{PD}$ is perpendicular to $d$. Then the tangent line to the parabola at $P$ makes equal angles with $\overline{PF}$ and $\overline{PD}$.*

*Proof.* If $P = (x_0, y_0)$, then $D = (-c, y_0)$. Since $F = (c, 0)$, we can compute the gradient of $DF$ to be $\frac{-y_0}{2c}$. Theorem 1.12 tells us that the gradient of the tangent line is $\frac{2c}{y_0}$, so the tangent line is perpendicular to $DF$. Let the intersection of the tangent line and $DF$ be $M$. Since $\triangle DPF$ is isosceles, $MP$ must coincide with the perpendicular bisector of $\overline{DF}$. Hence $\triangle DPM$ must be congruent to $\triangle FPM$ and so $\angle DPM = \angle FPM$. □

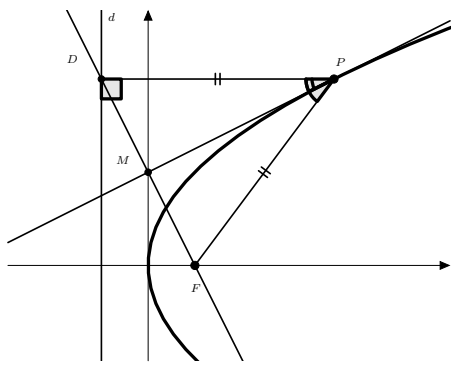


FIGURE 1.6. Proof of Theorem 1.16.

**Exercise 1.17.** *Let $Q = (q, 0)$ be a fixed point with $q > 0$. A line through $Q$ intersects a parabola defined by $y^2 = 4px$ ($p > 0$) at two points $A$ and $B$. Find an expression for the value of $|QA| \cdot |QB|$ and hence show that the minimum value is given by $4pq$.*

**Exercise 1.18.** *Let $A$ and $B$ be two points on a parabola. The line segment $\overline{AB}$ is also called a* **chord**. *If this chord passes through $F$ the focus, prove that the tangent lines at $A$ and $B$ intersect at right angles.*

**Exercise 1.19.** *Prove that the mid-points of chords through the focus of the parabola $y^2 = 4cx$ $(c > 0)$ lie on a second parabola with focus $(\frac{3c}{2}, 0)$ and directrix $x = \frac{c}{2}$.*

## 2. Ellipse

**Exercise 2.1.** *Let $A, F_1$ and $F_2$ be three points that form a triangle $\triangle F_2F_1A$. Find the locus of all points $P$ such that the triangle $\triangle F_2F_1P$ has the same area as $\triangle F_2F_1A$.*

The locus of $P$ is the line through $A$ parallel to $F_1F_2$ as well as its reflection in $F_1F_2$. We denote the line $F_1F_2$ by $\ell$ and recall the notation $|A_\ell|$ denotes the shortest distance from $A$ to $\ell$. Then the two parallel lines can be described as

$$\{P : |P_\ell| = |A_\ell|\} = \{P : |P_\ell| = k\}$$

since $|A_\ell|$ equals some fixed value $k$. We remark that the locus of a circle is given by the analogous form $\{P : |PF| = k\}$ where $k$ is the radius and $F$ the centre.

**Exercise 2.2.** *Let $A, F_1$ and $F_2$ be three points that form a triangle $\triangle F_2F_1A$. Find all points $P$ such that the triangle $\triangle F_2F_1P$ has the same* perimeter *as $\triangle F_2F_1A$.*

The locus of $P$ is $\{P : |PF_2| + |PF_1| = |AF_2| + |AF_1| = k\}$ for some real constant $k$, which defines an ellipse geometrically.

**Definition 2.3** (Geometric)**.** *An* **ellipse** *is the locus of points whose sum of distances to the* **foci** *$F_1$ and $F_2$ is a constant, i.e.*

$$\{P : |PF_1| + |PF_2| = k\}.$$

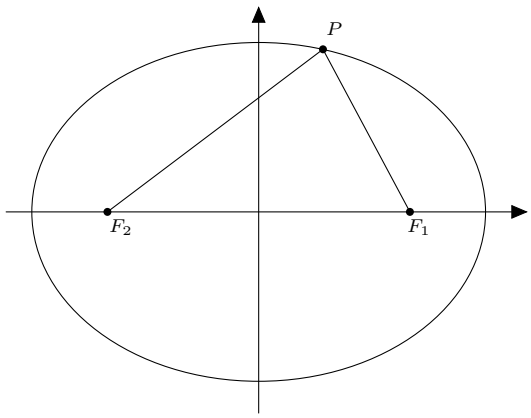


Figure 2.1. An ellipse.

We further define a **chord** as the line segment formed by any two distinct points on an ellipse. The **major axis** is defined as the chord that lies on the line $F_1F_2$, while the **minor axis** is defined as the chord that lies on the perpendicular bisector of $\overline{F_1F_2}$. The four **vertices** are the points of intersections of the major and minor axes with the ellipse.

**Theorem 2.4.** *The major and minor axes correspond to the largest and respectively smallest distance between anti-podal points of the ellipse.*

*Proof.* Although the result is geometrically evident, we shall provide a rigorous proof using Apollonius' Theorem. The theorem states that in any triangle $\triangle ABC$, if $\overline{AD}$ is the median, i.e. $D$ bisects $\overline{BC}$, then

$$|AB|^2 + |AC|^2 = 2(|AD|^2 + |BD|^2).$$

Now let $P$ be an arbitrary point on a ellipse defined by $\{P : |PF_1| + |PF_2| = k\}$ and let $O$ be the midpoint of $\overline{F_1F_2}$. If we let $|PF_1| = x$, then we have

$$\begin{aligned} 2|OP|^2 + 2|OF_1|^2 &= |PF_1|^2 + |PF_2|^2 \\ &= x^2 + (k-x)^2 \\ &= 2\left(x - \frac{k}{2}\right)^2 + \frac{k^2}{2} \\ \implies |OP|^2 &= \left(x - \frac{k}{2}\right)^2 + \frac{k^2}{4} - |OF_1|^2. \end{aligned}$$

So $|OP|$ is minimum when $x = \frac{k}{2}$. In other words, $P$ is on the perpendicular bisector of $\overline{F_1F_2}$, proving that the minor axis is the smallest distance. On the other hand, $|OP|$ is maximized when $x$ is maximized, i.e. when $P$ lies on the ray from $F_1$ to $F_2$. □

