DESIGNING TASKS FOR CONJECTURING AND PROVING IN NUMBER THEORY

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The purpose of this study is to develop design principles for crafting tasks that will encourage conjecturing and proving in the context of elementary number theory at the undergraduate level. From the analyses of the written work of 46 prospective mathematics teachers on a task designed according to these principles, we think that there is potential to build on and refine from these principles for other undergraduate mathematics courses.

INTRODUCTION

> Paul Erdös, one of the greatest mathematicians of the twentieth century, and certainly the most eccentric … believed that the meaning of life was to prove and conjecture. (Schechter, 2000)

Most mathematicians would agree that making conjectures and then proving them is an indispensable component of practicing mathematics. The acts of conjecturing and proving also have immense educational value. The NCTM *Principles and Standards for School Mathematics* states that school programs at all levels should enable students to "recognize reasoning and proof as fundamental aspects of mathematics; make and investigate mathematical conjectures; develop and evaluate mathematical arguments and proofs." (NCTM, 2000, p. 56). This is echoed by Lin et al. (2012, p. 308) who argued that "tasks of conjecturing and proving should be designed to be embedded into any level of mathematics classes in order to enhance students' conceptual understanding, procedural fluency, or problem solving." In addition, it would seem that for these acts of conjecturing and proving to be actualised in schools, it is even more imperative that prospective mathematics teachers should learn them in their mathematics training. This paper describes an attempt to develop design principles for crafting tasks that will encourage conjecturing and proving in an elementary number theory course for undergraduate prospective teachers.

BACKGROUND

It is widely accepted that the act of proving enhances students' mathematical concepts and reasoning (Hanna, 2000), however the enactment in the curriculum sometimes result in students possessing a distorted view of what constitutes a mathematical proof. Selden (2012) suggests that the requirement to construct two-column geometry proofs may be partially responsible for some students' perception that proofs are always constructed in a linear fashion. Hoyles (1997) argues that students see little meaning and purpose in the act of proving mathematical statements, especially those which they already assumed to be true. Schoenfeld, after a series of studies exploring students' understanding of geometry, formulated these erroneous students' beliefs: (1) The processes of formal mathematics (e.g. "proof") have little or nothing to do with discovery or invention. (2) Students who understand the subject matter can solve assigned mathematics problems in five minutes or less. (3) Only geniuses are capable of discovering, creating, or really understanding mathematics (1988, p. 151).

One possible remedy to address these wrong perceptions is to provide students with opportunities to formulate and prove their own conjectures (Lin et al., 2012). As teacher educators, we recognize that correcting these perceptions in prospective teachers is a crucial step in arresting the propagation of these erroneous beliefs. The design of suitable tasks to elicit these dispositions from the prospective teachers is an important step towards this end.

This study arose from the efforts of the first author – henceforth referred to in the first person singular pronoun – to design tasks to promote conjecturing and proving in an undergraduate elementary number theory course for prospective teachers. It is widely accepted that elementary number theory provides an appropriate context for undergraduates to learn proofs and engage in conjecturing (Ferrari, 2002; Selden & Selden 2002; Zazkis & Campbell, 2006). Thus I incorporated into the course problem solving tasks that explicitly required the prospective teachers to engage in conjecturing and proving. The tasks were designed according to these principles: (1) In line with the content emphasis of the course, the problem should require the content and techniques typical to undergraduate number theory courses; (2) The problem should lend itself to the motivation for prospective teachers to actively propose conjectures that is part of the process of solving the problem; in other words, we avoid problems that are too closed-ended – such as the conventional proof problems where the statement to be proven is given and thus there is no room for conjecturing; (3) the problem should be set at the right 'level'; it should not be deemed too inaccessible for most of the students to the point that they do not feel encouraged to even try conjecturing; on the other hand, there should be sufficient cognitive demand in the problem to render the task of solving meaningful; (4) the problem should be unfamiliar to the prospective teachers and not easily found in public media. This is to reduce the likelihood of prospective teachers resorting to duplicating solutions found elsewhere and as such blunt their motivation to attack the problem through conjecturing and proving for themselves.

