# Weighted generalized crank moments for $k$-colored partitions and Andrews-Beck type congruences

Bernard L.S. Lin[a,1], Lin Peng[a], Pee Choon Toh[b,*]

[a] *School of Science, Jimei University, Xiamen 361021, P.R. China*
[b] *Mathematics and Mathematics Education, National Institute of Education, Nanyang Technological University, Singapore 637616, Singapore*

**Abstract**

Recently, Beck studied a new partition statistic which involves counting the total number of parts of a partition with certain rank or crank. Andrews proved two of Beck's conjectures related to ranks. Chern subsequently proved several results involving weighted rank and crank moments and deduced a number of similar Andrews-Beck type congruences. In this paper, we show that some of Chern's results can be explained by a simple combinatorial argument, and extend this approach to the study of $k$-colored partitions. As a consequence, we derive a large number of new Andrews-Beck type congruences for $k$-colored partitions.

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## 1. Introduction

A partition $\lambda$ of a positive integer $n$ is a weakly decreasing sequence of positive integers whose sum is $n$. Each term of the sequence is called a part of the partition. We use $\lambda \vdash n$ to denote a partition $\lambda$ of $n$ and $|\lambda|$ to denote the sum of all parts, which equals $n$. We also use $\ell(\lambda)$ to denote the largest part of $\lambda$ and $\#(\lambda)$ to denote the number of parts of $\lambda$. Finally, we use $p(n)$ to denote

*Corresponding author
*Email addresses:* linlsjmu@163.com (Bernard L.S. Lin), peng2618@163.com (Lin Peng), peechoon.toh@nie.edu.sg (Pee Choon Toh)

the number of partitions of $n$. Ramanujan famously discovered that

$$p(5n+4) \equiv 0 \pmod 5,$$
$$p(7n+5) \equiv 0 \pmod 7,$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Elementary proofs of the modulo 5 and 7 congruences can be found in Chapter 2 of Berndt's excellent book [6]. In an attempt to provide combinatorial explanations for the modulo 5 and 7 congruences, Dyson [10] defined the *rank* of a partition $\lambda$ by

$$\text{rank}(\lambda) := \ell(\lambda) - \#(\lambda).$$

Subsequently, Atkin and Swinnerton-Dyer [5] proved that the rank could explain Ramanujan's modulo 5 and 7 congruences.

In 1988, Andrews and Garvan [4] defined the *crank* of a partition which explained all three of Ramanujan's congruences combinatorially. The crank is defined by

$$\text{crank}(\lambda) := \begin{cases} \ell(\lambda) & \text{if } \omega(\lambda) = 0, \\ \mu(\lambda) - \omega(\lambda) & \text{if } \omega(\lambda) > 0, \end{cases}$$

where $\omega(\lambda)$ denotes the number of parts of size 1 in $\lambda$ and $\mu(\lambda)$ denotes the number of parts in $\lambda$ larger than $\omega(\lambda)$. It is worth mentioning that Andrews and Garvan related the crank to Garvan's work [13] on *vector partitions*.

Recently, George Beck introduced a new partition statistic by considering the total number of parts of the partitions of $n$ with a certain rank. Let $NT(m, j, n)$ be the total number of parts of the partitions of $n$ with rank congruent to $m$ modulo $j$. Andrews [3] proved the following Andrews-Beck type congruence which was originally conjectured by Beck.

**Theorem 1.1.** *For any $n \geq 0$, if $i = 1, 4$,*

$$NT(1,5,5n+i) + 2NT(2,5,5n+i) - 2NT(3,5,5n+i) - NT(4,5,5n+i) \equiv 0 \pmod 5, \tag{1.1}$$

*which is equivalent to*

$$\sum_{m=1}^{4} mNT(m,5,5n+i) \equiv 0 \pmod 5.$$

Chern [7] gave a different proof of the above theorem by showing that the weighted first rank moment is related to the ordinary second rank moment. Namely,

$$\sum_{\lambda\vdash n} \#(\lambda)\mathrm{rank}(\lambda) = -\frac{1}{2}\sum_{\lambda\vdash n}\mathrm{rank}^2(\lambda). \tag{1.2}$$

Subsequently, Chern devoted another paper [8] to the study of weighted higher order rank and crank moments using analytic methods. For example, Chern established the following interesting theorem.

**Theorem 1.2** (Chern)**.** *For any $j>0$,*

$$\sum_{\lambda\vdash n} \#(\lambda)\mathrm{rank}^{2j-1}(\lambda) = -\frac{1}{2}\sum_{\lambda\vdash n}\mathrm{rank}^{2j}(\lambda), \tag{1.3}$$

$$\sum_{\lambda\vdash n} \omega(\lambda)\mathrm{crank}^{2j-1}(\lambda) = -\frac{1}{2}\sum_{\lambda\vdash n}\mathrm{crank}^{2j}(\lambda). \tag{1.4}$$

One of the aims of this paper is to provide combinatorial arguments to explain Theorem 1.2. We first observe that mapping each partition $\lambda$ to its conjugate $\lambda'$ gives

$$\sum_{\lambda\vdash n}\ell(\lambda) = \sum_{\lambda'\vdash n}\#(\lambda') = \sum_{\lambda\vdash n}\#(\lambda). \tag{1.5}$$