**Exercise 2.5.** *Use Pythagoras' Theorem (three times) to prove Apollonius' Theorem.*

We now obtain the familiar cartesian equation of an ellipse from its geometric definition. For simplicity, let us fix $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ for some $c > 0$. This means the major and minor axes lie on the $x$ and $y$ axes respectively. By symmetry, the vertices are at $(\pm a, 0)$ and $(0, \pm b)$ for some $a > b > 0$. Now when $P = (a, 0)$,

$$|PF_1| + |PF_2| = a - c + (a + c) = 2a.$$

On the other hand, when $P = (0, b)$, by Pythagoras' Theorem,

$$|PF_1| + |PF_2| = 2\sqrt{c^2 + b^2} = 2a.$$

Hence $c = \sqrt{a^2 - b^2}$. Since $|PF_1| + |PF_2| = 2a$, we have for any point $P = (x, y)$ on the ellipse,

$$\begin{aligned} \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} &= 2a \\ \implies (x+c)^2 + y^2 &= \left(2a - \sqrt{(x-c)^2 + y^2}\right)^2 \\ &= 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 \\ \implies 4xc - 4a^2 &= -4a\sqrt{(x-c)^2 + y^2} \\ \implies \left(a^2 - cx\right)^2 &= a^2(x-c)^2 + a^2y^2 \\ \implies a^4 + c^2x^2 &= a^2x^2 + a^2c^2 + a^2y^2 \\ \implies a^2(a^2 - c^2) &= (a^2 - c^2)x^2 + a^2y^2 \\ \implies a^2b^2 &= b^2x^2 + a^2y^2 \\ \implies 1 &= \frac{x^2}{a^2} + \frac{y^2}{b^2}. \end{aligned}$$

We record the above as the following theorem.

**Theorem 2.6.** *An ellipse defined by* $\{P : |PF_1| + |PF_2| = 2a\}$ *with foci* $F_1 = (c, 0)$ *and* $F_2 = (-c, 0)$ *satisfies the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

*where* $b = \sqrt{a^2 - c^2}$.

We remark that the **focal length** of the ellipse defined as $|F_1F_2| = 2c = 2\sqrt{a^2 - b^2}$. Furthermore, if $c \to 0$, then $a \to b$ and the circle is the limiting form of an ellipse. In the case of a circle, there is a unique focus and every chord (i.e. diameter) is both a major and minor axis. The analogous results hold if we fix the major axis on the $y$-axis.

**Exercise 2.7.** *An ellipse defined by* $\{P : |PF_1| + |PF_2| = 2b\}$ *with foci* $F_1 = (0, c)$ *and* $F_2 = (0, -c)$ *satisfies the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

*where* $a = \sqrt{b^2 - c^2}$.

The equations in Theorem 2.6 and Exercise 2.7 are the cartesian equations of ellipses in standard position. We can also perform translation to obtain what is commonly known as the standard form of an ellipse

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1.$$

In the above standard form, location of the major axis, and hence the foci, is implicit and depend of the relative sizes of $a$ and $b$.

**Exercise 2.8.** *Use GeoGebra to graph the curve* $3x^2 + 5xy + 4y^2 - 8x + 10y = 5$. *Observe that the curve is an ellipse not in standard position.*

**Exercise 2.9.** *Let*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b > 0,$$

*represent an ellipse in standard position and* $F$ *be a focus. Find the length of the chord parallel to the minor axis and passing through* $F$. *This chord is called a* **latus rectum**, *which means "straight side" in Latin.*

**GeoGebra Example 2.10.** *We can verify the geometric property of an ellipse defined by the equation* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(1) Create a sliders $0 \le a, b \le 5$.
(2) In the Input window, enter
```
x^2/a^2+y^2/b^2=1
```
(3) Vary the values of $a$ and $b$ and observe how the ellipse changes.
(4) Set $a$ to a value larger than $b$ and observe that major axis is on the $x$-axis.
(5) Create the foci $F_1 = (\sqrt{a^2 - b^2}, 0)$ and $F_2 = (-\sqrt{a^2 - b^2}, 0)$ by entering
```
F_1=(sqrt(a^2-b^2),0)
F_2=(-x(F_1),0)
```
(6) Create a point $P$ on the ellipse and use the segment tool to construct $\overline{PF_1}$ and $\overline{PF_2}$.
(7) Use the distance tool measure the length of these two segments, naming them $f$ and $g$ respectively.
(8) In the Input window, enter
```
dev=f+g-2*a
```

(9) The value of **dev** appears in the Algebra window and should always equals 0 as long as $a > b$. Vary $P$ and $a$ to check.

**Example 2.11.** *If the major axis of an ellipse given by the equation*

$$\frac{x^2}{4+t} + \frac{y^2}{t^2-2} = 1$$

*lies on the $y$-axis, find the range of values of $t$.*

*Solution.* We require $4 + t > 0 \implies t > -4$ and

$$t^2 - 2 > 0 \implies t < -\sqrt{2} \text{ or } t > \sqrt{2}.$$

Furthermore if the major axis is on the $y$-axis, we need $\sqrt{t^2-2} \geq \sqrt{4+t}$. In other words,

$$t^2 - t - 6 \geq 0 \implies t \geq 3 \text{ or } t \leq -2.$$

Combining the two gives $t \geq 3$ or $-4 < t \leq -2$.

We remark that in the special case of a circle ($t = 3$ and $-2$), every possible diameter can be considered a major axis. We can also use GeoGebra to verify our calculations. □

**GeoGebra Example 2.12.** *We use GeoGebra to explore the properties of an ellipse with foci $F_1 = (5, 0)$ and $F_2 = (3, 0)$ and major axis of length 4.*

(1) Since the foci lie on the major axis, one of the vertex of this ellipse must be $(6, 0)$.
(2) Create $F_1$, $F_2$ and the vertex $(6, 0)$.
(3) Use the ellipse tool to plot the required ellipse.
(4) Create an arbitrary point $P$ on this ellipse and construct the segment $\overline{PF_2}$.
(5) Vary the position of $P$ and try to deduce a relation between $|PF_2|$ and the coordinates of $P$.

In the above example, it appears that $|PF_2|$ is always half the value of the $x$-coordinate of $P$. In other words, any point $P$ is always twice as far away from the $y$-axis as its distance to $F_2$. If we define $d$ as the line $x = 0$, then

$$|Pd| = 2|PF_2|.$$

This is reminiscent of the geometric definition of a parabola in terms of a focus and a directrix, with the exception of the factor of 2. To see this clearly, we can define $D = (0, y(P))$, i.e. $D$ is on the directrix $x = 0$ and has the same $y$-coordinate as $P$. We continue to explore this concept with the next example.