In crafting these design principles, I relied on prior experiences teaching this course. It is heartening to note that these principles were in line with the characterisation *of open problems* as described by Furinghetti and Paola (2003), as well as several of the principles proposed by Lin et al. (2012). While the principles they stated were generic in nature, my motivation in deriving the design principles were for their specific relevance in the teaching of undergraduate-level number theory.

DATA SOURCE AND THE PROBLEM SOLVING TASK

The aim of our study is to find out whether a problem crafted based on the design principles stated in the previous section will be efficacious, that is, whether it will bring about productive conjectures and motivation for proving these conjectures in the prospective teachers' attempts at the problem.

This study took place in a first year undergraduate number theory course for 46 prospective teachers. The undergraduate programme for these prospective teachers is structured in such a way that they first learn mathematics content before they learn the pedagogical aspects concerning the teaching of the subject. Thus, during their first year, the academic profiles of these prospective teachers are typical to that of an undergraduate mathematics major. Prior to this course, these prospective teachers had already read introductory calculus and introductory linear algebra. In addition, the first two weeks of this 13 week number theory course were devoted to methods of proof.

Our data is taken from the following problem solving task assigned to the prospective teachers near the end of the course:

> Problem: An L-Shaped number is one that can be written as a difference of two squares. For example, $3 = 2^2-1^2$ and $21 = 5^2-2^2$ are L-Shaped numbers but *1* and *2* are not. Note that we do not consider *0* as a square. Can you describe as completely as possible, which natural numbers are L-Shaped numbers? (You should include proofs as necessary.)

A diagram illustrating the geometric interpretation of L-shaped numbers (Figure 1) accompanied the description of the task.

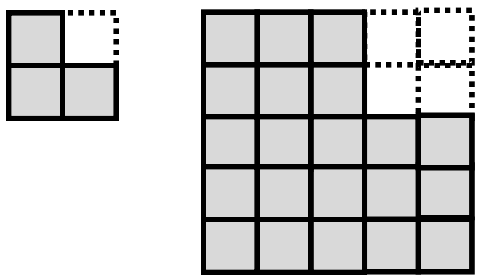


Figure 1: Geometric interpretation of L-Shaped numbers.

The prospective teachers were given two weeks to complete the task on a *practical worksheet*, an instructional scaffold designed to develop problem solving disposition based on Pólya's problem solving model (Pólya, 1945) and Schoenfeld's problem solving framework (Schoenfeld, 1985). It contains sections that explicitly guide students to use the Pólya stages. The practical worksheet was used as part of the prospective teachers' task as it provided a useful aid for their conjecturing and proving in the process of solving the problem. An account of how the practical worksheet was used to develop problem solving disposition in a previous iteration of the same number theory course can be found in Toh et al. (in press).

It is clear that the task coheres with the design principles mentioned in the previous section: (1) an essential step in the complete solution require parity arguments which is a typical technique in number theory courses – and this point will be elaborated in the context of discussion of solutions later; (2) the open nature of the problem requires the subjects' active proposal of conjectures; (3) there are multiple entry levels into the problems – such as proceeding geometrically first, or just listing examples to observe a pattern – and thus encourage the prospective teachers to make good attempts at conjecturing and proving; (4) To the best of my knowledge, this problem is not found in the open media. In fact, this problem is a substantial adaptation from another problem I came across in the Singapore Mathematical Olympiads.

ANALYSIS OF PROSPECTIVE TEACHERS' WORKSHEETS

All except one prospective teacher submitted their solution attempts. Their worksheets were analysed and coded according to whether they made one or more of three conjectures that are productive towards the complete solution of the problem, and whether they were able to provide a valid proof of their conjectures.