We thus have

$$\begin{aligned}
-\frac{1}{2}\sum_{\lambda\vdash n}\mathrm{rank}^{2j}(\lambda) &= -\frac{1}{2}\sum_{\lambda\vdash n}\big(\ell(\lambda)-\#(\lambda)\big)\big(\ell(\lambda)-\#(\lambda)\big)^{2j-1}\\
&= \frac{1}{2}\sum_{\lambda\vdash n}\#(\lambda)\big(\ell(\lambda)-\#(\lambda)\big)^{2j-1} - \frac{1}{2}\sum_{\lambda\vdash n}\ell(\lambda)\big(\ell(\lambda)-\#(\lambda)\big)^{2j-1}\\
&= \frac{1}{2}\sum_{\lambda\vdash n}\#(\lambda)\big(\ell(\lambda)-\#(\lambda)\big)^{2j-1} - \frac{1}{2}\sum_{\lambda'\vdash n}\#(\lambda')\big(\#(\lambda')-\ell(\lambda')\big)^{2j-1}\\
&= \frac{1}{2}\sum_{\lambda\vdash n}\#(\lambda)\big(\ell(\lambda)-\#(\lambda)\big)^{2j-1} + \frac{1}{2}\sum_{\lambda\vdash n}\#(\lambda)\big(\ell(\lambda)-\#(\lambda)\big)^{2j-1}\\
&= \sum_{\lambda\vdash n}\#(\lambda)\mathrm{rank}^{2j-1}(\lambda).
\end{aligned}$$

This proves (1.3). The proof of (1.4) is more complicated and will be described in Section 4.

Building on these ideas, we turn our attention to the *generalized crank* defined by Fu and Tang [11] for $k$-colored partitions, where $k \geq 2$. A $k$-colored partition $\pi$ of $n$ is a $k$-tuple of partitions,

$$\pi := (\pi_1, \pi_2, \ldots, \pi_k)$$

with $|\pi_1|+|\pi_2|+\cdots+|\pi_k| = n$. We use the notation $\pi \vDash n$ if $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ is a $k$-colored partition of $n$. Let $p_k(n)$ denote the number of $k$-colored partitions of $n$ with the corresponding generating function of $p_k(n)$ given by

$$\sum_{n=0}^{\infty} p_k(n)q^n = \frac{1}{(q;q)_\infty^k}, \tag{1.6}$$

where

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \qquad \text{and} \qquad (a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j).$$

Fu and Tang defined a generalized crank for $k$-colored partitions by

$$\text{crank}_k(\pi) := \#(\pi_1) - \#(\pi_2). \tag{1.7}$$

In terms of these generalized cranks, we have the following new results relating the weighted $(2j-1)$-th generalized crank moment to the ordinary $2j$-th moment for $k$-colored partitions.

**Theorem 1.3.** *For any $j > 0$ and $k \geq 2$,*

$$\sum_{\pi \vDash n} \#(\pi_1)\text{crank}_k^{2j-1}(\pi) = \frac{1}{2}\sum_{\pi \vDash n} \text{crank}_k^{2j}(\pi). \tag{1.8}$$

We also have the corresponding result for symmetrized generalized crank moments.

**Theorem 1.4.** *For any $j > 0$ and $k \geq 2$,*

$$\sum_{\pi \vDash n} \#(\pi_1)(\text{crank}_k(\pi)+j-1)_{2j-1} = \frac{1}{2}\sum_{\pi \vDash n}(\text{crank}_k(\pi)+j-1)_{2j}, \tag{1.9}$$

*where we adopted the falling factorial notation*

$$(x)_r := x(x-1)\cdots(x-r+1).$$

In the rest of this section, we will present some results for the $k = 2$ case of 2-colored partitions, otherwise known as bipartitions. Gandhi [12, Eq. (4.9)] attributed the following congruences to Ramanathan [18]. For any $n \geq 0$,

$$p_2(5n+2) \equiv p_2(5n+3) \equiv p_2(5n+4) \equiv 0 \pmod{5}. \tag{1.10}$$

Hammond and Lewis [15] subsequently introduced a birank function in order to provide combinatorial explanations for these congruences. For a bipartition $\pi = (\pi_1, \pi_2) \vDash n$, the birank (which is the $k = 2$ case of the generalized crank) is defined as

$$\text{birank}(\pi) := \#(\pi_1) - \#(\pi_2).$$

Define $NB_2(m, j, n)$ as the total number of parts of $\pi_1$ in each bipartition $\pi$ of $n$ with birank($\pi$) congruent to $m$ modulo $j$. In other words,

$$NB_2(m, j, n) := \sum_{\substack{\pi \vDash n \\ \text{birank}(\pi) \equiv m \pmod{j}}} \#(\pi_1). \tag{1.11}$$

We shall establish the following Andrews-Beck type congruence.

**Theorem 1.5.** *For any $n \geq 0$, if $i = 0, 2, 3$ or $4$,*

$$\sum_{m=1}^{4} m NB_2(m, 5, 5n+i) \equiv 0 \pmod{5}. \tag{1.12}$$

We illustrate Theorem 1.5 for bipartitions of 3.

| Bipartitions of 3 | birank (mod 5) | Sum of $\#(\pi_1)$ |
|---|---|---|
| (2, 1), (1, 2) | 0 | 2 |
| (3, –), (1+1, 1) | 1 | 3 |
| (2+1, –), (–, 1+1+1) | 2 | 2 |
| (1+1+1, –), (–, 2+1) | 3 | 3 |
| (1, 1+1), (–, 3) | 4 | 1 |

Consequently,

$$\begin{aligned} &NB_2(1,5,3) + 2NB_2(2,5,3) + 3NB_2(3,5,3) + 4NB_2(4,5,3) \\ &= 3 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 1 = 20. \end{aligned}$$

In Section 2, we shall prove Theorems 1.3 and 1.4, and present a large number of Andrews-Beck type congruences for other values of $k$. In Section 3, we consider alternative analytic proofs and investigate congruences and conjectures associated with higher order moments. Finally in Section 4, we will provide a combinatorial proof of Chern's (1.4).

## 2. Weighted generalized crank moments

We begin with a proof of our main results.

