ON THE CRANK FUNCTION OF CUBIC PARTITION PAIRS

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Abstract. We study a crank function $M(m,n)$ for cubic partition pairs. We show that the function $M(m,n)$ explains a cubic partition pair congruence and we also obtain various arithmetic properties regarding $M(m,n)$. In particular, using the Θ-operator, we confirm a conjecture on the sign pattern of $c(n)$, the number of cubic partition pairs of n, weighted by the parity of the crank.

1. Introduction and Statements of Results

In a series of papers ([3], [4] and [5]) H.-C. Chan studied congruence properties for a certain partition function $a(n)$ which is defined by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q;q)_\infty (q^2;q^2)_\infty}.$$

Here and throughout, we use the q-product notation:

$$(a;q)_\infty := \prod_{n=1}^{\infty}(1-aq^{n-1}) \quad \text{and} \quad (a_1,\dots,a_k;q)_\infty := (a_1;q)_\infty \cdots (a_k;q)_\infty.$$

H.-C. Chan [3] proved the congruence

$$a(3n+2) \equiv 0 \pmod{3} \tag{1.1}$$

using an identity which arose from Ramanujan's cubic continued fraction and hence $a(n)$ is also known as the cubic partition function. Combinatorially, $a(n)$ counts the number of 2-color partitions of n with colors r and b subject to the restriction that the color b appears only in even parts. We term such partitions as cubic partitions.

H. Zhao and Z. Zhong [21] subsequently investigated congruences for the partition function

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2 (q^2;q^2)_\infty^2}.$$

Here $b(n)$ counts the number of cubic partition pairs, that is, the number of partitions (λ_1, λ_2), where λ_1 and λ_2 are cubic partitions such that the total sum of parts in both λ_1 and λ_2 equals

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to n. Zhao and Zhong [21, Theorem 3.2] proved that

$$b(5n+4) \equiv 0 \pmod{5}, \tag{1.2}$$

$$b(7n+\ell) \equiv 0 \pmod{7}, \text{if } \ell = 2,\ 3,\ 4, \text{ or } 6. \tag{1.3}$$

Since partitions are combinatorial objects, it is natural to look for combinatorial explanations for congruences satisfied by various types of partition functions. In two landmark papers [1] and [10], G.E. Andrews and F.G. Garvan used the idea of the crank of a partition to provide combinatorial explanations of Ramanujan's congruences for the ordinary partition function. Using analogous methods, the first author introduced a crank function for cubic partitions [11] to explain (1.1). It is clear from their definitions that the generating function for cubic partition pairs is the square of the generating function for cubic partitions. It is thus natural to consider if the square of the crank function for cubic partitions would also serve as a crank function for cubic partition pairs. We define such a crank function $M(m,n)$ by

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} M(m,n) z^m q^n := \left(\frac{(q;q)_\infty (q^2;q^2)_\infty}{(zq,q/z;q)_\infty (zq^2,q^2/z;q^2)_\infty}\right)^2. \tag{1.4}$$

It turns out that $M(m,n)$ does indeed explain (1.2) but not (1.3). Specifically, if we define

$$M(k,j,n) := \sum_{m\equiv k \pmod{j}} M(m,n),$$

then the following holds.

Theorem 1.1. *For all nonnegative integers n,*

$$M(0,5,5n+4) \equiv M(1,5,5n+4) \equiv M(2,5,5n+4) \equiv \cdots \equiv M(4,5,5n+4) \pmod{5}.$$

We remark that the first author had previously proposed two other crank functions for cubic partition pairs [12], one for each of the congruences (1.2) and (1.3).

Motivated by the work of D. Choi, S.-Y. Kang and J. Lovejoy [7], the first author [13] also studied the function $c(n)$ which counts the number of cubic partition pairs of n, weighted by the parity of the crank. The generating function for $c(n)$ is

$$\begin{aligned}\sum_{n=0}^{\infty} c(n) q^n &:= \sum_{n=0}^{\infty}\left(\sum_{m=-\infty}^{\infty} (-1)^m M(m,n)\right) q^n \\ &= \left(\frac{(q;q)_\infty (q^2;q^2)_\infty}{(-q;q)_\infty^2 (-q^2;q^2)_\infty^2}\right)^2 \\ &= \frac{(q;q)_\infty^6 (q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^4}. \end{aligned}\tag{1.5}$$

In [13] it was shown that, in a similar way of the proof (1.2) and (1.3), the function $c(n)$ also satisfies congruences modulo 5 and 7, namely

$$c(5n+4) \equiv 0 \pmod{5} \quad \text{and} \quad c(7n+2) \equiv 0 \pmod{7}.$$

In fact, there are congruences for $c(n)$ modulo every odd integer greater than 1.

Theorem 1.2. *For an odd integer $m > 1$, if $\ell \in \{1, 2, \dots, m-1\}$ is the unique integer satisfying $4\ell \equiv 1 \pmod{m}$, then*

$$c(mn + \ell) \equiv 0 \pmod{m}.$$

In the same paper [13], there is a conjecture on the sign pattern for $c(n)$. We prove that the conjecture is true.