**Example 2.13.** *Let $F = (0, 0)$ and $d$ be the line $x = 4$. Find the locus of points whose distance from $d$ is thrice the distance from $F$, i.e.*

$$\{P : |Pd| = 3|PF|\}.$$

*Solution.* From definition, we can see that $(1, 0)$ satisfy the equation and so does $(-2, 0)$. We further guess that there may be points on the $y$-axis. Let $P = (0, y)$ and $|Pd| = 4$, so we have $y = \pm\frac{4}{3}$. These four points are insufficient to determine curve. In general, just as two points determine a line, three (non-collinear) points determine a circle, five distinct points are required to determine a general conic section. It is instructive to use plot these four points and use the conics tool in GeoGebra to see how different types of conic sections all intersect these four points.

Returning to the example, we represent $P = (x, y)$ and work out the cartesian equation explicitly.

$$3\sqrt{x^2+y^2} = 4 - x \implies 8\left(x + \frac{1}{2}\right)^2 + 9y^2 = 18.$$

□

**Definition 2.14** (Directrix-Eccentricity-Focus)**.** *An ellipse is the locus of points whose distance from a focus $F$ and to a directrix $d$ are in a constant ratio less than 1. This ratio is termed the* **eccentricity***. The ellipse can be described as*

$$\left\{P : \frac{|PF|}{|Pd|} = e,\ 0 < e < 1\right\}.$$

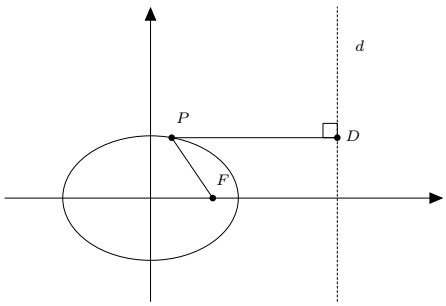


FIGURE 2.2. Directrix-Eccentricity-Focus definition of an ellipse.

The Directrix-Eccentricity-Focus properties of conic sections were first recorded in the works of Pappus of Alexandria (290 $\sim$ 350 AD). The eccentricity, $e$, is a measure of how "uncircular" the ellipse is, i.e. as $e \to 0$, the ellipse becomes a circle. Furthermore, we recognize that when $e = 1$ gives us the definition of a parabola. We shall see later that the same locus with $e > 1$ describes a hyperbola, so the Directrix-Eccentricity-Focus definition is a uniform definition that works for all non-degenerate conic sections.

We shall work out the cartesian equation from the Directrix-Eccentricity-Focus definition. Let $F = (c, 0)$ and directrix $d$ be given by the line $x = k$, $k \neq c$. We substitute $P = (x, y)$ into $\frac{|PF|}{|Pd|} = e$ to get

$$(x - c)^2 + y^2 = e^2(k - x)^2. \tag{2.1}$$

If $e = 0$, only the point $F$ satisfies the equation. To see what is meant by the eccentricity as a measure of how "uncircular" the ellipse is, we set $e = \frac{1}{k}$ and let $k \to \infty$. In the limit, we obtain

$$(x - c)^2 + y^2 = 1.$$

We return to equation (2.1). By expanding and regrouping terms we obtain

$$x^2(1 - e^2) - 2x(c - ke^2) + y^2 = e^2k^2 - c^2.$$

We note that if $e = 1$, the equation above describes a parabola. If $e \neq 1$, we can divide throughout by the factor $(1 - e^2)$ and complete the square to obtain the following equation of an ellipse.

$$\left(x - \frac{c - ke^2}{1 - e^2}\right)^2 + \frac{y^2}{1 - e^2} = \frac{e^2(k - c)^2}{(1 - e^2)^2}. \tag{2.2}$$

For standard position, we can set $k = c/e^2$. The equation simplifies to

$$\frac{e^2x^2}{c^2} + \frac{e^2y^2}{c^2(1 - e^2)} = 1.$$

We can now set $a = c/e$ and $b = a\sqrt{1 - e^2}$ to obtain the usual cartesian equation in Theorem 2.6. Furthermore, all our computations are reversible, i.e. we can begin from the cartesian equation and work backwards to $\frac{|PF|}{|Pd|} = e$ to obtain the focus,

directrix and eccentricity in terms of the parameters $a$ and $b$. We record the various relationships below.

**Theorem 2.15.** *If $a > b > 0$, the ellipse satisfying the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

*has a focus $F_1 = (c, 0)$ where $c = \sqrt{a^2 - b^2}$ and directrix $d$ given by $x = \frac{a}{e}$ where eccentricity $e$ is given by*

$$e = \sqrt{1 - \frac{b^2}{a^2}}.$$

*An alternative description in terms of these parameters is*

$$\left\{ P : \frac{|PF_1|}{|Pd|} = e, \ \ 0 < e < 1 \right\}.$$

Two more important observations are in order. Firstly, by symmetry, we could use the focus $F_2 = (-c, 0)$ and directrix $d'$ given by $x = -\frac{a}{e}$, with the same values of $c$ and $e$. The second observation is that the foci defined here are exactly the same foci in Definition 2.3 and Theorem 2.6. For if

$$|PF_1| = e|P_d| \quad \text{and} \quad |PF_2| = e|P_{d'}|,$$

then the sum

$$|PF_1| + |PF_2| = e\left(|P_d| + |P_{d'}|\right) = e\left(\frac{a}{e} - \left(-\frac{a}{e}\right)\right) = 2a.$$

We saw previously how the perpendicular bisector was used to construct a parabola directly from its focus and directrix. For an ellipse defined by $|PF| = e|P_d|$, we use the facts that $|P_d| = m$ is the locus of two lines parallel to $d$ and $|PF| = em$ is the locus of a circle with radius $em$. Thus $P$ can be located as the intersection of these lines with the circle.