*Proof of Theorem 1.3.* For $k$-colored partitions, we consider the map

$$\pi = (\pi_1, \pi_2, \dots, \pi_k) \mapsto \delta = (\delta_1, \delta_2, \dots, \delta_k) = (\pi_2, \pi_1, \dots, \pi_k)$$

which interchanges partitions $\pi_1$ and $\pi_2$. It is clear that

$$\sum_{\pi \vDash n} \#(\pi_1) = \sum_{\delta \vDash n} \#(\delta_2) = \sum_{\pi \vDash n} \#(\pi_2). \tag{2.1}$$

We thus have

$$\begin{aligned}
\frac{1}{2}\sum_{\pi \vDash n} \mathrm{crank}_k^{2j}(\pi) &= \frac{1}{2}\sum_{\pi \vDash n} \big(\#(\pi_1) - \#(\pi_2)\big)\big(\#(\pi_1) - \#(\pi_2)\big)^{2j-1} \\
&= \frac{1}{2}\sum_{\pi \vDash n} \#(\pi_1)\big(\#(\pi_1) - \#(\pi_2)\big)^{2j-1} - \frac{1}{2}\sum_{\pi \vDash n} \#(\pi_2)\big(\#(\pi_1) - \#(\pi_2)\big)^{2j-1} \\
&= \frac{1}{2}\sum_{\pi \vDash n} \#(\pi_1)\big(\#(\pi_1) - \#(\pi_2)\big)^{2j-1} + \frac{1}{2}\sum_{\pi \vDash n} \#(\pi_2)\big(\#(\pi_2) - \#(\pi_1)\big)^{2j-1} \\
&= \frac{1}{2}\sum_{\pi \vDash n} \#(\pi_1)\big(\#(\pi_1) - \#(\pi_2)\big)^{2j-1} + \frac{1}{2}\sum_{\delta \vDash n} \#(\delta_1)\big(\#(\delta_1) - \#(\delta_2)\big)^{2j-1} \\
&= \sum_{\pi \vDash n} \#(\pi_1)\mathrm{crank}_k^{2j-1}(\pi).
\end{aligned}$$

This completes the proof. $\square$

*Proof of Theorem 1.4.* Using the same map $\pi \mapsto \delta$ as before, we have

$$\begin{aligned}
&\sum_{\pi\vDash n}(\mathrm{crank}_k(\pi)+j-1)_{2j}+\sum_{\delta\vDash n}(\mathrm{crank}_k(\delta)+j-1)_{2j}\\
&=\sum_{\pi\vDash n}\big((\mathrm{crank}_k(\pi)+j-1)_{2j}+(-\mathrm{crank}_k(\pi)+j-1)_{2j}\big)\\
&=\sum_{\pi\vDash n}2\,\mathrm{crank}_k(\pi)\big(\mathrm{crank}_k(\pi)+j-1\big)_{2j-1}\\
&=2\sum_{\pi\vDash n}\#(\pi_1)\big(\mathrm{crank}_k(\pi)+j-1\big)_{2j-1}-2\sum_{\pi\vDash n}\#(\pi_2)\big(\mathrm{crank}_k(\pi)+j-1\big)_{2j-1}\\
&=2\sum_{\pi\vDash n}\#(\pi_1)\big(\mathrm{crank}_k(\pi)+j-1\big)_{2j-1}-2\sum_{\delta\vDash n}\#(\delta_1)\big(-\mathrm{crank}_k(\delta)+j-1\big)_{2j-1}\\
&=2\sum_{\pi\vDash n}\#(\pi_1)\big(\mathrm{crank}_k(\pi)+j-1\big)_{2j-1}+2\sum_{\delta\vDash n}\#(\delta_1)\big(\mathrm{crank}_k(\delta)+j-1\big)_{2j-1}\\
&=4\sum_{\pi\vDash n}\#(\pi_1)\big(\mathrm{crank}_k(\pi)+j-1\big)_{2j-1}.
\end{aligned}$$

In the second equality above, we used the identity

$$(m+j-1)_{2j}+(-m+j-1)_{2j}=2m(m+j-1)_{2j-1}, \tag{2.2}$$

while another identity

$$(-m+j-1)_{2j-1}=-(m+j-1)_{2j-1}, \tag{2.3}$$

was used for the second sum in the fifth equality. □

We now turn our attention to Andrews-Beck type congruences. For $k \geq 2$, define $NB_k(m,j,n)$ as the total number of parts of $\pi_1$ in each $k$-colored partition $\pi$ of $n$ with $\mathrm{crank}_k(\pi)$ congruent to $m$ modulo $j$. We have

$$NB_k(m,j,n) := \sum_{\substack{\pi\vDash n\\ \mathrm{crank}_k(\pi)\equiv m \pmod j}} \#(\pi_1). \tag{2.4}$$

In addition to Theorem 1.5, the following Andrews-Beck type congruences also hold.