Theorem 1.3. *For all nonnegative integers n,*

$$c(4n+1) < 0, \quad c(4n+2) > 0, \quad c(4n+3) > 0, \quad c(4n+4) < 0.$$

Furthermore there are other interesting arithmetic properties for the crank function modulo 4. In particular, we have the following congruence.

Theorem 1.4. *For all nonnegative integers n,*

$$M(0, 4, 2n+1) \equiv M(1, 4, 2n+1) \pmod{4}.$$

The rest of the paper is organized as follows. In Section 2, we discuss some important preliminary results. In Section 3, we prove Theorem 1.1 and other related results using the theory of modular forms. Theorems 1.2 to 1.4 and other arithmetic properties of the crank function $M(m, n)$ are proved in Section 4. In Section 5, we describe a connection between $c(n)$ and other partition functions in the literature.

2. Preliminary results

We begin with a few preliminary results.

Lemma 2.1.

$$\begin{aligned}\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} &= \frac{(q^8;q^8)_\infty^5}{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2} - 2q\frac{(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty} \\ &= \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2} - 4q\frac{(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}.\end{aligned}$$

Proof. Using the Jacobi triple product identity [2, p. 10], we have

$$\begin{aligned}\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} \\ &= \sum_{n=-\infty}^{\infty} q^{4n^2} - 2\sum_{n=-\infty}^{\infty} q^{(4n+1)^2} \qquad (2.1) \\ &= \frac{(q^8;q^8)_\infty^5}{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2} - 2q\frac{(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}.\end{aligned}$$

For the second assertion, since

$$\sum_{n=-\infty}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{4n^2} + 2\sum_{n=-\infty}^{\infty} q^{(4n+1)^2},$$

we can rewrite (2.1) as

$$\frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} = \sum_{n=-\infty}^{\infty} q^{n^2} - 4\sum_{n=-\infty}^{\infty} q^{(4n+1)^2}$$
$$= \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2} - 4q\frac{(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}. \qquad \square$$

Lemma 2.2.
$$\frac{(q;q)_\infty^4}{(q^2;q^2)_\infty^2} + \frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^4(q^4;q^4)_\infty^4} = 2\frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^4(q^8;q^8)_\infty^4}.$$

Proof. Using the Jacobi triple product identity, we can see that the left hand side of the identity is

$$\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2} + \sum_{m,n=-\infty}^{\infty} q^{m^2+n^2} = 2\sum_{m\equiv n \pmod 2} q^{m^2+n^2}$$
$$= 2\sum_{m,n=-\infty}^{\infty} q^{(m+n)^2+(m-n)^2}$$
$$= 2\sum_{m,n=-\infty}^{\infty} q^{2m^2+2n^2}.$$

In the above, we have used the fact that $j = m+n$ and $k = m-n$ is a bijection between the sets

$$\{(m,n)\in\mathbb{Z}^2\} \quad \text{and} \quad \{(j,k)\in\mathbb{Z}^2 \mid j\equiv k \pmod 2\}. \qquad \square$$

Now we describe some basic properties of modular forms. For more details, consult [15] or [16]. We first define the modular group $\Gamma(1) := SL_2(\mathbb{Z})$ and, for a positive integer N, its level N congruence subgroups

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \pmod N \right\}$$

and

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) : a \equiv b \equiv 1 \pmod N \right\}.$$

An element $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ acts on $\mathbb{H}$, the complex upper half plane, by

$$M\tau = \frac{a\tau+b}{c\tau+d}.$$

For a fixed real number r, a function $F(\tau)$ that is meromorphic in $\mathbb{H}$ is said to be a modular form of weight r with respect to Γ, a subgroup of $\Gamma(1)$, if

(1) $F(\tau)$ satisfies $F(M\tau) = (c\tau+d)^r F(\tau)$, for any $\tau\in\mathbb{H}$ and $M\in\Gamma$;
(2) there exists a fundamental domain R for Γ such that $F(\tau)$ has at most finitely many poles in R;
(3) $F(\tau)$ is meromorphic at each cusp.

Let $\mathcal{M}_r(\Gamma)$ denote the space of modular forms of weight r on Γ, where Γ is a subgroup of $\Gamma(1)$ of finite index. The following valence formula [16, Theorem 4.1.4] enables us to prove identities involving modular forms by checking a finite number of terms. (See [8] or [16, p. 95] for the definition of $\mathrm{Ord}_\Gamma(f;\tau)$.)

Theorem 2.3 (Valence formula). *If $F \in \mathcal{M}_r(\Gamma)$ and $f \neq 0$, then*

$$\sum_{\tau \in R} \mathrm{Ord}_\Gamma(F;\tau) = \frac{r}{12}[\Gamma(1) : \Gamma],$$

where R is a fundamental domain for Γ.

We also define two operators [15, p. 23 and 28] on modular forms, namely

$$\Theta f(q) := q\frac{\mathrm{d}}{\mathrm{d}q} f(q),$$
$$V_n F(\tau) := F(n\tau).$$

The modular forms of interest to us consist of products or quotients of Dedekind η-functions or the generalized Dedekind η-functions, which we define below.