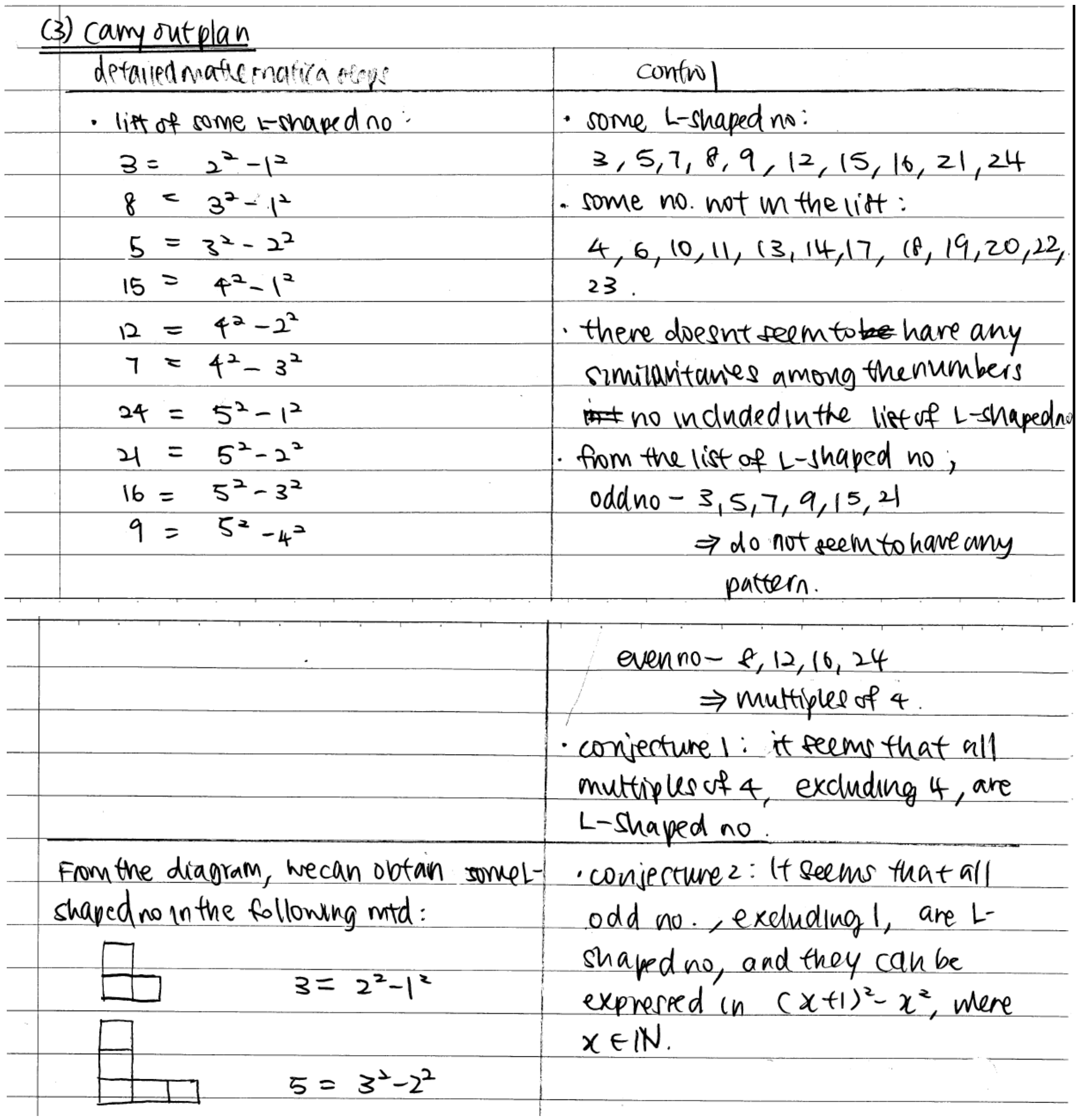
Possible Conjectures	Made the conjecture	Provided valid proof
All odd numbers, with the exception of *1*, are L-shaped numbers	43	34
All even numbers which are multiples of *4*, with the exception of *4*, are L-shaped numbers	39	22
All even numbers which are *not* multiples of *4* are *not* L-shaped numbers	15	3

Table 1: Conjectures made and proved by prospective teachers.

About conjecturing

Most of the prospective teachers began by listing examples of L-shaped numbers and attempted to seek patterns from the list. All but two of the prospective teachers managed to observe that all odd numbers that are greater than *1*, are L-shaped. A total of 39 prospective teachers also made the second conjecture that every even multiple of *4*, with the exception of *4*, are L-shaped.[1] Figure 2 provides an example of a prospective teacher who wrote the two conjectures clearly. It is noteworthy that – as we anticipated under Design Principle (3) – entries made into the problem include the technique of listing and geometrical approaches.

[1] A mathematical point: Both *1* and *4* are not considered L-shaped numbers because of the explicit requirement that *0* is not a square. This additional constraint caused some difficulties for a small number of student teachers. However, since we were interested in the broad conjectures made by the student teachers, we did not make any distinction between students who included or ignored these two exceptional cases.