**Theorem 2.1.** *For any $n \geq 0$,*

$$NB_3(1,3,3n+2)+2NB_3(2,3,3n+2)\equiv 0 \pmod 3. \tag{2.5}$$

**Theorem 2.2.** *For any $n \geq 0$, if $i = 0$ or $7$,*

$$\sum_{m=1}^{10} mNB_3(m, 11, 11n + i) \equiv 0 \pmod{11}. \tag{2.6}$$

**Theorem 2.3.** *For any $n \geq 0$, if $i = 0$ or $15$,*

$$\sum_{m=1}^{16} mNB_3(m, 17, 17n + i) \equiv 0 \pmod{17}. \tag{2.7}$$

**Theorem 2.4.** *For any $n \geq 0$, if $i = 0, 3$ or $4$,*

$$\sum_{m=1}^{4} mNB_4(m, 5, 5n + i) \equiv 0 \pmod{5}. \tag{2.8}$$

**Theorem 2.5.** *For any $n \geq 0$, if $i = 0, 2, 4, 5$ or $6$,*

$$\sum_{m=1}^{6} mNB_4(m, 7, 7n + i) \equiv 0 \pmod{7}. \tag{2.9}$$

**Theorem 2.6.** *For any $n \geq 0$, if $i = 0$ or $8$,*

$$\sum_{m=1}^{10} mNB_5(m, 11, 11n + i) \equiv 0 \pmod{11}. \tag{2.10}$$

**Theorem 2.7.** *For any $n \geq 0$, if $i = 0$ or $9$,*

$$\sum_{m=1}^{10} mNB_7(m, 11, 11n + i) \equiv 0 \pmod{11}. \tag{2.11}$$

**Theorem 2.8.** *For any $n \geq 0$, if $i = 0$ or $11$,*

$$\sum_{m=1}^{16} mNB_9(m, 17, 17n + i) \equiv 0 \pmod{17}. \tag{2.12}$$

**Theorem 2.9.** *For any $n \geq 0$, if $i = 0$ or $17$,*

$$\sum_{m=1}^{18} mNB_9(m, 19, 19n + i) \equiv 0 \pmod{19}. \tag{2.13}$$

**Theorem 2.10.** *For any $n \geq 0$, if $i = 0$ or $9$,*

$$\sum_{m=1}^{22} mNB_9(m, 23, 23n + i) \equiv 0 \pmod{23}. \tag{2.14}$$

We remark that in addition to the above list, there are a large number of other Andrews-Beck type congruences, including some infinite families, for values of $k \geq 5$. The key result to establish these Andrews-Beck type congruences is the following.

**Theorem 2.11.** *For any* $k \geq 2$,

$$\sum_{\pi \vDash n} \text{crank}_k^2(\pi) = \frac{2n}{k} p_k(n). \tag{2.15}$$

Theorem 2.11 was stated without proof in [11, Eq. (4.2)], and so for completeness, we will provide an analytic proof in the next section. For now, we note that invoking the $j = 1$ case of Theorem 1.3, we have

$$\sum_{\pi \vDash n} \#(\pi_1)\text{crank}_k(\pi) = \frac{n}{k} p_k(n). \tag{2.16}$$

Hirschhorn [16, Eq. (1.3)] recently showed that

$$p_3(3n + 2) \equiv 0 \pmod{9}.$$

Hence

$$NB_3(1, 3, 3n + 2) + 2NB_3(2, 3, 3n + 2) \equiv \sum_{\pi \vDash (3n+2)} \#(\pi_1)\text{crank}_3(\pi) \equiv 0 \pmod{3},$$

which proves Theorem 2.1.

Suppose now that $\ell$ is any odd prime that does not divide $k$, by (2.16) we have an infinite family of congruences,

$$\sum_{m=1}^{\ell-1} mNB_k(m, \ell, \ell n) \equiv \sum_{\pi \vDash (\ell n)} \#(\pi_1)\text{crank}_k(\pi) \equiv 0 \pmod{\ell}. \tag{2.17}$$

This settles the $i = 0$ case in each of Theorem 1.5 and Theorems 2.2 to 2.10. The remaining cases can be proved by establishing the following congruences for $p_k(n)$ which are analogues of (1.10).

**Theorem 2.12.** *For any $n \geq 0$,*

$$p_3(11n+7) \equiv 0 \pmod{11}, \tag{2.18}$$

$$p_3(17n+15) \equiv 0 \pmod{17}, \tag{2.19}$$

$$p_4(5n+3) \equiv p_4(5n+4) \equiv 0 \pmod{5}, \tag{2.20}$$

$$p_4(7n+2) \equiv p_4(7n+4) \equiv p_4(7n+5) \equiv p_4(7n+6) \equiv 0 \pmod{7}, \tag{2.21}$$

$$p_5(11n+8) \equiv 0 \pmod{11}, \tag{2.22}$$

$$p_7(11n+9) \equiv 0 \pmod{11}, \tag{2.23}$$

$$p_9(17n+11) \equiv 0 \pmod{17}, \tag{2.24}$$

$$p_9(19n+17) \equiv 0 \pmod{19}, \tag{2.25}$$

$$p_9(23n+9) \equiv 0 \pmod{23}. \tag{2.26}$$

*Proof.* We generalize the definition of $p_k(n)$ to allow negative integers $k$. A large number of congruences satisfied by $p_k(n)$ for specific values of negative $k$ were established by Newman [17]. For example,

$$p_{-8}(11n+7) \equiv 0 \pmod{11}. \tag{2.27}$$

In general, if $\ell$ is a prime that does not divide $k$, we have

$$\sum_{n=0}^{\infty} p_{k-\ell}(n)q^n = \frac{(q;q)_\infty^\ell}{(q;q)_\infty^k} \equiv (q^\ell;q^\ell)_\infty \sum_{n=0}^{\infty} p_k(n)q^n \pmod{\ell}. \tag{2.28}$$

We can then conclude (2.18) holds with $\ell = 11$ and $k = 3$ by appealing to (2.27). Similarly, each of (2.19) and (2.22) to (2.26) holds respectively from the following six congruences

$$p_{-14}(17n+15) \equiv 0 \pmod{17}, \tag{2.29}$$

$$p_{-6}(11n+8) \equiv 0 \pmod{11}, \tag{2.30}$$

$$p_{-4}(11n+9) \equiv 0 \pmod{11}, \tag{2.31}$$

$$p_{-8}(17n+11) \equiv 0 \pmod{17}, \tag{2.32}$$

$$p_{-10}(19n+17) \equiv 0 \pmod{19}, \tag{2.33}$$

$$p_{-14}(23n+9) \equiv 0 \pmod{23}. \tag{2.34}$$

Four of the congruences, namely (2.30) to (2.32) and (2.34), can be deduced from Newman [17]. Cooper, Hirschhorn and Lewis subsequently extended Newman's result, and both (2.29) and (2.33) can be deduced from [9, Thm. 1].