Definition 2.4. *For $\tau \in \mathbb{H}$ and positive integers n, m, we define the Dedekind η-function as*

$$\eta(n\tau) := \eta_n = q^{\frac{n}{24}}(q^n;q^n)_\infty$$

and the generalized Dedekind η-function as

$$\eta_{n,m}(\tau) := \eta_{n,m} = q^{P_2(\frac{m}{n})\frac{n}{2}}(q^m, q^{n-m}; q^n)_\infty,$$

where $q = \exp(2\pi i\tau)$, $P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$ is the second Bernoulli function, and $\{t\} := t - [t]$ is the fractional part of t.

The following two theorems allow us to determine when an η-quotient and a generalized η-quotient is a modular form.

Theorem 2.5 (Theorem 1.64 in [15]). *For a positive integer N, let $F(\tau) := \prod_{\substack{n|N \\ n>0}} \eta(n\tau)^{r_n}$ be an η-quotient . If*

(1) $\sum_{n|N} r_n = 2k$;
(2) $\sum_{n|N} nr_n \equiv 0 \pmod{24}$;
(3) $\sum_{n|N} \frac{N}{n} r_n \equiv 0 \pmod{24}$,

then $F(\tau) \in \mathcal{M}_k(\Gamma_1(N))$.

Theorem 2.6 (Theorem 3 in [17]). *For a positive integer N, let $F(\tau) := \prod_{n|N,\ 0\le m<n} \eta_{n,m}^{r_{n,m}}(\tau)$, where $r_{n,m}$ are integers. If*

$$\sum_{n|N,\ 0\le m<n} nP_2\left(\frac{m}{n}\right) r_{n,m} \equiv 0 \pmod{2}$$

and

$$\sum_{n|N,\ 0\le m<n} \frac{N}{n} P_2(0) r_{n,m} \equiv 0 \pmod{2},$$

then $F(\tau) \in \mathcal{M}_0(\Gamma_1(N))$.

Finally, the following theorem allows us to decide when generalized Dedekind η-quotients are holomorphic.

Lemma 2.7 (Lemma 2.10 of [8]). *Let ℓ, m and n be positive integers. Then, for a cusp $k = \frac{\lambda}{\mu\epsilon}$ for $\Gamma_1(N)$, where $\epsilon \mid N$ and $(\lambda, N) = (\lambda, \mu) = (\mu, N) = 1$,*

$$\mathrm{ord}(\eta_{n,m}; k) + \mathrm{ord}(\eta_n; k) \geq 0 \quad \textit{and} \quad \mathrm{ord}(\eta_{\ell n,m}; k) + \ell\, \mathrm{ord}(\eta_n; k) \geq 0,$$

where $\mathrm{ord}(f;\tau)$ is the order of vanishing at τ.

3. Crank function $M(m, n)$ modulo 5

In this section, we shall use the theory of modular forms to prove Theorem 1.1 and other arithmetic properties of $M(m, n)$ modulo 5. By setting $z = \zeta := \exp(2\pi i/5)$ in (1.4), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{j=0}^{4} M(j, 5, n)\zeta^j q^n &= \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^2}{(\zeta q, \zeta^{-1}q; q)_\infty^2 (\zeta q^2, \zeta^{-1}q^2; q^2)_\infty^2} \\
&= \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^2 (\zeta^2 q^2, \zeta^{-2}q^2, q^2; q^2)_\infty^2}{(\zeta q, \zeta^{-1}q; q)_\infty^2 (q^{10};q^{10})_\infty^2} \\
&= \frac{(-\zeta q, -\zeta^{-1}q, q; q)_\infty^2 (q^2;q^2)_\infty^4}{(q^{10};q^{10})_\infty^2} \\
&\equiv \frac{(-\zeta q, -\zeta^{-1}q, q; q)_\infty^2}{(q^2;q^2)_\infty (q^{10};q^{10})_\infty} \pmod 5.
\end{aligned}$$

Recall from Kolberg [14] that

$$\frac{1}{(q^2;q^2)_\infty} \equiv F_0(q^{10}) + q^2 F_1(q^{10}) + 2q^4 F_2(q^{10}) + 3q^6 F_3(q^{10}) \pmod 5, \tag{3.1}$$

where

$$\begin{aligned}
F_0(q) &:= \frac{(q;q)_\infty}{(q, q^4; q^5)_\infty^3}, \\
F_1(q) &:= \frac{(q^5;q^5)_\infty}{(q, q^4; q^5)_\infty}, \\
F_2(q) &:= \frac{(q^5;q^5)_\infty}{(q^2, q^3; q^5)_\infty}, \\
F_3(q) &:= \frac{(q;q)_\infty}{(q^2, q^3; q^5)_\infty^3}.
\end{aligned}$$