**GeoGebra Example 2.16.** *We use the directrix-eccentricity-focus property, $|PF| = e|P_d|$, to construct an ellipse.*

(1) Create sliders $0 \leq e \leq 5$ and $-5 \leq k \leq 20$.
(2) Create the directrix $x = k$ and focus $F = (0, 0)$.
(3) Create a slider $0 \leq m \leq 100$, where $m$ is the value of $|Pd|$. This means that $P$ lies on one of the lines $x = k + m$ or $x = k - m$. Create these two auxiliary lines.
(4) On the other hand, $P$ lies on a circle of radius $em$ with centre $F$. Use the circle with center and radius tool to create a circle with centre at $F$ and radius $em$.
(5) Vary the values of $e, k$ and $m$ so that four possible intersections of this circle with the previous two lines gives $P$.
(6) Use the locus tool to draw the locus of each of the four points depending the variable $m$.
(7) Declutter the plot by hiding the circle and the two auxiliary parallel lines. Right-click on the circle and uncheck show object.
(8) Adjust $e$ to a value near 0 and observe how the ellipse is almost a circle. Increase $k$ to increase the "radius".
(9) In fact, our construction works well for values $e \geq 1$. The ellipse morphs into a parabola at $e = 1$ and a hyperbola as $e$ increases.

We consider yet another possible way to obtain an ellipse. We begin with the circle $x^2 + y^2 = a^2$ and apply scaling of factor $\frac{b}{a}$ parallel to the $y$-axis. If $Q$ is a

point of the circle, it can be parametrized as $Q = (a\cos t, a\sin t)$ for $-\pi < t \leq \pi$. Let $P$ be the corresponding point on the ellipse. The coordinates of $P$ is given by

$$P = (a\cos t, b\sin t).$$

It is important to note that $t$ is the angle between $\overline{OQ}$ and the $x$-axis and *not equal* $\theta$, the angle between $\overline{OP}$ and the $x$-axis. The two angles are related by

$$\tan(\theta) = \frac{b}{a}\tan(t).$$

The angle $t$ is sometimes called the **eccentric angle** for the point $P$ on the ellipse.

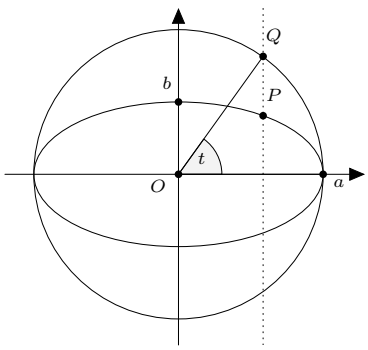


FIGURE 2.3. An ellipse as a scaled circle.

Using standard techniques in calculus, we can compute the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by

$$\text{Area} = 4\int_0^a y\,dx = 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}\,dx.$$

Using the parametrization $x = a\cos t$, which means $\frac{dx}{dt} = -a\sin t$. So

$$\begin{aligned}\text{Area} &= 4\int_{\frac{\pi}{2}}^0 b\sqrt{1-\cos^t}(-a\sin t)\,dt \\ &= 4ab\int_0^{\frac{\pi}{2}} \sin^2 t\,dt \\ &= \pi ab.\end{aligned}$$

Likewise, we can use the arc length formula to write down an expression for the perimeter of the ellipse. However this integral is not easily evaluated.

$$\begin{aligned}\text{Perimeter} &= 4\int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= 4a\int_0^{\frac{\pi}{2}} \sqrt{1 + \frac{b^2\cos^2 t}{a^2\sin^2 t}}\sin t\,dt \\ &= 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - e^2\cos^2 t}\,dt\end{aligned}$$

where $e$ is the eccentricity.

It turns out that for an ellipse in standard position, the expression $|PF_1|$ has an elegant expression.

**Example 2.17.** *Let* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *be an ellipse with* $a > b > 0$ *and let* $F_1$ *be the right focus. Show that for an arbitrary point* $P = (a\cos t, b\sin t)$,

$$|PF_1| = a(1 - e\cos t).$$

*Solution.*

$$\begin{aligned}
|PF_1| &= \sqrt{(a\cos t - \sqrt{a^2-b^2})^2 + b^2\sin^2 t} \\
&= \sqrt{a^2\cos^2 t - 2a(\cos t)\sqrt{a^2-b^2} + a^2 - b^2 + b^2(1-\cos^2 t)} \\
&= \sqrt{\left(\sqrt{a^2-b^2}\cos t - a\right)^2} \\
&= a\sqrt{\left(\sqrt{1-\frac{b^2}{a^2}}\cos t - 1\right)^2} \\
&= a(1 - e\cos t).
\end{aligned}$$

□

It is actually not necessary to use the parametric equations and we can show that if $P = (x_0, y_0)$ then $|PF_1| = a - ex_0$. A similar calculation shows that $|PF_2| = a(1 + e\cos t) = a + ex_0$ and consequentially we recover

$$|PF_1| + |PF_2| = 2a.$$

As in the case of the parabola, we wish to study the conditions for a straight line, $y = mx + k$, to intersect an ellipse at exactly one point. Combining with the cartesian equation of an ellipse in standard position, we have

$$\frac{x^2}{a^2} + \frac{(mx+k)^2}{b^2} = 1$$

which simplifies to

$$(a^2m^2 + b^2)x^2 + 2a^2mkx + a^2(k^2 - b^2) = 0.$$

For a unique solution, we set the discriminant

$$4a^4m^2k^2 - 4a^2(a^2m^2 + b^2)(k^2 - b^2) = 0,$$

which reduces to

$$k^2 = a^2m^2 + b^2.$$

**Theorem 2.18.** *The line*

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

*intersects the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *at exactly one point.*

**Exercise 2.19.** *Let* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *be an ellipse and* $P = (x_0, y_0)$ *be a point on the ellipse. Show that the equation of the tangent line at* $P$ *is given by*

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$

**Exercise 2.20.** *Show that if the tangents to two points of an ellipse,* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *intersect at right angles, the intersection lies on the* **director circle**, *which is a circle with radius* $\sqrt{a^2+b^2}$ *and centred at the origin.*

Similar to a parabola, an ellipse also satisfies a **reflection property** which can be phrased as "light emitted from one focus is reflected at an elliptic mirror to pass through the other focus." Ellipsoidal reflector spotlights used in stage lighting are designed based on the reflection property.

**Theorem 2.21** (Reflection Property)**.** *Given an ellipse with foci $F_1$ and $F_2$. Let $P$ be a point on the ellipse and $N$ be on the major axis such that $PN$ is normal to the tangent line of the ellipse at $P$. Then*

$$\angle F_2PN = \angle NPF_1.$$

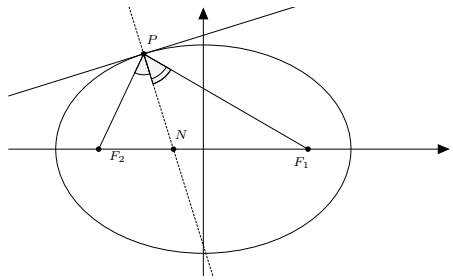


FIGURE 2.4. Reflection property for an ellipse in standard position.