To prove (2.21), we note that

$$\begin{aligned}(q^7;q^7)_\infty \sum_{n=0}^{\infty} p_4(n)q^n &\equiv (q;q)_\infty^3 \pmod 7 \\ &= \sum_{j=0}^{\infty} (-1)^j (2j+1) q^{j(j+1)/2} \end{aligned} \tag{2.35}$$

by Jacobi's identity [6, Thm. 1.3.9]. Since $\frac{j(j+1)}{2}$ is only congruent to $0, 1, 3, 6 \pmod 7$, there are no powers of $q$ that are congruent to $2, 4$ or $5 \pmod 7$. We are left with the coefficients of $q^{7n+6}$. In this case, $\frac{j(j+1)}{2} \equiv 6 \pmod 7$ if and only if $j \equiv 3 \pmod 7$ which means that $2j+1$ is divisible by 7, and so the coefficients are congruent to 0 modulo 7.

Finally, (2.20) is a direct consequence of (1.10) since

$$\sum_{n=0}^{\infty} p_4(n)q^n = \left( \sum_{n=0}^{\infty} p_2(n)q^n \right)^2. \tag{2.36}$$

This completes the proof. □

We remark that (2.21) is a special case of an observation by Andrews [2, Thm. 1] that for every prime $\ell > 3$, there are $(\ell+1)/2$ values of $b$ such that

$$p_{\ell-3}(\ell n + b) \equiv 0 \pmod{\ell}. \tag{2.37}$$

This immediately yields another infinite family of Andrews-Beck type congruences of the form

$$\sum_{m=1}^{\ell-1} mNB_{\ell-3}(m, \ell, \ell n + b) \equiv 0 \pmod{\ell}. \tag{2.38}$$

Finally, if we fix $\ell$ and let $k$ vary, each of the Andrews-Beck type congruences that we have proved can be viewed as the first case of an infinite family. For example, having established that Theorem 2.4 holds, then it is immediate that

$$\sum_{m=1}^{4} mNB_{4+5j}(m, 5, 5n + i) \equiv 0 \pmod 5 \tag{2.39}$$

also holds for the same values of $i$ and $j \geq 1$.

## 3. Analytic proofs and higher order moments

Previously, we adopted a combinatorial approach to prove Theorems 1.3 and 1.4 which relate the weighted and ordinary moments for generalized cranks for $k$-colored partitions. In this section, we provide alternative analytic proofs and consider some implications for the higher order moments.

For $k \geq 2$, the generating function for generalized cranks of $k$-colored partitions [11, Eq. (1.1)] is given by

$$\frac{(q;q)_\infty^{2-k}}{(qz;q)_\infty (q/z;q)_\infty}.$$

The next result gives us a useful representation of the generating function for the weighted generalized crank.

**Lemma 3.1.**

$$\frac{(q;q)_\infty^{2-k}}{(qxz;q)_\infty (q/z;q)_\infty} = \frac{(q;q)_\infty^{1-k}}{(qx;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{(qx;q)_{n+1}}{(q;q)_n} \times \left(1 + \sum_{m=1}^{\infty} q^{m(n+1)} (x^m z^m + z^{-m})\right). \tag{3.1}$$

*Proof.* From the $q$-binomial theorem [1, p. 17] or [6, p. 8], we have

$$\frac{1}{(qxz;q)_\infty (q/z;q)_\infty} = \sum_{r=0}^{\infty} \frac{(qxz)^r}{(q;q)_r} \cdot \sum_{s=0}^{\infty} \frac{(q/z)^s}{(q;q)_s} := \sum_{m=-\infty}^{\infty} A(m) z^m.$$

For $m \geq 0$, the coefficient

$$\begin{aligned}
A(m) &= \sum_{s=0}^{\infty} \frac{(qx)^{s+m}}{(q;q)_{s+m}} \frac{q^s}{(q;q)_s} \\
&= \frac{q^m x^m}{(q;q)_m} \sum_{s=0}^{\infty} \frac{(q^2x)^s}{(q^{m+1};q)_s (q;q)_s} \\
&= \frac{q^m x^m}{(q;q)_m} \cdot \frac{1}{(q^{m+1};q)_\infty (q^2x;q)_\infty} \sum_{s=0}^{\infty} \frac{(-1)^s q^{ms+s(s+1)/2} (q^2x;q)_s}{(q;q)_s} \\
&= \frac{q^m x^m}{(q;q)_\infty (qx;q)_\infty} \sum_{s=0}^{\infty} \frac{(-1)^s q^{ms+s(s+1)/2} (qx;q)_{s+1}}{(q;q)_s}.
\end{aligned}$$

Heine's transformation [1, p. 19] was used to establish the penultimate equality above. Similarly, for $m \geq 1$, we find that

$$A(-m) = \frac{q^m}{(q;q)_\infty (qx;q)_\infty} \sum_{s=0}^{\infty} \frac{(-1)^s q^{ms+s(s+1)/2}(qx;q)_{s+1}}{(q;q)_s}.$$

It follows that

$$A(0) + \sum_{m=1}^{\infty} (A(m)z^m + A(-m)z^{-m})$$
$$= \frac{1}{(q;q)_\infty (qx;q)_\infty} \sum_{s=0}^{\infty} \frac{(-1)^s q^{s(s+1)/2}(qx;q)_{s+1}}{(q;q)_s} \left(1 + \sum_{m=1}^{\infty} q^{m(s+1)}x^m z^m + q^{m(s+1)}z^{-m}\right).$$

This completes the proof. □

We remark that Lemma 3.1 is equivalent to [8, Thm. 3.1], although our method of proof differs from his.