We shall now dissect $(-\zeta q, -\zeta^{-1}q, q; q)_\infty$. Using the Jacobi triple product identity, we find that

$$
\begin{aligned}
(-\zeta q, -\zeta^{-1}q, q; q)_\infty &= \frac{1}{1+\zeta}\sum_{n=-\infty}^{\infty} \zeta^{-n} q^{n(n+1)/2} \\
&= \sum_{n=-\infty}^{\infty} q^{5n(5n+1)/2} + \frac{\zeta^2+\zeta^4}{1+\zeta} q \sum_{n=-\infty}^{\infty} q^{(25n^2+15n)/2} + \frac{2\zeta^3}{1+\zeta} q^3 \sum_{n=0}^{\infty} q^{(25n^2+25n)/2} \\
&= A_0(q^5) + (\zeta+\zeta^{-1}-1)qA_1(q^5) - 2(\zeta+\zeta^{-1})q^3A_3(q^5),
\end{aligned}
$$

where

$$
\begin{aligned}
A_0(q) &:= (-q^2, -q^3, q^5; q^5)_\infty, \\
A_1(q) &:= (-q, -q^4, q^5; q^5)_\infty, \\
A_3(q) &:= \frac{(q^{10}; q^{10})_\infty^2}{(q^5; q^5)_\infty}.
\end{aligned}
$$

Therefore, the contribution to q^{5n+4} terms from

$$\frac{(-\zeta q, -\zeta^{-1}q, q; q)_\infty^2}{(q^2; q^2)_\infty}$$

equals to

$$
\begin{aligned}
&2q^4A_0(q^5)^2F_2(q^{10}) + (3+\zeta^{-2}-2\zeta^{-1}-2\zeta+\zeta^2)q^4A_1(q^5)^2F_1(q^{10}) \\
&-4(2+\zeta^{-2}-\zeta^{-1}-\zeta+\zeta^2)q^4A_1(q^5)A_3(q^5)F_0(q^{10}) - 12(\zeta^{-1}+\zeta)q^9A_0(q^5)A_3(q^5)F_3(q^{10}).
\end{aligned}
$$

Defining $M_{ij,5}(n) := M(i,5,n) - M(j,5,n)$, we observe that

$$\sum_{n=0}^{\infty} M_{01,5}(5n+4)q^n \equiv \frac{2A_0(q)^2F_2(q^2) - 2A_1(q)A_3(q)F_0(q^2) + 2qA_0(q)A_3(q)F_3(q^2)}{(q^2; q^2)_\infty} \pmod 5,$$

$$\sum_{n=0}^{\infty} M_{02,5}(5n+4)q^n \equiv \frac{2A_0(q)^2F_2(q^2) + 2A_1(q)^2F_1(q^2) - 4A_1(q)A_3(q)F_0(q^2)}{(q^2; q^2)_\infty} \pmod 5.$$

Our aim is to show that each of right hand sides of the equations above is in fact congruent to 0 modulo 5, which means

$$M(0,5,5n+4) \equiv M(1,5,5n+4) \equiv M(2,5,5n+4) \pmod 5.$$

Since $M(m,n) = M(-m,n)$, the above suffices to prove Theorem 1.1.

Lemma 3.1. *We have*

$$
\begin{aligned}
A_0(q)^2F_2(q^2) + qA_0(q)A_3(q)F_3(q^2) &= A_1(q)A_3(q)F_0(q^2), \\
A_0(q)^2F_2(q^2) + A_1(q)^2F_1(q^2) &= 2A_1(q)A_3(q)F_0(q^2).
\end{aligned}
$$

Proof. By Definition 2.4, we find that the first identity equals to

$$\frac{\eta_5^2\eta_{10}\eta_{10,4}}{\eta_{5,2}^2} + \frac{\eta_2\eta_{10}^2}{\eta_{5,2}\eta_{10,4}^2} = \frac{\eta_2\eta_{10}^2}{\eta_{5,1}\eta_{10,2}^2}.$$

To obtain a holomorphic form, we multiply throughout by $\eta_1\eta_2\eta_5^3\eta_{10}^4\eta_{5,1}\eta_{5,2}^2\eta_{10,2}^2\eta_{10,4}^4$ and rearrange to obtain $F(\tau)$ given by

$$F(\tau) = \eta_1\eta_2\eta_5^5\eta_{10}^5\eta_{5,1}\eta_{10,2}^2\eta_{10,4}^5 + \eta_1^2\eta_2^4\eta_5^2\eta_{10}^4 - \eta_1\eta_2^2\eta_5^3\eta_{10}^6\eta_{5,2}^2\eta_{10,4}^4.$$