We first prove the following result from geometry.

**Theorem 2.22** (Angle Bisector Theorem)**.** *In $\triangle ABC$ with $D$ on $\overline{BC}$,$\overline{AD}$ bisects $\angle BAC$ if and only if $\frac{|AB|}{|BD|} = \frac{|AC|}{|CD|}$.*

*Proof.* To prove $\angle BAD = \angle CAD$, we extend $\overline{BA}$ to a point $C'$ such that $|AC'| = |AC|$. Since

$$\frac{|AB|}{|BD|} = \frac{|AC|}{|CD|} = \frac{|AC'|}{|CD|} = \frac{|BC'|}{|BC|},$$

we see that $\triangle BAD$ is similar to $\triangle BC'C$. Thus $AD$ is parallel to $C'C$ and

$$\angle BAD = \angle BC'C = \angle BCC' = \angle CAD.$$

The converse can be proven in a similar manner by extending $\overline{BA}$ to a point $C'$ such that $C'C$ is parallel to $AD$. □

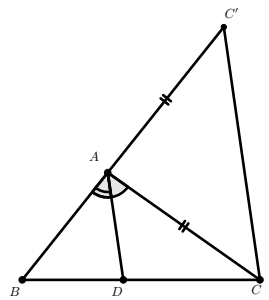


FIGURE 2.5. Proof of Angle Bisector Theorem

*Proof of Theorem 2.21.* We will show that $\frac{|PF_2|}{|NF_2|} = \frac{|PF_1|}{|NF_1|}$. Recall from Example 2.17 that $|PF_1| = a - ex_0$ and $|PF_2| = a + ex_0$.

Next we can use Theorem 2.18 or differentiation to find that the gradient of the tangent line at $P = (x_0, y_0)$ is given by $-\frac{x_0 b^2}{y_0 a^2}$. So the gradient of $PN$ is given by $\frac{a^2 y_0}{b^2 x_0}$. Let $N = (x_1, 0)$, then

$$\frac{y_0}{x_0 - x_1} = \frac{a^2 y_0}{b^2 x_0} \implies x_1 = x_0 \left(1 - \frac{b^2}{a^2}\right) = e^2 x_0,$$

where $e$ is the eccentricity. This means that

$$|NF_1| = c - x_1 = ae - e^2 x_0 = e(a - ex_0),$$

and similarly $|NF_2| = e(a + ex_0)$. Thus the ratio

$$\frac{|PF_2|}{|NF_2|} = \frac{|PF_1|}{|NF_1|} = \frac{1}{e}.$$

□

The Directrix-Eccentricity-Focus property of an ellipse can be reformulated into a polar equation. Suppose that focus $F_1$ is now at the origin and the directrix, $d$, is $x = k$ for some $k > 0$. Then

$$|PF_1| = e|P_d| \implies r = e(k - r\cos(\theta)).$$

Solving for $r$ gives the following.

**Theorem 2.23** (Polar form with respect to the focus)**.** *Let $F_1 = (0, 0)$ be a focus and $x = k$, $k > 0$ be the directrix. The polar form of the resulting ellipse with eccentricity $e$ is*

$$r = \frac{ek}{1 + e\cos\theta}.$$

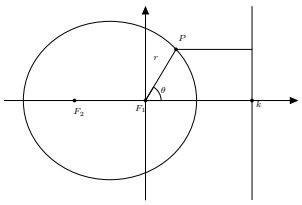


FIGURE 2.6. Polar form of an ellipse.

**Exercise 2.24.** *Let $F_1 = (0, 0)$ and $k > 0$. For each of the three possible directrices: $x = -k$, $y = k$ and $y = -k$, find the corresponding polar form of an ellipse with eccentricity $e$.*

**Example 2.25.** *Given an ellipse in standard position. The polar form of the resulting ellipse with respect to the centre is*

$$r = \frac{ab}{\sqrt{a^2 \sin^2\theta + b^2 \cos^2\theta}}.$$

*Solution.* Let $P = (a\cos t, b\sin t)$ be a point on the ellipse with polar coordinates $(r, \theta)$. Recall that $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{b\sin t}{a\cos t}$. So

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{b^2\sin^2 t}{a^2\cos^2 t} + 1 \implies \frac{a^2\cos^2 t}{\cos^2\theta} = r^2.$$

On the other hand,

$$\frac{a^2\sin^2\theta}{b^2\cos^2\theta} + 1 = \frac{\sin^2 t}{\cos^2 t} + 1 \implies \frac{a^2\sin^2\theta + b^2\cos^2\theta}{b^2\cos^2\theta} = \frac{1}{\cos^2 t}.$$

□

**Exercise 2.26.** *The eccentricity of an ellipse in its standard position is* $\frac{\sqrt{2}}{2}$. *Its foci on the x-axis are denoted by* $F_1$ *and* $F_2$. *For any point* $Q$ *on the ellipse, find the coordinates of the point* $Q$ *when* $\angle F_2QF_1$ *is maximum.*

**Exercise 2.27.** *In the diagram* $\overline{AB}$ *is the major axis of an ellipse in standard position. The tangent to the ellipse at* $P$ *intersects the tangents to vertices* $A$ *and* $B$ *at* $C$ *and* $D$ *respectively. Show that* $|PF_1||PF_2| = |PC||PD|$.

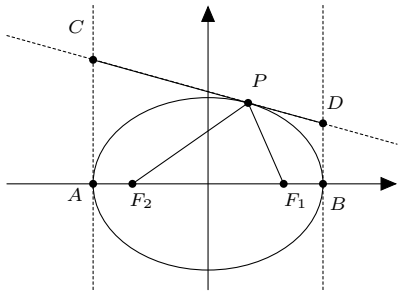


FIGURE 2.7. Exercise 2.27.