We observe that

$$\sum_{n=0}^{\infty} \sum_{\pi \vDash n} x^{\#(\pi_1)} z^{\text{crank}_k(\pi)} q^n = \frac{(q;q)_\infty^{2-k}}{(qxz;q)_\infty (q/z;q)_\infty}, \tag{3.2}$$

and thus

$$\sum_{n=0}^{\infty} \sum_{\pi \vDash n} x^{\#(\pi_1)} \text{crank}_k^j(\pi) q^n = D_j \left( \frac{(q;q)_\infty^{2-k}}{(qxz;q)_\infty (q/z;q)_\infty} \right) \Bigg|_{z=1},$$

where

$$D_j(f(z)) := z \frac{\partial}{\partial z} D_{j-1}(f(z)).$$

The representation in Lemma 3.1 gives us

$$D_j \left( \frac{(q;q)_\infty^{2-k}}{(qxz;q)_\infty (q/z;q)_\infty} \right) \Bigg|_{z=1} = \frac{(q;q)_\infty^{1-k}}{(qx;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \frac{(qx;q)_{n+1}}{(q;q)_n}$$
$$\times \sum_{m=1}^{\infty} q^{m(n+1)} (m^j x^m + (-m)^j). \tag{3.3}$$

When $j$ is even, setting $x = 1$ yields the even moments, which we record as

$$\sum_{n=0}^{\infty} \sum_{\pi \vDash n} \text{crank}_k^{2j}(\pi) q^n = \frac{1}{(q;q)_\infty^k} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} (1 - q^{n+1}) \times \sum_{m=1}^{\infty} q^{m(n+1)} 2m^{2j}. \tag{3.4}$$

On the other hand, if $j$ is odd, we can differentiate (3.3) with respect to $x$ and then set $x = 1$. This gives us the weighted odd moments

$$\sum_{n=0}^{\infty}\sum_{\pi\models n}\#(\pi_1)\mathrm{crank}_k^{2j-1}(\pi)q^n = \frac{1}{(q;q)_\infty^k}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}(1-q^{n+1}) \times \sum_{m=1}^{\infty} m^{2j}q^{m(n+1)}. \tag{3.5}$$

Comparing (3.5) and (3.4) yields an alternative proof of Theorem 1.3. An alternative proof of Theorem 1.4 can also be obtained in an analogous manner.

Next, we prove Theorem 2.11 which was stated without proof in [11].

*Proof of Theorem 2.11.* Applying

$$\sum_{m=1}^{\infty} m^2 z^m = \frac{z(1+z)}{(1-z)^3} \tag{3.6}$$

to (3.4), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{\pi\models n}\mathrm{crank}_k^2(\pi)q^n &= \frac{2}{(q;q)_\infty^k}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}\frac{q^{n+1}(1+q^{n+1})}{(1-q^{n+1})^2}\\ &= \frac{2}{(q;q)_\infty^k}\sum_{n=1}^{\infty}(-1)^{n-1} q^{n(n+1)/2}\frac{(1+q^{n})}{(1-q^{n})^2}.\end{aligned}$$

From [14, Cor. 3.4], we have

$$\sum_{n=1}^{\infty}(-1)^{n-1} q^{n(n+1)/2}\frac{(1+q^{n})}{(1-q^{n})^2} = \sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}.$$

Hence,

$$\sum_{n=0}^{\infty}\sum_{\pi\models n}\mathrm{crank}_k^2(\pi)q^n = \frac{2}{(q;q)_\infty^k}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}.$$

On the other hand, applying $q\dfrac{\partial}{\partial q}$ to the generating function of $p_k(n)$ yields

$$\sum_{n=0}^{\infty} np_k(n)q^n = \frac{1}{(q;q)_\infty^k}\sum_{n=1}^{\infty}\frac{knq^n}{1-q^n} = \frac{k}{(q;q)_\infty^k}\sum_{n=1}^{\infty}\frac{q^n}{(1-q^n)^2}.$$

It follows that

$$\sum_{\pi\models n}\mathrm{crank}_k^2(\pi) = \frac{2n}{k}p_k(n).$$

This completes the proof. □

All the Andrews-Beck type congruences in the previous section arose from (2.16), the identity for the weighted first moment of the generalized crank. Other interesting congruences can also be deduced from the higher order moments. By differentiating (3.6) and applying to (3.5), we have expressions for the weighted third moment,

$$\sum_{n=0}^{\infty}\sum_{\pi\vDash n}\#(\pi_1)\mathrm{crank}_k^3(\pi)q^n = \frac{1}{(q;q)_\infty^k}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n+1)/2}\frac{1+11q^n+11q^{2n}+q^{3n}}{(1-q^n)^4}, \tag{3.7}$$

and weighted fifth moment,

$$\sum_{n=0}^{\infty}\sum_{\pi\vDash n}\#(\pi_1)\mathrm{crank}_k^5(\pi)q^n$$
$$= \frac{1}{(q;q)_\infty^k}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n+1)/2}\frac{1+57q^n+302q^{2n}+302q^{3n}+57q^{4n}+q^{5n}}{(1-q^n)^6}. \tag{3.8}$$

Let us first consider the weighted third moment. Since $j^3 \equiv j \pmod 3$, any modulo 3 congruence satisfied by the weighted first moment, for example (2.5), extends to the third moment. For a non-trivial congruence, set $k=5$ and consider (3.7) modulo 5.

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{\pi\vDash n}\#(\pi_1)\mathrm{crank}_5^3(\pi)q^n &\equiv \frac{1}{(q^5;q^5)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n+1)/2}\frac{1+q^n+q^{2n}+q^{3n}}{(1-q^n)^4}\\ &\equiv \frac{1}{(q^5;q^5)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n+1)/2}\frac{(1+q^n+q^{2n}+q^{3n})(1-q^n)}{(1-q^n)^5}\\ &\equiv \frac{1}{(q^5;q^5)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n+1)/2}\frac{(1-q^{4n})}{(1-q^{5n})} \pmod 5.\end{aligned} \tag{3.9}$$

Now $\frac{j(j+1)}{2}$ is only congruent to $0,1,3 \pmod 5$ and likewise $\frac{j(j+1)}{2}+4j$. Thus there are no powers of $q$ which are congruent to 2 or 4 (mod 5) in (3.9). We summarize this as the next result.