Here, we used the identities $\eta_5\eta_{5,1}\eta_{5,2} = \eta_1$ and $\eta_{10}\eta_{10,2}\eta_{10,4} = \eta_2$. To prove the first identity, it now suffices to prove $F(\tau) = 0$. By Lemma 2.7, we see that $F(\tau)$ is holomorphic. By Theorem 2.6, we find that

$$V_{12}\left(\eta_{5,1}\eta_{10,2}^2\eta_{10,4}^5\right) \quad \text{and} \quad V_{12}\left(\eta_{5,2}^2\eta_{10,4}^4\right)$$

are in $\mathcal{M}_0(\Gamma_1(1440))$. Moreover, by Theorem 2.5, we find that

$$V_{12}\left(\eta_1\eta_2\eta_5^5\eta_{10}^5\right), V_{12}\left(\eta_1^2\eta_2^4\eta_5^2\eta_{10}^4\right) \quad \text{and} \quad V_{12}\left(\eta_1\eta_2^2\eta_5^3\eta_{10}^6\right)$$

are in $\mathcal{M}_6(\Gamma_1(1440))$. Therefore, we observe that

$$F(12\tau) \in \mathcal{M}_6(\Gamma_1(1440)).$$

Thus, by Theorem 2.3, we can prove $F(12\tau)$ vanishes if the first 663553 terms are zero. In other words, $F(\tau) = 0$ if the first 55297 coefficients are zero. This was checked using MAGMA.

For the second identity, we need to prove that

$$\eta_5^2\eta_{10}\eta_{10,4}\eta_{5,2}^{-2} + \eta_5^2\eta_{10}\eta_{10,2}\eta_{5,1}^{-2} - 2\eta_2\eta_{10}^2\eta_{5,1}^{-1}\eta_{10,2}^{-2} = 0.$$

As the proof follows in the same way as the first proof, we omit it. □

Using similar methods, we can also prove the following.

Theorem 3.2. *For all nonnegative integers n,*

$$M(0,5,5n+3) \equiv M(1,5,5n+3) \pmod 5,$$
$$M(1,5,5n) \equiv M(2,5,5n) \pmod 5.$$

Proof. The first congruence follows from the identity

$$A_0(q)A_1(q)F_1(q^2) + qA_1(q)A_3(q)F_2(q^2) = A_0(q)A_3(q)F_0(q^2) + 2qA_3(q)^2F_1(q^2),$$

which can be proved in the same way as Lemma 3.1. The second congruence follows from

$$A_0(q)A_1(q)F_2(q^2) + qA_1(q)A_3(q)F_3(q^2) = A_0(q)A_3(q)F_1(q^2) + 2qA_3(q)^2F_2(q^2).$$

□

4. Proofs of Theorems 1.2 to 1.4

In this section, we consider the properties of the function $c(n)$ defined in (1.5) We first note that the generating function for $c(n)$ can be expressed as an η-quotient.

$$f_c(\tau) := \frac{\eta_4^6\eta_8^2}{\eta_{16}^4} = \sum_{n=0}^{\infty} c(n)q^{4n-1}.$$

We shall show that $f_c(\tau)$ is the q-derivative of another η-quotient.

Lemma 4.1.

$$\Theta\left(\frac{\eta_1^2\eta_8}{\eta_2\eta_{16}^2}\right) = -\frac{\eta_4^6\eta_8^2}{\eta_{16}^4}.$$

Proof. We let $h(q)$ denote the eta-quotient $\frac{\eta_1^2\eta_8}{\eta_2\eta_{16}^2}$. Performing logarithmic differentiation on $h(q)$ with respect to q gives us

$$\begin{aligned}-q\frac{\mathrm{d}}{\mathrm{d}q}\log h(q) &= 1+2\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}+\sum_{n=1}^{\infty}\frac{8nq^{8n}}{1-q^{8n}}-2\sum_{n=1}^{\infty}\frac{16nq^{16n}}{1-q^{16n}}\\ &= \frac{(q^2;q^2)_\infty(q^4;q^4)_\infty^6(q^8;q^8)_\infty}{(q;q)_\infty^2(q^{16};q^{16})_\infty^2}. \end{aligned} \tag{4.1}$$

The last equality in (4.1) is due to the following identity

$$\sum_{n=-\infty}^{\infty}\frac{xq^n}{(1-xq^n)^2}-\sum_{n=-\infty}^{\infty}\frac{yq^n}{(1-yq^n)^2}=\frac{x(xy,q/xy,y/x,qx/y;q)_\infty(q;q)_\infty^4}{(x,q/x,y,q/y;q)_\infty^2}. \tag{4.2}$$

The identity (4.2) is equivalent to the classical result which expresses the difference of two Weierstrass $\wp$-functions as an infinite product [20, p. 451]. Proofs of this identity may be found in [9, Section 18] or [6]. In fact, identity (4.1) appears in [6, p. 608]. Now from (4.1), we can deduce that

$$\begin{aligned}-q\frac{\mathrm{d}h(q)}{\mathrm{d}q} &= h(q)\frac{(q^2;q^2)_\infty(q^4;q^4)_\infty^6(q^8;q^8)_\infty}{(q;q)_\infty^2(q^{16};q^{16})_\infty^2}\\ &= \frac{\eta_4^6\eta_8^2}{\eta_{16}^4}.\end{aligned}$$

□

Remark. The theory of modular forms says that the Θ-operator maps modular functions of weight 0 to modular forms of weight 2 [15, Prop. 2.11]. However, unlike in Lemma 4.1, there is no guarantee that η-quotients correspond to η-quotients under this map. Another example where the Θ-operator maps an η-quotient to another η-quotient is given later in Lemma 5.1 and a number of other examples can be found in [9, Section 33].