**Exercise 2.28.** *Cut out a paper disc and mark out an arbitrary interior point* $X$ *distinct from the centre of the disc. Now pick a random point* $P$ *on the circumference of the disc and and fold the disc (with one crease) such that* $P$ *touches* $X$. *Use a pencil to mark out the crease. Repeat for as many points* $P$ *as possible. What do you observe? Explain.*

**GeoGebra Example 2.29.** *We visualize the previous exercise.*

(1) Create a point $A$.
(2) Construct a circle of radius 3 with centre $A$.
(3) Create a point $X$ in the interior of the circle.
(4) Create a point $P$ on the circle.
(5) Construct the perpendicular bisector of $X$ and $P$.
(6) Right-click the perpendicular bisector and toggle trace on.
(7) Right-click on $P$ and toggle animation on.
(8) To stop the animation right-click on $P$ in the Algebra window and toggle animation off.
(9) Vary the position of $X$. What happens when $X$ is on the circle? What if $X$ is outside the circle?

**Exercise 2.30.** *In the diagram the tangents to an ellipse in standard position at points* $P$ *and* $Q$ *meet at* $R$. *The point* $N$ *is the foot of the perpendcular from* $R$ *to the x-axis. Prove that if* $N$ *lies inside the ellipse,* $\angle PNR = \angle QNR$.

## 3. Hyperbola

Recall that an ellipse is defined as the locus of points $P$ such that $|PF_1|+|PF_2| = k$. What happens if $k \leq |F_1F_2|$? If $k = |F_1F_2|$, then the locus is the line segment $\overline{F_1F_2}$. We can think of this as a degenerate ellipse. However if $k < |F_1F_2|$, there will be no solutions for $P$ because of the the triangle inequality, namely,

$$|F_1F_2| \leq |PF_1| + |PF_2|.$$

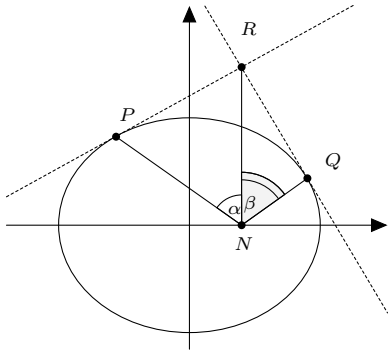


FIGURE 2.8. Exercise 2.30.

We now consider the locus of $P$ defined by

$$\{P : |PF_1| - |PF_2| = k, \;\; k \geq 0\}.$$

By the triangle inequality,

$$|PF_1| - |PF_2| \leq |F_1F_2|.$$

Thus $k \leq |F_1F_2|$. If $k = |F_1F_2|$, the locus is a ray from $F_2$ in the direction of $\overrightarrow{F_1F_2}$. On the other hand, if $k = 0$, the locus is the perpendicular bisector of $\overline{F_1F_2}$.

**GeoGebra Example 3.1.** *We study the remaining values of $k$ with GeoGebra.*

(1) Create $F_1$, $F_2$ such that $|F_1F_2| = 2$ and slider $-5 \leq k \leq 5$.
(2) Create slider $0 \leq r \leq 20$.
(3) Construct the circles $|PF_2| = r$ and $|PF_1| = r + k$.
(4) The two intersections of the circles give feasible points for $P$.
(5) Use the locus tool to draw the locus of $P$ depending on $r$.
(6) Vary $k$ to observe the locus.
(7) Set $k$ as a value between 1 and 2. Right-click $r$ and turn animation on.

We observe that the locus is that of a hyperbola. In fact, the animation was inspired by the interference pattern of circular waves made by dropping two stones into a still pool of water. As seen in the GeoGebra example, we obtain the right branch of the hyperbola when $k < 0$. This motivates the following definition.

**Definition 3.2** (Geometric)**.** *A* **hyperbola** *is the locus of points where the distances to the* **foci** $F_1$ *and* $F_2$ *satisfies*

$$\{P : \big||PF_1| - |PF_2|\big| = k\}.$$

If we set $F_1 = (c, 0)$ and $F_2 = (-c, 0)$ for some positive $c$, we know from our previous discussion that $0 < k < 2c$. For symmetry we replace $k$ by $2a$, where

$0 < a < c$. If $P = (x, y)$ is a point on the hyperbola, we have

$$\begin{aligned}
\sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} &= \pm 2a \\
\implies (x-c)^2+y^2 &= \left(\pm 2a + \sqrt{(x+c)^2+y^2}\right)^2 \\
&= 4a^2 \pm 4a\sqrt{(x+c)^2+y^2} + (x+c)^2 + y^2 \\
\implies -4xc - 4a^2 &= \pm 4a\sqrt{(x+c)^2+y^2} \\
\implies \left(cx+a^2\right)^2 &= a^2(x+c)^2 + a^2y^2 \\
\implies (c^2-a^2)x^2 - a^2y^2 &= a^2(c^2-a^2) \\
\implies \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} &= 1.
\end{aligned}$$

**Theorem 3.3.** *A hyperbola defined by* $\{P : \big||PF_1| - |PF_2|\big| = 2a\}$ *with foci* $F_1 = (c, 0)$ *and* $F_2 = (-c, 0)$ *satisfies the equation*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

*where* $b = \sqrt{c^2 - a^2}$.

We say that such a hyperbola is in standard position with **transverse axis** on $x$-axis. If we name the vertices $A = (a, 0)$, $A' = (-a, 0)$, then the transverse axis is actually the line segment $\overline{AA'}$. There are also two **asymptotes** described by

$$y = \pm\frac{b}{a}x.$$

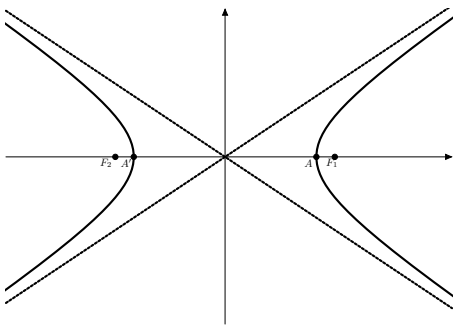


FIGURE 3.1. Hyperbola in standard position with transverse axis on $x$-axis.

**Exercise 3.4.** *Show that a hyperbola defined by* $\{P : \big||PF_1| - |PF_2|\big| = 2b\}$ *with foci* $F_1 = (0, c)$ *and* $F_2 = (0, -c)$ *satisfies the equation*

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1,$$

*where* $a = \sqrt{c^2 - b^2}$.