(3) carry out plan

detailed mathematical steps	control
• list of some L-shaped no :	• some L-shaped no:
3 = 2^2-1^2	3, 5, 7, 8, 9, 12, 15, 16, 21, 24
8 = 3^2-1^2	• some no. not in the list :
5 = 3^2-2^2	4, 6, 10, 11, 13, 14, 17, 18, 19, 20, 22,
15 = 4^2-1^2	23.
12 = 4^2-2^2	• there doesnt seem to have any
7 = 4^2-3^2	similaritaries among the numbers
24 = 5^2-1^2	no included in the list of L-shaped no
21 = 5^2-2^2	• from the list of L-shaped no,
16 = 5^2-3^2	odd no – 3, 5, 7, 9, 15, 21
9 = 5^2-4^2	⇒ do not seem to have any
	pattern.
	even no – 8, 12, 16, 24
	⇒ multiples of 4.
	• conjecture 1 : it seems that all
	multiples of 4, excluding 4, are
	L-shaped no.
From the diagram, we can obtain some L-shaped no in the following mtd:	• conjecture 2 : It seems that all
	odd no., excluding 1, are L-
$3 = 2^2-1^2$	shaped no, and they can be
	expressed in $(x+1)^2 - x^2$, where
$5 = 3^2-2^2$	$x \in \mathbb{N}$.

Figure 2: Prospective teacher formulating two conjectures.

About proofs

Almost 80% of the 43 prospective teachers managed to prove their conjecture that all odd numbers are L-shaped. For the second conjecture, a smaller albeit still significant 56% of those who made the conjecture provided valid proofs. Examples of correct proofs of the two conjectures are given in Figure 3.

Among the prospective teachers who failed to produce a valid proof of their conjectures, more than 75% chose to tackle the problem from the algebraic definition of an L-shaped number as $n = a^2 – b^2$. They then proceeded to consider all the possible parities of a and b which lead to the conclusion that L-shaped numbers are either odd or multiples of 4. These are actually the converses of the first two conjectures shown in Table 1; in other words, instead of proving that *every* odd number and *every* multiple of *4* is an L-shaped number, they showed that an L-shaped number must be *some* odd number or *some* multiple of *4*. It is possible that some prospective teachers were not aware of the differences. Another explanation for this discrepancy between proof and conjecture is perhaps the lack of sufficient resources to prove their conjectures. They focused on the definition of L-shaped numbers and attempted to deduce whatever implications they could, and stopped once they arrived at some plausible conclusions, without checking whether their conclusions were aligned to their conjectures. This is in line with the observations of Selden et al. (2010) of some students' preference for immediately examining the hypothesis without considering the conclusion to be proved.