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Lemma 4.1 implies that

$$(4n-1)\Big|c(n)$$

for all positive integers n. Therefore, if $4\ell-1$ is a multiple of m, we find that

$$m\Big|(4mn+4\ell-1),$$

and thus $c(mn+\ell)\equiv 0 \pmod{m}$. □

Next we examine the sign of $c(n)$.

Proof of Theorem 1.3. We can rewrite the first equality in Lemma 2.1 as

$$\frac{\eta_1^2\eta_8}{\eta_2\eta_{16}^2}=\frac{\eta_8^6}{\eta_4^2\eta_{16}^4}-2.$$

Since the Θ-operator preserves sign, Lemma 4.1 allows us to conclude that the sign of $c(n)$ always equal the sign of $d(n)$ which is defined by

$$\begin{aligned}\sum_{n=0}^{\infty} d(n)q^n &:= \frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^2(q^4;q^4)_\infty^4}\\ &= \frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty^2}\sum_{n=-\infty}^{\infty} q^{n^2}\\ &= \frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty^2}\left(\sum_{n=-\infty}^{\infty} q^{4n^2} + 2\sum_{n=-\infty}^{\infty} q^{(4n+1)^2}\right)\\ &= \frac{(q^2;q^2)_\infty}{(q^4;q^4)_\infty^2}\left(\frac{(q^8;q^8)_\infty^5}{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2} + 2q\frac{(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}\right).\end{aligned}$$

One can observe that the signs of the Fourier coefficients for both

$$\frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\left(\frac{(q^4;q^4)_\infty^5}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^2}\right) \quad \text{and} \quad \frac{(q;q)_\infty}{(q^2;q^2)_\infty^2}\left(\frac{(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty}\right)$$

are alternating, and this is sufficient to complete the proof of Theorem 1.3. Moreover, the alternating property is due to the first product in each of the above. To see this, we replace q by $-q$ and observe that

$$\frac{(-q;-q)_\infty}{((-q)^2;(-q)^2)_\infty^2} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}$$

which is strictly positive. □

In the remainder of this section, we study the properties of $M(m,n)$ modulo 4. Our first result is the following.

Theorem 4.2. *For all nonnegative integers n,*

$$M(1,4,n) = M(3,4,n).$$

Proof. Substituting $z=i$ into (1.4), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\left((M(0,4,n)-M(2,4,n)) + i(M(1,4,n)-M(3,4,n))\right)q^n\\ &= \sum_{n=0}^{\infty}\left(\sum_{m=-\infty}^{\infty} i^m M(m,n)\right)q^n\\ &= \left(\frac{(q;q)_\infty(q^2;q^2)_\infty}{(iq,q/i;q)_\infty(iq^2,q^2/i;q^2)_\infty}\right)^2\\ &= \frac{(q;q)_\infty^2(q^2;q^2)_\infty^4}{(q^8;q^8)_\infty^2}.\end{aligned}$$

Since the final expression is purely real, we conclude that $M(1,4,n)=M(3,4,n)$. □

An immediate consequence is the following.

Corollary 4.3. *The number of cubic partition pairs with odd cranks is always even.*

As a result of Theorem 4.2 and (1.5), we have

$$\sum_{n=0}^{\infty}\left(M(0,4,n)-M(2,4,n)\right)q^n=\frac{(q;q)_\infty^2(q^2;q^2)_\infty^4}{(q^8;q^8)_\infty^2},$$

$$\sum_{n=0}^{\infty}\left(M(0,4,n)+M(2,4,n)-2M(1,4,n)\right)q^n=\frac{(q;q)_\infty^6(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^4}.$$

So if we define $M_{01,4}(n):=M(0,4,n)-M(1,4,n)$, then

$$F(q):=\sum_{n=0}^{\infty}2M_{01,4}(n)q^n=\frac{(q;q)_\infty^2(q^2;q^2)_\infty^4}{(q^8;q^8)_\infty^2}+\frac{(q;q)_\infty^6(q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^4}.$$

To prove Theorem 1.4, it suffices to show

$$M_{01,4}(2n+1)\equiv 0 \pmod 4.$$

To this end, we first note that

$$\begin{aligned}(q;q)_\infty^2-(-q;-q)_\infty^2&=(q;q)_\infty^2-\frac{(q^2;q^2)_\infty^6}{(q;q)_\infty^2(q^4;q^4)_\infty^2}\\&=-4q\frac{(q^2;q^2)_\infty(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty},\end{aligned}\tag{4.3}$$

where we have used the second equality of Lemma 2.1. On the other hand, Lemma 2.2 allows us to write

$$\begin{aligned}(q;q)_\infty^4+(-q;-q)_\infty^4&=(q;q)_\infty^4+\frac{(q^2;q^2)_\infty^{12}}{(q;q)_\infty^4(q^4;q^4)_\infty^4}\\&=2\frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^4}.\end{aligned}\tag{4.4}$$