In the above, we say that the hyperbola is in standard position with transverse axis $\overline{BB'}$ on $y$-axis, where $B = (0, b)$ and $B' = (0, -b)$. In addition, the asymptotes are also described by

$$y = \pm\frac{b}{a}x.$$

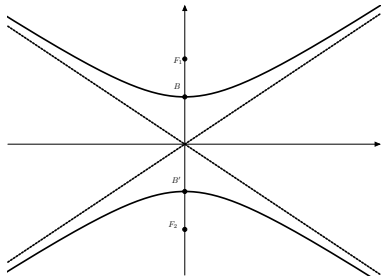


FIGURE 3.2. Hyperbola in standard position with transverse axis on $y$-axis.

The properties of the hyperbola often parallel those of the ellipse with the exception of the change in sign. A useful *heuristic* is to take a result for an ellipse and replace $b^2$ by $-b^2$ for the corresponding hyperbola with transverse axis on $x$-axis.

**Exercise 3.5.** *Show that the line*

$$y = mx \pm \sqrt{a^2m^2 - b^2},$$

*provided it exists, intersects the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *at exactly one point.*

**Exercise 3.6.** *Let* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *be a hyperbola and* $P = (x_0, y_0)$ *be a point on the hyperbola. Show that the equation of the tangent line at* $P$ *is given by*

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1.$$

**Exercise 3.7.** *Show that if the tangents to two points of a hyperbola,* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, *intersect at right angles, the intersection lies on the* **director circle**, *which is a circle with radius* $\sqrt{a^2 - b^2}$ *and centred at the origin.*

Just as the ellipse can be parameterized by ordinary trigonometric functions $x = a\cos(t)$ and $y = b\sin(t)$, the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be parameterized by a pair of **hyperbolic functions**.

**Definition 3.8.** *The* **hyperbolic cosine** *and* **hyperbolic sine** *functions are defined for all real* $t$ *in terms of the exponential function as follows.*

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \quad \textit{and} \quad \sinh(t) = \frac{e^t - e^{-t}}{2}.$$

**Exercise 3.9.** *Let* $t \in \mathbb{R}$. *Show that* $\cosh(t) \geq 1$, *while* $\sinh(t)$ *takes on every possible real value, i.e. its range is* $\mathbb{R}$.

**Exercise 3.10.** *Prove the differentiation formulas:*

$$\frac{d\cosh(t)}{dt} = \sinh(t) \quad \textit{and} \quad \frac{d\sinh(t)}{dt} = \cosh(t).$$

**Exercise 3.11.** *Show that the following identity holds for every* $t \in \mathbb{R}$.

$$\cosh^2(t) - \sinh^2(t) = 1.$$

The results of the previous exercise suggest the use of hyperbolic cosine and sine functions as parameterizations of the hyperbola. However, since $\cosh(t) \geq 1$, the left and right branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ have to be parameterized respectively as

$$(x, y) = \big(-a\cosh(t), b\sinh(t)\big) \quad \text{and} \quad (x, y) = \big(a\cosh(t), b\sinh(t)\big).$$

**Exercise 3.12.** *Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be a hyperbola and $P = (x_0, y_0)$ be a point on the hyperbola. For each of the cases, $x_0 > 0$ and $x_0 < 0$, use the parametrization for hyperbolas to show that the gradient of the tangent line at $P$ is given by $\dfrac{b^2 x_0}{a^2 y_0}$.*

Recall that the parameter $t$ in the parametrization $(x, y) = (a\cos(t), b\sin(t))$ for an ellipse represents the eccentric angle. Likewise, the parameter $t$ in the parametrization $(x, y) = (a\cosh(t), b\sinh(t))$ has an interesting geometric interpretation in terms of the area of what is known as a hyperbolic sector.

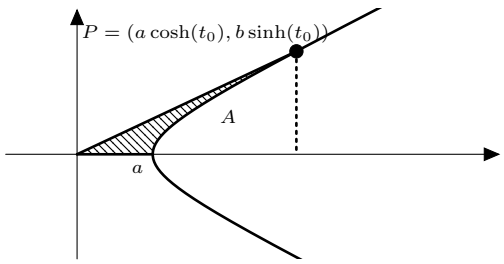


FIGURE 3.3. Hyperbolic sector with area $\frac{abt_0}{2}$.

We compute explicitly the area of the hyperbolic sector. Let $P = (x_0, y_0)$, where $x_0 > 0$ and $y_0 > 0$, be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. We denote the area bounded by the hyperbola, the $x$-axis and the vertical line $x = x_0$ as $A$. We first note that the area of the triangle with vertices at the origin, $P$ and $(x_0, 0)$ is given by

$$\begin{aligned}
\frac{x_0 y_0}{2} &= \frac{a\cosh(t_0) \cdot b\sinh(t_0)}{2} \\
&= \frac{ab}{2} \cdot \frac{e^{t_0} + e^{-t_0}}{2} \cdot \frac{e^{t_0} - e^{-t_0}}{2} \\
&= \frac{ab\sinh(2t_0)}{4}.
\end{aligned}$$

In the above, we used the parametrization $(x_0, y_0) = (a\cosh(t_0), b\sinh(t_0))$ and used definitions to compute a double-angle identity for hyperbolic sine. On the other hand, we can compute $A$ directly with the integral

$$\begin{aligned}
A = \int_a^{x_0} y\,dx &= \int_a^{x_0} b\sqrt{\frac{x^2}{a^2} - 1}\,dx \\
&= \int_0^{t_0} b\sqrt{\cosh^2(t) - 1} \cdot a\sinh(t)\,dt \\
&= ab\int_0^{t_0} \sinh^2(t)\,dt \\
&= \frac{ab}{4}\int_0^{t_0} e^{2t} - 2 + e^{-2t}\,dt \\
&= \frac{ab}{8}\left(e^{2t_0} - e^{-2t_0}\right) - \frac{abt_0}{2}.
\end{aligned}$$

The first term above is exactly $\frac{ab\sinh(2t_0)}{4}$ and so we conclude that the shaded area in the above Figure is $\frac{abt_0}{2}$.