**Theorem 3.1.** *For any $n \geq 0$, if $i = 2$ or 4,*

$$\sum_{m=1}^{4} m^3 NB_5(m,5,5n+i) \equiv 0 \pmod 5. \tag{3.10}$$

Next we consider the weighted fifth moment. In this case, we have $j^5 \equiv j \pmod 5$ and also $j^5 \equiv j \pmod 3$. Thus congruences modulo 3 or 5 for the weighted first moment extend to the weighted fifth moment. The next non-trivial case occurs when we consider (3.8) modulo 7.

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{\pi\vDash n}\#(\pi_1)\mathrm{crank}_k^5(\pi)q^n \\
&\qquad\equiv \frac{1}{(q;q)_\infty^k}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n+1)/2}\frac{1+q^n+q^{2n}+q^{3n}+q^{4n}+q^{5n}}{(1-q^n)^6} \\
&\qquad\equiv \frac{1}{(q;q)_\infty^k}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n+1)/2}\frac{(1-q^{6n})}{(1-q^{7n})} \pmod 7. \qquad (3.11)
\end{aligned}$$

The expression $\frac{j(j+1)}{2}$ is only congruent to $0,1,3,6 \pmod 7$ and likewise $\frac{j(j+1)}{2}+6j$. So if $k=7$, there are no powers of $q$ which are congruent to $2,4$ or $5 \pmod 7$ in (3.11). This gives rise to another Andrews-Beck type congruence.

**Theorem 3.2.** *For any $n \geq 0$, if $i = 2, 4$ or $5$,*

$$\sum_{m=1}^{6} m^5 NB_7(m,7,7n+i) \equiv 0 \pmod 7. \qquad (3.12)$$

Suppose now that $k=4$, (3.11) becomes

$$\sum_{n=0}^{\infty}\sum_{\pi\vDash n}\#(\pi_1)\mathrm{crank}_4^5(\pi)q^n \equiv \frac{(q;q)_\infty^3}{(q^7;q^7)_\infty}\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n+1)/2}\frac{(1-q^{6n})}{(1-q^{7n})} \pmod 7. \qquad (3.13)$$

As before, both $\frac{j(j+1)}{2}$ and $\frac{j(j+1)}{2}+6j$ contain only powers of $q$ congruent to $0,1,3,6 \pmod 7$. On the other hand, as discussed in (2.35), $(q;q)_\infty^3$ when expanded as a series modulo 7 contains only powers of $q$ congruent to $0,1$ or $3 \pmod 7$. Thus the right side of (3.13), as a product of two $q$-series, has no powers of $q$ congruent to $5 \pmod 7$, resulting in another congruence.

**Theorem 3.3.** *For any $n \geq 0$,*

$$\sum_{m=1}^{6} m^5 NB_4(m,7,7n+5) \equiv 0 \pmod 7. \qquad (3.14)$$

The three Andrews-Beck type congruences that we have proved appear to be just the tip of the iceberg. We conclude with a selected list of congruences that are conjectured to hold.

**Conjecture 3.1.** *For any* $n \geq 0$*,*

$$\sum_{m=1}^{6} m^3 NB_4(m, 7, 7n+6) \equiv 0 \pmod{7}. \tag{3.15}$$

**Conjecture 3.2.** *For any* $n \geq 0$*,*

$$\sum_{m=1}^{10} m^3 NB_5(m, 11, 11n+8) \equiv 0 \pmod{11}. \tag{3.16}$$

**Conjecture 3.3.** *For any* $n \geq 0$*,*

$$\sum_{m=1}^{12} m^3 NB_5(m, 13, 13n+12) \equiv 0 \pmod{13}. \tag{3.17}$$

**Conjecture 3.4.** *For any* $n \geq 0$*, if* $i = 0$ *or* $3$*,*

$$\sum_{m=1}^{16} m^3 NB_7(m, 17, 17n+i) \equiv 0 \pmod{17}. \tag{3.18}$$

**Conjecture 3.5.** *For any* $n \geq 0$*, if* $i = 0, 9$ *or* $16$*,*

$$\sum_{m=1}^{18} m^3 NB_7(m, 19, 19n+i) \equiv 0 \pmod{19}. \tag{3.19}$$

**Conjecture 3.6.** *For any* $n \geq 0$*,*

$$\sum_{m=1}^{10} m^5 NB_7(m, 11, 11n+9) \equiv 0 \pmod{11}. \tag{3.20}$$

**Conjecture 3.7.** *For any* $n \geq 0$*,*

$$\sum_{m=1}^{12} m^5 NB_{11}(m, 13, 13n+9) \equiv 0 \pmod{13}. \tag{3.21}$$

**Conjecture 3.8.** *For any* $n \geq 0$*,*

$$\sum_{m=1}^{18} m^5 NB_{11}(m, 19, 19n+13) \equiv 0 \pmod{19}. \tag{3.22}$$

## 4. Combinatorial proof of (1.4)

We first define an involution $\varphi$ on an ordinary partition $\lambda = \lambda_1 + \lambda_2 + \cdots$. Suppose first that both $\omega(\lambda)$ and $\mu(\lambda)$ are nonzero. Then $\lambda$ contains a $\mu(\lambda) \times \omega(\lambda)$ block, which we label as $A$, as well as a $\mu(\lambda) \times 1$ column immediately to the right of $A$, in its Ferrers diagram. (See Figure 1.) The map $\varphi$ interchanges this $\mu(\lambda) \times 1$ column with the $\omega(\lambda) \times 1$ column of parts of size 1, switches blocks $B$ and $C$, and then conjugates all three blocks $A$, $B$ and $C$. The tail end consisting of the difference of the two largest parts, $\lambda_1 - \lambda_2$, is left intact.