Combining (4.3) and (4.4), we get

$$\begin{aligned}&(q;q)_\infty^6-(-q;-q)_\infty^6\\&=((q;q)_\infty^2-(-q;-q)_\infty^2)((q;q)_\infty^4+(q;q)_\infty^2(-q;-q)_\infty^2+(-q;-q)_\infty^4)\\&=-4q\frac{(q^2;q^2)_\infty(q^{16};q^{16})_\infty^2}{(q^8;q^8)_\infty}\left(2\frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^2(q^8;q^8)_\infty^4}+\frac{(q^2;q^2)_\infty^6}{(q^4;q^4)_\infty^2}\right)\\&=-4q\left(2\frac{(q^4;q^4)_\infty^{10}(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty(q^8;q^8)_\infty^5}+\frac{(q^2;q^2)_\infty^7(q^{16};q^{16})_\infty^2}{(q^4;q^4)_\infty^2(q^8;q^8)_\infty}\right).\end{aligned}$$

As

$$F(q)-F(-q)=\sum_{n=0}^{\infty}4M_{01,4}(2n+1)q^{2n+1},$$

we have arrived at

$$\begin{aligned}-\sum_{n\geq 0}M_{01,4}(2n+1)q^n&=\frac{(q;q)_\infty^5(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty^3}+2\frac{(q;q)_\infty(q^2;q^2)_\infty^6(q^8;q^8)_\infty^2}{(q^4;q^4)_\infty^5}+\frac{(q;q)_\infty^9(q^8;q^8)_\infty^2}{(q^2;q^2)_\infty^6(q^4;q^4)_\infty}\\&=:D_1(q)+2D_2(q)+D_3(q).\end{aligned}$$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Note that the parities of $D_1(q)$, $D_2(q)$, and $D_3(q)$ are the same as

$$D_j(q) \equiv (q;q)_\infty (q^8;q^8)_\infty \pmod 2$$

from the binomial theorem. Moreover, since

$$(q^2;q^2)_\infty^2 \equiv (q;q)_\infty^4 \pmod 4,$$

we deduce that

$$\begin{aligned} D_3(q) &= \frac{(q;q)_\infty^9 (q^8;q^8)_\infty^2}{(q^2;q^2)_\infty^6 (q^4;q^4)_\infty} \\ &\equiv \frac{(q;q)_\infty^5 (q^8;q^8)_\infty^2}{(q^2;q^2)_\infty^4 (q^4;q^4)_\infty} \pmod 4 \\ &\equiv \frac{(q;q)_\infty^5 (q^8;q^8)_\infty^2}{(q^4;q^4)_\infty^3} = D_1(q) \pmod 4. \end{aligned}$$

Let $d_j(n)$ be the n-th Fourier coefficient of $D_j(q)$. Suppose that $d_2(n)$ is even. Then $d_1(n) \equiv d_3(n) \equiv 0$ or $2 \pmod 4$, and thus

$$M_{01,4}(2n+1) = d_1(n) + 2d_2(n) + d_3(n) \equiv 0 \pmod 4.$$

Suppose on the other hand that $d_2(n)$ is odd. Then $d_1(n) \equiv d_3(n) \equiv 1$ or $3 \pmod 4$, and so

$$M_{01,4}(2n+1) = d_1(n) + 2d_2(n) + d_3(n) \equiv 1+2+1 \text{ or } 3+2+3 \pmod 4. \qquad \square$$

We end this section with the observation that

$$\begin{aligned} \sum_{n=1}^{\infty} 2(M(1,4,n) - M(2,4,n))q^n &= \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^4}{(q^8;q^8)_\infty^2} - \frac{(q;q)_\infty^6 (q^2;q^2)_\infty^2}{(q^4;q^4)_\infty^4} \\ &\equiv \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^4}{(q^8;q^8)_\infty^2} - \frac{(q;q)_\infty^2 (q^2;q^2)_\infty^4}{(q^4;q^4)_\infty^4} \equiv 0 \pmod 4 \end{aligned}$$

which yields the following.

Corollary 4.4. *For all nonnegative integers n,*

$$M(1,4,n) \equiv M(2,4,n) \pmod 2.$$

5. Connection to other partition functions

There is some surprising connection between $c(n)$ and other partition functions. To explain this connection, we first define a companion function $\tilde{c}(n)$ by its generating function

$$f_{\tilde{c}}(\tau) := \frac{\eta_4^{10}\eta_{16}^4}{\eta_8^{10}} = \sum_{n=0}^{\infty} \tilde{c}(n) q^{4n-1}.$$

Analogous to $f_c(z)$ defined in the previous section, $f_{\tilde{c}}(\tau)$ is also a weight 2 modular form and arises from applying the Θ-operator to some modular function of weight 0.

Lemma 5.1.

$$\Theta\left(\frac{\eta_1^2\eta_4^2\eta_{16}^2}{\eta_2\eta_8^5}\right) = -2\frac{\eta_4^{10}\eta_{16}^4}{\eta_8^{10}}.$$

As a result, $\tilde{c}(n)$ satisfies the following analogue of Theorem 1.2.