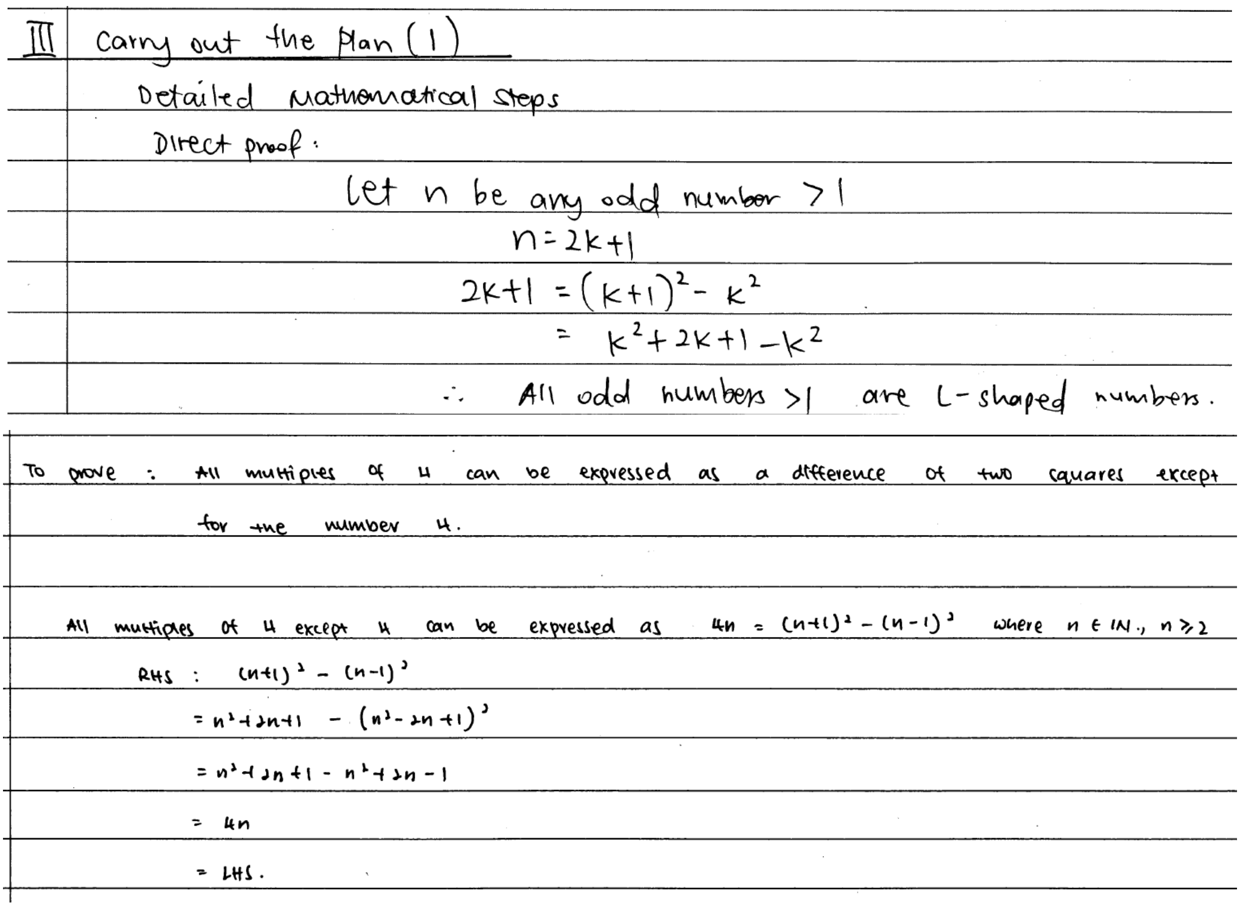


Figure 3: Examples of correct proofs of the two conjectures.

Only 15 out of 45 prospective teachers explicitly stated, in some form or other, the third conjecture that even numbers which are *not* multiples of *4* are *not* L-shaped. Proving this conjecture – together with the previous two – would have completed the solution to the problem. We believe there are two plausible reasons for the relatively small number of students who stated this conjecture: the first is related to the problem of distinguishing a statement and its converse, as discussed earlier; the second was the given instruction to "*… describe as completely as possible, which natural numbers are L-Shaped numbers?*" Prospective teachers may interpret it literally that they need not consider those numbers which are not L-shaped.

DISCUSSION

We set out to study how prospective teachers would respond to a problem that was meant to encourage conjecturing and proving. We crafted the problem based on the design principles as stated in an earlier section of this paper. As seen from Table 1, we note that most of the prospective teachers were able to formulate correctly the first two conjectures and a majority managed to provide a valid proof. We derive encouragement from this result. We interpret this finding to mean that there is potential in these design principles in developing problems that will be helpful for prospective teachers to practise conjecturing and proving. In future research, we intend to replicate these principles and perhaps refine them to elicit better responses from the prospective teachers.

We also notice that the reason a significant proportion of prospective teachers failed to provide a correct proof was due to their attempts at proving the converse instead. This finding reveals a gap in prospective teachers' ability to make a distinction between necessary and sufficient conditions of a mathematical statement. From the perspective of teacher educators, there is a need to respond to this phenomenon. The responses can be in these forms: (1) In the regular teaching of mathematics courses, there should be more opportunities for prospective teachers to make judgments of statements and their converses; (2) in the design of the problems, we should be cognisant of these gaps in their knowledge. The errors made in confusing the necessary and sufficient conditions are opportunities for us to address these deficiencies. In general, we can include this in the list of design principles for problems: We should take into account prospective teachers' errors in the choice of problems so that their solution attempts would reveal the errors and thus provide a motivation for us to address them accordingly.

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