WHAT IS AN IRRATIONAL NUMBER?

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We learnt in school that a real number can be classified as either a *rational* or an *irrational* number. A number is rational, not because it is logical or reasonable but rather that it is the *ratio* of two integers. To be precise, a number is rational if it can be expressed as $\frac{a}{b}$, where a and b are integers, with the additional requirement that b is not zero.

What about irrational numbers? Correspondingly, these are numbers that *cannot* be expressed as a ratio of two integers. We should tread with caution when dealing with definitions that characterize the *absence* of certain properties. If a child asks "what is a cat?", it is more helpful to say a cat has four legs and whiskers than to say a cat has no wings and does not bark, or worse, to say a cat is not a dog.

A concept defined by clear conditions like a rational number is relatively easy to understand. We can produce many examples like $\frac{1}{2}$, $\frac{2}{3}$, $\frac{-3}{4}$ or $\frac{17}{11}$. In fact, any integer n is a rational number, since $n = \frac{n}{1}$. But how do we understand a number that *cannot* be expressed as a ratio of integers? It begs the question *what then can we express it as*? Textbooks inform us that $\sqrt{2}$ and π are examples of irrational numbers but usually do not explain why such numbers cannot be expressed as a ratio of integers. The reason for the omission is probably because proving that a number is irrational can be quite difficult and may involve sophisticated mathematics. A famous example is that of the Euler-Mascheroni constant,

$$\gamma = 0.5772156649\ldots.$$

At the time of writing, no mathematician in the world has managed to prove conclusively that this number is irrational!

So are we resigned to accept that we cannot have a good idea of what is an irrational number? Fortunately, there is an alternative way to think about such numbers. In the example above, although understanding the nature of the Euler-Mascheroni constant is difficult, it is still possible to represent it using decimal digits. This leads us to consider the decimal representations of some rational and irrational numbers. (It is worth pointing out that we are assuming that every real number has at least one decimal representation.)

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$$\begin{aligned}
\frac{1}{2} &= 0.5,\\
\frac{2}{3} &= 0.6666666666\ldots,\\
\frac{17}{11} &= 1.5454545454\ldots,\\
\sqrt{2} &= 1.4142135623\ldots,\\
\pi &= 3.1415926535\ldots.
\end{aligned}$$

Besides $\frac{1}{2}$, the other four numbers all have decimal representations that appear to go on indefinitely. We have only displayed the first ten digits (without rounding) after the decimal point. You are free to try it out on your own calculator but your results may differ slightly because your calculator has limited precision and makes rounding adjustments.

Apart from the never ending string of decimals, what other patterns do you observe? Science proceeds along the cycle of Observation–Hypothesis–Validation. Likewise, mathematics progresses through the cycle of Observation–Conjecture–Proof. Mathematicians conjecture (which is another word for guess or hypothesize) that certain properties or results are true, usually through studying sufficient data or examples, and then attempt to prove the conjecture. If the proof is accepted as rigorous and correct, the conjecture becomes a theorem.

Returning to the previous five decimal representations, a reasonable guess could be that the decimal representation of a rational number either stops or terminates at some point like $\frac{1}{2}$ or it goes on indefinitely but eventually exhibits a repeating pattern like that of $\frac{2}{3}$ and $\frac{17}{11}$. On the other hand, the decimal representations of irrational numbers goes on indefinitely without any discernible pattern. Before proceeding further, it is prudent for us to verify our conjecture with more data. First of all, we based our guess on only the first ten digits. So, we should use a computer or other available means to check further. The following shows the decimal representation of the four numbers displayed to twenty digits.

$$\begin{aligned}
\frac{2}{3} &= 0.66666666666666666666\ldots,\\
\frac{17}{11} &= 1.54545454545454545454\ldots,\\
\sqrt{2} &= 1.41421356237309504880\ldots,\\
\pi &= 3.14159265358979323846\ldots.
\end{aligned}$$

The pattern appears to hold. Next, we should check the decimal representations of other rational and irrational numbers. Once we are sufficiently confident that our guess is valid, we should write it down precisely so that we can attempt to prove it. There are a number of ways to state our conjecture and the following is one.

Conjecture 1. *If the decimal representation of a real number terminates or is eventually repeating, then that number is a rational number.*

We shall not prove Conjecture 1 but provide a couple of examples to illustrate why it is true. First of all, consider a real number $x = 3.141$ where the decimal representation terminates after three digits. This means that x can be expressed as $\frac{3141}{1000}$, a rational number. It is not hard to see that this idea extends to other real numbers with a decimal representation that terminates.

Now, consider a real number $y = 7.123123123\ldots$, where the three digits 123 repeat indefinitely. Multiplying y by 1000 and putting the two together,

$$\begin{aligned} 1000y &= 7123.123123123\ldots, \\ y &= \quad\;\; 7.123123123\ldots. \end{aligned}$$

If we take the difference, the digits occurring after the decimal point should cancel out and we end up with $999y = 7116$. Equivalently $y = \frac{7116}{999}$, which is a rational number. Again this idea can be extended to any decimal representation that is eventually repeating.

A rigourous proof of Conjecture 1 using the same ideas can be found in a number of undergraduate mathematics textbooks. We thus upgrade our conjecture and record it as a theorem.

Theorem 2. *If the decimal representation of a real number terminates or is eventually repeating, then that number is a rational number.*

What about irrational numbers? Instead of first posing it as a conjecture, we state the result directly.

Theorem 3. *The decimal representation of an irrational number neither terminates, nor does it eventually repeat.*

We shall prove Theorem 3 using a technique called *proof by contradiction.* This is akin to 'playing devil's advocate'. In other words, when arguing a certain point of view, we take on the opposite position for the sake of debate. The proof of Theorem 3 is as follows. Consider an arbitrary irrational number which we shall call x. We wish to show that the decimal representation of x neither terminates, nor does it eventually repeat. We argue by supposing that it does. That is, the decimal representation of x either terminates or is eventually repeating. Now Theorem 2 states that in either case x is a rational number and this would contradict the fact that x is an irrational number. The only way to resolve this apparent contradiction is to accept that our initial supposition about the decimal representation of x, i.e. it terminates or eventually repeats, was untrue. Hence, Theorem 3 is true.

Two remarks are in order. First, our proof of Theorem 3 may read like we are just playing with words and twisting facts around. It is in fact rigorous and necessary. This is because irrational numbers are defined by what they are not and so we are unable to argue based on properties that an irrational number possesses. The second remark is to point out what we have not proven. In our discussion, we have not shown that the decimal expansion of a rational number either terminates or eventually repeats. This is the converse of Theorem 2 and you are encouraged to think along the lines of long division to show that this is in fact true.

Although irrational numbers remain a difficult concept to understand and we did not get any closer to why $\sqrt{2}$ or π is irrational, nonetheless Theorem 3 provides us with an alternative way to characterize irrational numbers. Instead of saying it cannot be written as a ratio of two integers, we can now say that it is a number whose decimal representation continues indefinitely but does not eventually repeat. We challenge you to come up with an example of an irrational number based on this characterization. (Hint: all you need are the digits 0 and 1.)

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