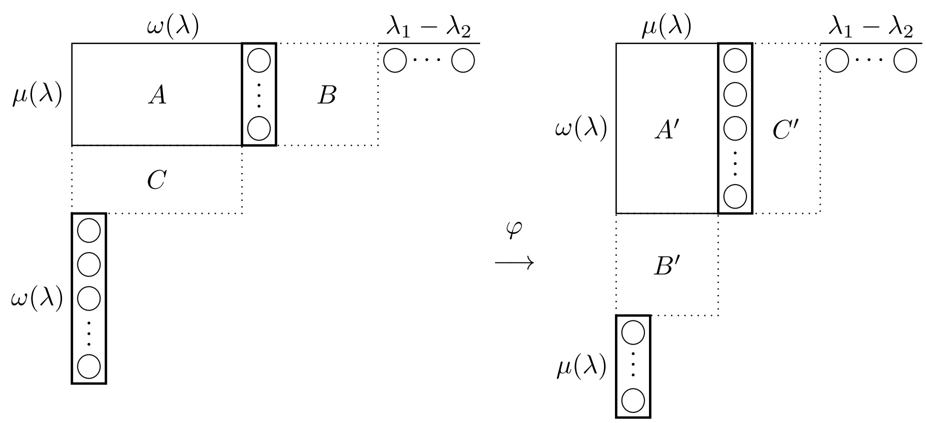


Figure 1: Involution $\varphi$

It is clear that $\mu(\varphi(\lambda)) = \omega(\lambda)$ and $\omega(\varphi(\lambda)) = \mu(\lambda)$, thus $\varphi$ is an involution satisfying $\text{crank}(\varphi(\lambda)) = -\text{crank}(\lambda)$. It remains to consider the cases where either of $\omega(\lambda)$ or $\mu(\lambda)$ is zero. If $\omega(\lambda) = 0$, we remove the largest part $\lambda_1$, create $\lambda_1$ many parts of size 1 and call this resulting partition $\varphi(\lambda)$. Note that $\mu(\varphi(\lambda)) = 0$ and thus $\text{crank}(\varphi(\lambda)) = -\lambda_1$. Finally, if $\mu(\lambda) = 0$, then $\omega(\lambda) \geq \lambda_1$, the largest part. We remove these parts of size 1, and insert a new largest part of size $\omega(\lambda)$, resulting in $\text{crank}(\varphi(\lambda)) = \omega(\lambda) = -\text{crank}(\lambda)$.

*Proof of* (1.4). We collect all the partitions of $n$ into these two sets

$$P_1(n) := \{\lambda \vdash n \mid \omega(\lambda) > 0, \mu(\lambda) > 0\},$$

$$P_2(n) := \{\lambda \vdash n \mid \omega(\lambda) = 0 \text{ or } \mu(\lambda) = 0\}.$$

For partitions in $P_1(n)$, we have

$$
\begin{aligned}
2\sum_{\lambda\in P_1(n)}\omega(\lambda)\mathrm{crank}^{2j-1}(\lambda) &= \sum_{\lambda\in P_1(n)}\omega(\lambda)\mathrm{crank}^{2j-1}(\lambda)+\sum_{\varphi(\lambda)\in P_1(n)}\omega(\varphi(\lambda))\mathrm{crank}^{2j-1}(\varphi(\lambda))\\
&= \sum_{\lambda\in P_1(n)}\omega(\lambda)\mathrm{crank}^{2j-1}(\lambda)-\sum_{\varphi(\lambda)\in P_1(n)}\mu(\lambda)\mathrm{crank}^{2j-1}(\lambda)\\
&= -\sum_{\lambda\in P_1(n)}(\mu(\lambda)-\omega(\lambda))\mathrm{crank}^{2j-1}(\lambda). \qquad (4.1)
\end{aligned}
$$

Now for partitions in $P_2(n)$, we first observe that

$$
\begin{aligned}
\sum_{\substack{\lambda\in P_2(n)\\ \omega(\lambda)>0}}\mathrm{crank}^{2j}(\lambda) &= \sum_{\substack{\varphi(\lambda)\in P_2(n)\\ \omega(\varphi(\lambda))=0}}\mathrm{crank}^{2j}(\varphi(\lambda))\\
&= \sum_{\substack{\lambda\in P_2(n)\\ \omega(\lambda)=0}}\mathrm{crank}^{2j}(\lambda). \qquad (4.2)
\end{aligned}
$$

Hence

$$
\begin{aligned}
-2\sum_{\lambda\in P_2(n)}\omega(\lambda)\mathrm{crank}^{2j-1}(\lambda) &= -2\sum_{\substack{\lambda\in P_2(n)\\ \omega(\lambda)>0}}\omega(\lambda)\mathrm{crank}^{2j-1}(\lambda)\\
&= 2\sum_{\substack{\lambda\in P_2(n)\\ \omega(\lambda)>0}}\mathrm{crank}^{2j}(\lambda)\\
&= \sum_{\substack{\lambda\in P_2(n)\\ \omega(\lambda)>0}}\mathrm{crank}^{2j}(\lambda)+\sum_{\substack{\lambda\in P_2(n)\\ \omega(\lambda)=0}}\mathrm{crank}^{2j}(\lambda)\\
&= \sum_{\lambda\in P_2(n)}\mathrm{crank}^{2j}(\lambda). \qquad (4.3)
\end{aligned}
$$

Combining (4.1) and (4.3) completes the proof. □

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