Theorem 5.2. *For an odd integer $m > 1$, if $\ell \in \{1, 2, \ldots, m-1\}$ is the unique integer satisfying $-4\ell \equiv 1 \pmod{m}$, then*

$$\tilde{c}(mn+\ell) \equiv 0 \pmod{m}.$$

In [19], L. Wang studied arithmetic properties of the $\mathrm{pod}_{-4}(n)$ function. This function counts the number of partition quadruples of n where the odd parts are distinct. Wang observed that a result from [18, Cor. 2.4] can be interpreted as

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(n) q^n := \frac{(q^2;q^2)_\infty^4}{(q;q)_\infty^4 (q^4;q^4)_\infty^4}$$
$$= \frac{(q^4;q^4)_\infty^{10}}{(q^2;q^2)_\infty^{10} (q^8;q^8)_\infty^4} + 4q \frac{(q^8;q^8)_\infty^4}{(q^2;q^2)_\infty^6 (q^4;q^4)_\infty^2}.$$

In particular, this means that

$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(2n) q^n = \frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^{10} (q^4;q^4)_\infty^4} = \left(\sum_{n=0}^{\infty} \tilde{c}(n) q^n \right)^{-1},$$
$$\sum_{n=0}^{\infty} \mathrm{pod}_{-4}(2n+1) q^n = 4 \frac{(q^4;q^4)_\infty^4}{(q;q)_\infty^6 (q^2;q^2)_\infty^2} = 4 \left(\sum_{n=0}^{\infty} c(n) q^n \right)^{-1}.$$

6. Concluding Remarks

It seems that there is a sign pattern with period 16. Namely, apart from a few initial exceptions, $M(0,4,n) > M(2,4,n)$ when $n \equiv 0, 1, 2, 3, 4, 9, 11, 14 \pmod{16}$ and $M(0,4,n) < M(2,4,n)$ for the remaining congruence classes. An asymptotic proof using the classical circle method should be possible.

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References

[1] G.E. Andrews, F.G. Garvan, *Dyson's crank of a partition*, Bull. Amer. Math. Soc. **18** (1988), 167–171.
[2] B.C. Berndt, *Number theory in the spirit of Ramanujan*, American Mathematical Society, Providence, RI, 2006.
[3] H.-C. Chan, *Ramanujan's cubic continued fraction and an analog of his "most beautiful identity"*, Int. J. Number Theory **6** (2010), 673–680.
[4] H.-C. Chan, *Ramanujan's cubic continued fraction and Ramanujan type congruences for a certain partition function*, Int. J. Number Theory **6** (2010), 819–834.
[5] H.-C. Chan, *Distribution of a certain partition function modulo powers of primes*, Acta Math. Sin. (Engl. Ser.) **27** (2011), 625–634.
[6] S.H. Chan, *Generalized Lambert series identities*, Proc. Lond. Math. Soc. **91** (2005), 598–622.
[7] D. Choi, S.-Y. Kang, and J. Lovejoy, *Partitions weighted by the parity of the crank*, J. Combin. Theory Ser. A **116** (2009), 1034–1046.
[8] Y.-S. Choi, *Tenth order mock theta functions in Ramanujan's Lost Notebook III*, Proc. London Math. Soc. (3) **94** (2006), 26–52.
[9] N.J. Fine, *Basic hypergeometric series and applications*, Mathematical Surveys and Monographs **27**, American Mathematical Society, Providence, RI, 1988.

[10] F.G. Garvan, *New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11*, Tran. Amer. Math. Soc. **305** (1988), 47–77.
[11] B. Kim, *A crank analog on a certain kind of partition function arising from the cubic continued fraction*, Acta Arith. **148** (2011), 1–19.
[12] B. Kim, *Partition statistics for cubic partition pairs*, Electron. J. Combin. **18** (2011), P128.
[13] B. Kim, *Cubic partition pairs weighted by the parity of the crank*, Korean J. Math. **23** (2015), 637–642.
[14] O. Kolberg, *Some identities involving the partition function*, Math. Scand. **5** (1957), 77–92.
[15] K. Ono, *Web of Modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, 102. Published for the Conference Board of the Mathematical Sciences, Washington, DC, by the American Mathematical Society, Providence, RI, 2004.
[16] R.A. Rankin, *Modular forms and functions*, Cambridge University Press, Cambridge, 1977
[17] S. Robins, *Generalized Dedekind η-products*, Contemp. Math. **166** (1994), 119-128.
[18] P.C. Toh, *Ramanujan type identities and congruences for partition pairs*, Discrete Math. **312** (2012), 1244–1250.
[19] L. Wang, *Arithmetic properties of partitions quadruples with odd parts distinct*, Bull. Aust. Math. Soc. **92** (2015), 353–364.
[20] E.T. Whittaker, G.N. Watson, *A course of modern analysis,* 4th ed., Cambridge University Press, Cambridge, 1927.
[21] H. Zhao and Z. Zhong, *Ramanujan type congruences for a certain partition function*, Electron. J. Combin. **18** (2011), P58.

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