While the hyperbolic functions are the natural choice for parameterizing hyperbolas, it is also possible to make do with the usual trigonometric functions. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ can be parameterized by

$$x = a\sec(\theta) \quad \text{and} \quad y = b\tan(\theta), \qquad 0 \leq \theta < 2\pi, \theta \neq \frac{\pi}{2}, \frac{3\pi}{2}.$$

**GeoGebra Example 3.13.** *We use GeoGebra to investigate the two possible parametrizations of the hyperbola.*

(1) Create sliders $-5 \leq a, b, \leq 5$ and sliders $-20 \leq t \leq 20$ and $0 \leq w \leq 2\pi$. The latter can be achieved by selecting the Angle option.
(2) In the Input window enter
```
x=a*cosh(t)
y=b*sinh(t)
```
(3) Intersect the two straight lines and use the locus tool to draw the locus of points depending on the parameter $t$.
(4) Note that only the right branch is given and the left branch can be drawn with another line $x = -a\cosh(t)$.
(5) Right-click on the slider $t$ and toggle animation on to observe how the points on the hyperbola vary with $t$
(6) Do the same with $x = a\sec(z)$ and $y = b\tan(z)$.
(7) Observe that only one parametrization is necessary and observe which values of $0 \leq w \leq 2\pi$ correspond to which section of the hyperbola, paying attention to the discontinuities.

Now let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be a hyperbola and $F_1$ be the right focus. If $P = (x_0, y_0)$ is a point on the hyperbola, the similar to the case for ellipses, $|PF_1|$ has a simple expression. We shall assume $x_0 > 0$ and use the parametrization $(x_0, y_0) = (a\cosh(t_0), b\sinh(t_0))$.

$$\begin{aligned}
|PF_1| &= \sqrt{(a\cosh(t_0) - \sqrt{a^2+b^2})^2 + b^2\sinh^2(t_0)} \\
&= \sqrt{a^2\cosh^2(t_0) - 2a\cosh(t_0)\sqrt{a^2+b^2} + a^2 + b^2 + b^2(\cosh^2(t_0) - 1)} \\
&= \sqrt{\left(\sqrt{a^2+b^2}\cosh(t_0) - a\right)^2} \\
&= c\cosh(t_0) - a.
\end{aligned}$$

**Exercise 3.14.** *Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be a hyperbola with foci $F_1$ and $F_2$. If $P = (x_0, y_0)$ is a point on the hyperbola, show that*

$$|PF_1| = \left|\frac{cx_0 - a^2}{a}\right| \quad \textit{and} \quad |PF_2| = \left|\frac{cx_0 + a^2}{a}\right|.$$

From the above exercise, we can also recover the geometric definition of a hyperbola as the curve with locus given by

$$\left\{P : \big||PF_1| - |PF_2|\big| = 2a\right\},$$

for some $a > 0$. In addition to the geometric definition, the hyperbola can also be defined in terms of a directrix-eccentricity-focus.

**Definition 3.15** (Directrix-Eccentricity-Focus)**.** *A hyperbola is the locus of points whose distance from the focus $F$ (focus) and the directrix $d$ are in a constant ratio greater than 1, i.e.* $\left\{P : \frac{|PF|}{|Pd|} = e,\ e > 1\right\}$.

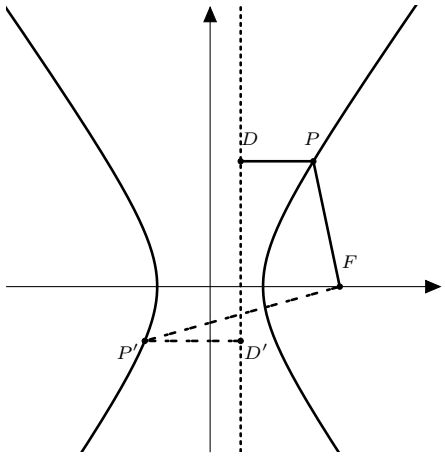


FIGURE 3.4. Directrix-Eccentricity-Focus definition of a hyperbola.

We can work out the cartesian equation of a hyperbola with eccentricity $e > 1$, $F = (c, 0)$ and directrix $x = \frac{c}{e^2}$. In fact, the working proceeds in exactly the same way as the case for ellipse in the previous section as we had only assumed $e \neq 1$. We arrive at Equation (2.2) which we rewrite as

$$\frac{e^2x^2}{c^2} - \frac{e^2y^2}{c^2(e^2-1)} = 1,$$

since $e^2 - 1 > 0$. Thus $a = \frac{c}{e}$ and $b = a\sqrt{(e^2-1)}$. The directrix can then be written as $x = \frac{a}{e}$ and $e = \sqrt{1 + \frac{b^2}{a^2}}$. Finally we also have $c = ae = \sqrt{a^2+b^2}$. By symmetry $F_2 = (-c, 0)$ and $x = -\frac{a}{e}$ also qualify as focus and directrix.

**Exercise 3.16.** *Let $F = (0, c)$, $c > 0$ and directrix $d$ be the line $y = k$. Show that equation $\frac{|PF|}{|Pd|} = e$ for $e > 1$ defines a hyperbola and determine the value of $e$, $c$ and $k$ in terms of the parameters for a hyperbola in standard position with transverse axis on $y$-axis.*

**Exercise 3.17.** *Given a hyperbola in standard position with transverse axis on $x$-axis, show that the polar form with respect to the centre is given by*

$$r = \frac{ab}{\sqrt{b^2\cos^2(\theta) - a^2\sin^2(\theta)}}.$$

The corresponding reflection property for a hyperbola can be phrased as "light coming from $F_1$ is reflected in a hyperbolic mirror in such a way that it appears to have come from $F_2$." An application of this is in the **Cassegrain reflector** which utilizes a primary parabolic reflector and a secondary hyperbolic mirror to bring the focal point to a location behind the primary reflector.

**Exercise 3.18** (Apollonius)**.** *Let $T$ be a point on a hyperbola in standard position with transverse axis on $x$-axis. The tangent line to the hyperbola at $T$ cuts the asymptotes at $P$ and $Q$. Show that*

(1) *$T$ is the midpoint of $\overline{PQ}$.*
(2) *If $O$ is the origin, show that $|OQ| \cdot |OP|$ is a constant and hence the triangle $OPQ$ has constant area.*

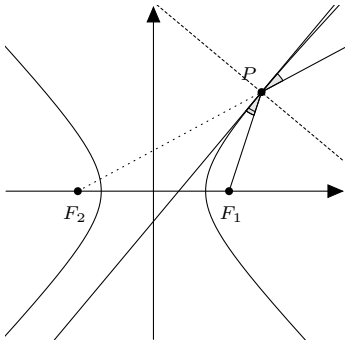


FIGURE 3.5. Reflection property of a hyperbola.

MATHEMATICS & MATHEMATICS EDUCATION, NATIONAL INSTITUTE OF EDUCATION, NANYANG TECHNOLOGICAL UNIVERSITY, 1 NANYANG WALK, SINGAPORE 637616
*E-mail address*: peechoon.toh@nie.edu.sg